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On Some Fixed Point Theorems Of Commuting Operators

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Abstract

In this paper, some fixed point theorems of commuting operators defined on a complete 2-metric space are established. These theorems are generalizations to non-expansive self-map of fixed point theorems that were proved by authors like *Rhoades*, *Lal* and *Singh*.

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Introduction:

In 1906, *Frechet* $([1],[2])$ introduced the notion of metric as an abstract generalization of length concept. In 1963, *Gahler* [3] introduced the notion of 2-metric as an abstract generalization of the notion of area function for Euclidean triangles. Many fixed point theorems appeared in 2-metric spaces in later years analogous to the fixed point theorems in metric spaces proved by various authors like *Iseki* [4], *Lal* and *Singh* [6], *Rhoades* [8] etc.

In this present work, we establish some fixed point theorems of commuting operators that are defined on a 2-metric space. In what follows X and $\mathbb{R}$ stand for a non-empty set and the real line respectively.

1. Preliminaries

In this section, we present some basic definitions which are needed for the further study of this paper.

1.1 Definition:

A point $x \in X$ is said to be a *fixed point* of a self-map $f : X \to X$ if $f(x) = x$.

1.2 Definition:

Let X be a non-empty set and $d : X \times X \times X \to \mathbb{R}$. For all x, y, z and u in X, if d satisfies the following conditions

(a) $d(x, y, z) = 0$ if at least two of x, y, z are equal

(b) $d(x, y, z) = d(x, z, y) = d(y, z, x) = ...$

(c) $d(x, y, z) \le d(x, y, u) + d(x, u, z) + d(u, y, z)$

Then d is called a *2-metric* on X and the pair (X, d) is called a *2-metric space.*

1.3 Definition:

Let (X, d) be a 2-metric space. A sequence $\{x_n\}$ in X is called a *Cauchy sequence,* if $d(x_m, x_n, a) \to 0$ as $m, n \to \infty$ for all $a \in X$.

1.4 Definition:

Let (X, d) be a 2-metric space. A sequence $\{x_n\}$ is said to *converge* to a point x in X if $\lim_{n \to \infty} d(x_n, x, a) = 0$ for every a in X.

1.5 Definition:

A 2-metric space (X, d) is said to be a *complete 2-metric space* if every Cauchy sequence in X *converges* in X.

1.6 Definition:

Let (X, d) be a 2-metric space. A mapping $T : X \to X$ is called a *non-expansive mapping* if $d(T(x), T(y), a) \le d(x, y, a)$ for every $x, y, a \in X$.

1.7 Definition:

Two operators T_1 and T_2 defined on a 2-metric space X into itself are said to *commute* if $T_1 T_2 = T_2 T_1$.

2. Some fixed point theorems for commuting Operators

This section is devoted to some fixed point theorems of commuting operators in a complete 2-metric space. We establish these fixed point theorems as generalizations of some fixed point theorems that are already proved by authors like *Rhoades*, *Lal* and *Singh*.

2.1 Theorem[6]:

Let (X,d) be a complete 2-metric space and T_1 and T_2 two self-maps on X such that for all x,y,a in X and positive integers p,q,

$$d\left(T_1^p(x),T_2^q(y),a\right)\le \alpha_1 d\left(x,T_1^p(x),a\right)+\alpha_2 d\left(y,T_2^q(y),a\right)$$

$+\alpha_3 d\left(x,T_2^q(y),a\right)+\alpha_4 d\left(y,T_1^p(x),a\right)+\alpha_5 d\left(x,y,a\right)$, where $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ and α_5 are non-negative constants such that $\sum_{i=1}^{5}\alpha_i<1$ and $(\alpha_1-\alpha_2)(\alpha_3-\alpha_4)\ge 0$. Then T_1 and T_2 have a unique common fixed point in X.

The following theorem 2.2 is an extension and generalization of the above theorem 2.1 to non-expansive self-map T which commutes with the self-maps T_1 and T_2.

2.2 Theorem:

Let p and q be any two positive integers. Let T,T_1 and T_2 be three operators on a complete 2-metric space (X,d) into itself. If

(a) $d\left(T_1^p(x),T_2^q(y),a\right)\le \alpha_1 d\left(T(x),T_1^p\left(T(x)\right),a\right)+\alpha_2 d\left(T(y),T_2^q\left(T(y)\right),a\right)+\alpha_3 d\left(x,y,a\right)+\alpha_4 d\left(T(x),T_2^q\left(T(y)\right),a\right)+\alpha_5 d\left(T(y),T_1^p\left(T(x)\right),a\right)$

For every $x,y,a\in X$, for each $\alpha_i\ge 0$ and $\sum_{i=1}^{5}\alpha_i<1$, $(\alpha_1-\alpha_2)(\alpha_3-\alpha_4)\ge 0$

(b) $d\left(T(x),T(y),a\right)\le d(x,y,a)$ for every $x,y,a\in X$.

(c) $T\,T_i=T_i\,T$ for $i=1,2$

Then there is a unique common fixed point of T,T_1 and T_2 in X.

Proof:

Using conditions (b) and (c), we have

$$d\left(T_1^p(x),T_2^q(y),a\right)\le \alpha_1 d\left(T(x),T_1^p\left(T(x)\right),a\right)+\alpha_2 d\left(T(y),T_2^q\left(T(y)\right),a\right)$$

$$+\alpha_3 d\left(x,y,a\right)+\alpha_4 d\left(T(x),T_2^q\left(T(y)\right),a\right)$$

$$+\alpha_5 d\left(T(y),T_1^p\left(T(x)\right),a\right)$$

$$=\alpha_1 d\left(T(x),T\left(T_1^p(x)\right),a\right)+\alpha_2 d\left(T(y),T\left(T_2^q(y)\right),a\right)$$

$$+\alpha_3 d\left(x,y,a\right)+\alpha_4 d\left(T(x),T\left(T_2^q(y)\right),a\right)$$

$$+\alpha_5 d\left(T(y),T\left(T_1^p(x)\right),a\right)$$

$$\le \alpha_1 d\left(x,T_1^p(x),a\right)+\alpha_2 d\left(y,T_2^q(y),a\right)+\alpha_3 d\left(x,y,a\right)$$

$+\alpha_4 d\left(x, T_2^q(y), a\right) + \alpha_5 d\left(y, T_1^p(x), a\right)$

Then by theorem-2.1, there exist a point x_0 in X, which is a unique common fixed point of T_1 and T_2.

Now we prove that x_0 is a unique common fixed point of T, T_1 and T_2.

We have $d\left(x_0, T\left(x_0\right), a\right) = d\left(T_1^p(x_0), T\left(T_2^q(x_0)\right), a\right)$

$= d\left(T_1^p\left(x_0\right), T_2^q\left(T(x_0)\right), a\right)$

$\leq \alpha_1 d\left(T\left(x_0\right), T_1^p\left(T(x_0)\right), a\right) + \alpha_2 d\left(T^2(x_0), T_2^q\left(T^2(x_0)\right), a\right)$

$+ \alpha_3 d\left(x_0, T\left(x_0\right), a\right) + \alpha_4 d\left(T\left(x_0\right), T_2^q\left(T^2(x_0)\right), a\right) +$

$\alpha_5 d\left(T^2(x_0), T_1^p\left(T(x_0)\right), a\right)$

$= \alpha_1 d\left(T\left(x_0\right), T\left(x_0\right), a\right) + \alpha_2 d\left(T^2\left(x_0\right), T^2\left(x_0\right), a\right) +$

$\alpha_3 d\left(x_0, T\left(x_0\right), a\right) + \alpha_4 d\left(T\left(x_0\right), T^2(x_0), a\right) +$

$\alpha_5 d\left(T^2(x_0), T\left(x_0\right), a\right)$

$= \alpha_3 d\left(x_0, T\left(x_0\right), a\right) + \alpha_4 d\left(T\left(x_0\right), T^2\left(x_0\right), a\right) +$

$\alpha_5 d\left(T^2(x_0), T\left(x_0\right), a\right)$

$= \alpha_3 d\left(x_0, T\left(x_0\right), a\right) + (\alpha_4 + \alpha_5) d\left(T^2(x_0), T(x_0), a\right)$

$\leq \alpha_3 d\left(x_0, T\left(x_0\right), a\right) + (\alpha_4 + \alpha_5) d\left(T(x_0), x_0, a\right)$

$= (\alpha_3 + \alpha_4 + \alpha_5) d\left(x_0, T(x_0), a\right)$

$\Rightarrow (1 - \alpha_3 - \alpha_4 - \alpha_5) d\left(x_0, T(x_0), a\right) \leq 0$ for every a in X

$\Rightarrow d\left(x_0, T(x_0), a\right) = 0$ for every a in X

$\Rightarrow x_0 = T(x_0)$.

Hence x_0 is a common fixed point of T, T_1 and T_2.

Let y_0 be another common fixed point of T, T_1 and T_2 in X.

Then $d\left(x_0, y_0, a\right) = d\left(T_1^p(x_0), T_2^q(y_0), a\right)$.

$\leq \alpha_1 d\left(T(x_0), T_1^p\left(T(x_0)\right), a\right) + \alpha_2 d\left(T(y_0), T_2^q\left(T(y_0)\right), a\right)$

$+ \alpha_3 d\left(x_0, y_0, a\right) + \alpha_4 d\left(T(x_0), T_2^q\left(T(y_0)\right), a\right)$

$+ \alpha_5 d\left(T(y_0), T_1^p\left(T(x_0)\right), a\right)$

$= (\alpha_3 + \alpha_4 + \alpha_5) d\left(x_0, y_0, a\right)$

$\Rightarrow (1 - \alpha_3 - \alpha_4 - \alpha_5) d\left(x_0, y_0, a\right) \leq 0$

$\Rightarrow d\left(x_0, y_0, a\right) = 0$ for every a in X

$\Rightarrow x_0 = y_0$.

Thus x_0 is a unique common fixed point for T, T_1 and T_2.
Hence T, T_1 and T_2 have a unique common fixed point in X.
The second condition means T is non-expansive. This by itself would not ensure a fixed point for T.

2.3 Theorem[8]&[9]:
Let (X,d) be complete 2-metric space. Let p and q be any two positive integers and $0<h<1$. If T_1 and T_2 are any two self-maps on X such that $d\left(T_1^p(x)T_2^q(y),a\right) \le h \max$

$$\left\{d\left(x,y,a\right), d\left(x,T_1^p(x),a\right), d\left(y,T_2^q(y),a\right), \frac{1}{2}\left(d\left(x,T_2^q(y),a\right)+d\left(y,T_1^p(x),a\right)\right)\right\}$$

for every $x, y, a \in X$ then there exists a unique common fixed point for T_1 and T_2 in X.
The following theorem 2.4 is an extension and generalization of the above theorem 2.3 to non-expansive self-map T which commutes with the self-maps T_1 and T_2.

2.4 Theorem:
Let p and q be any two positive integers and $0<h<1$. Let T, T_1 and T_2 be three operators on a complete 2-metric space (X,d) into itself. If

(a) $d\left(T_1^p(x),T_2^q(y),a\right) \le h \max\left\{ d\left(T(x),T_1^p\left(T(x)\right),a\right), d\left(T(y),T_2^q\left(T(y)\right),a\right),\right.$
$d\left(T(x),T(y),a\right), ½\left(d\left(T(y),T_1^p\left(T(x)\right),a\right)+\right.$
$\left.\left.d\left(T(x),T_2^q\left(T(y)\right),a\right)\right)\right\}$ for every $x, y, a \in X$

(b) $d\left(T(x),T(y),a\right) \le d\left(x,y,a\right)$ for every $x, y, a \in X$

(c) $TT_i = T_iT$ *and* $i=1,2$

Then there is a unique common fixed point for T, T_1 and T_2 in X.

Proof:
Using conditions (b) and (c), we have
$d\left(T_1^p(x),T_2^q(y),a\right) \le h \max\left\{ d\left(T(x),T_1^p\left(T(x)\right),a\right), d\left(T(y),T_2^q\left(T(y)\right),a\right),\right.$
$d\left(T(x),T(y),a\right), ½\left(d\left(T(y),T_1^p\left(T(x)\right),a\right)+\right.$
$\left.\left.d\left(T(x),T_2^q\left(T(y)\right),a\right)\right)\right\}$ for every $x, y, a \in X$
$= h \max\left\{ d\left(T(x),T\left(T_1^p(x)\right),a\right), d\left(T(y),T\left(T_2^q(y)\right),a\right)\right.$

$d\left(T(x),T(y),a\right)$, $\frac{1}{2}\left(d\left(T(y),T\left(T_1^{p}(x)\right),a\right)+\right.$

$\left.\left.d\left(T(x),T\left(T_2^{q}(y)\right),a\right)\right)\right\}$

$\leq h \max \left\{ d\left(T(x),T_1^{p}(x),a\right), d\left(T(y),T_2^{q}(y),a\right),\right.$

$\left.d\left(T(x),T(y),a\right), \frac{1}{2}\left(d\left(T(y),T_1^{p}(x),a\right)+d\left(T(x),T_2^{q}(y),a\right)\right)\right\}$

Then by theorem-2.3, there exists a point $x_0 \in X$ which is a unique common fixed point for T_1 and T_2.

Now we prove that x_0 is a unique common fixed point of T ,T_1 and T_2. Now we have

$d\left(x_0,T(x_0),a\right)=d\left(T_1^{p}(x_0),T\left(T_2^{q}(x_0)\right),a\right)$

$=d\left(T_1^{p}(x_0),T_2^{q}\left(T(x_0)\right),a\right)$

$\leq h\max\left\{d\left(T(x_0),T_1^{p}\left(T(x_0)\right),a\right), d\left(T^2(x_0),T_2^{q}\left(T^2(x_0)\right),a\right),\right.$

$d\left(T(x_0),T^2(x_0),a\right)$, $\frac{1}{2}\left(d\left(T^2(x_0),T_1^{p}\left(T(x_0)\right),a\right)\right.$ +

$\left.\left.d\left(T(x_0),T_2^{q}\left(T^2(x_0)\right),a\right)\right)\right\}$

$=h\max\left\{d\left(T(x_0),T\left(T_1^{p}(x_0)\right),a\right), d\left(T^2(x_0),T^2\left(T_2^{q}(x_0)\right),a\right),\right.$

$d\left(T(x_0),T^2(x_0),a\right)$, $\frac{1}{2}\left(d\left(T^2(x_0),T\left(T_1^{p}(x_0)\right),a\right)+\right.$

$\left.\left.d\left(T(x_0),T^2\left(T_2^{q}(x_0)\right),a\right)\right)\right\}$

$=h\max\left\{d\left(T(x_0),T(x_0),a\right), d\left(T^2(x_0),T^2(x_0),a\right),\right.$

$d\left(T(x_0),T^2(x_0),a\right)$, $\frac{1}{2}\left(d\left(T^2(x_0),T(x_0),a\right)+\right.$

$\left.\left.d\left(T(x_0),T^2(x_0),a\right)\right)\right\}$

$=h\, d\left(T(x_0),T^2(x_0),a\right)$

$\leq h\, d\left(x_0,T(x_0),a\right)$

$\Rightarrow\ (1-h)\, d\left(x_0,T(x_0),a\right)\leq 0$

$\Rightarrow\ d\left(x_0,T(x_0),a\right)=0$ for every a in X $\Rightarrow\ T(x_0)=x_0$

Hence x_0 is a common fixed of T,T_1 and T_2.

Let y_0 be another common fixed point of T,T_1 and T_2 in X.

Then we have

$d\left(x_0,y_0,a\right)= d\left(T_1^{p}(x_0),T_2^{q}(y_0),a\right)$

$\leq h \max \left\{ d\left(T(x_0),T_1^{p}\left(T(x_0)\right),a\right), d\left(T(y_0),T_2^{q}\left(T(y_0)\right),a\right),\right.$

$d\left(T(x_0),T(y_0),a\right), \frac{1}{2}\left(d\left(T(y_0),T_1^p\left(T(x_0)\right),a\right)+\right.$
$\left.\left.d\left(T(x_0),T_2^q\left(T(y_0)\right),a\right)\right)\right\}$
$= h \max \left\{ d(x_0,x_0,a), d(y_0,y_0,a), d(x_0,y_0,a), \frac{1}{2}\left(d(y_0,x_0,a)+\right.\right.$
$\left.\left.d(x_0,y_0,a)\right)\right\}$
$= h\, d(x_0,y_0,a)$
$\Rightarrow \quad d(x_0,y_0,a)=0$ for every a in $X \Rightarrow \quad x_0 = y_0$
Thus x_0 is a unique common fixed of T, T_1 and T_2 in X.

2.5 Remark:
If we define $T : X \to X$ by $T(x) = x$ for every x in X, then T has infinitely many fixed points in X. But according to theorems 2.2 and 2.4, the self maps T, T_1 and T_2 on X have a unique common fixed point in X together.

2.6 Theorem:
Let (X,d) be a complete 2-metric space and let T_1, T_2, T_3 and T_4 be four self-maps from X into itself satisfying the following conditions.

(a) $\left[d\left(T_1T_2(x),T_3T_4(y),a\right)\right]^2 \leq \alpha_1\left[d(x,y,a)\right]^2$
$+ \alpha_2\left[d\left(x,T_1T_2(x),a\right) d\left(y,T_3T_4(y),a\right)\right]$
$+ \alpha_3\left[d\left(x,T_1T_2(x),a\right) d\left(x,T_3T_4(y),a\right)\right]$
$+ \alpha_4\left[d\left(x,T_1T_2(x),a\right)\right]^2$
$+ \alpha_5\left[d\left(y,T_3T_4(y),a\right) d\left(x,T_3T_4(y),a\right)\right]$
$+ \alpha_6\left[d\left(y,T_3T_4(y),a\right)\right]^2$
$+ \alpha_7\left[d\left(x,T_3T_4(y),a\right) d\left(y,T_1T_2(x),a\right)\right]$
$+ \alpha_8\left[d(x,y,a) d\left(x,T_1T_2(x),a\right)\right]$
$+ \alpha_9\left[d(x,y,a) d\left(y,T_3T_4(y),a\right)\right]$
$+ \alpha_{10}\left[d(x,y,a) d\left(x,T_3T_4(y),a\right)\right]$

for every x, y, a in X and $\alpha_i \geq 0$.

(b) $\sum_{i=1}^{10} \alpha_i < 1$

(c) $T_1T_2 = T_2T_1$ and $T_3T_4 = T_4T_3$.

Then T_1, T_2, T_3 and T_4 have a unique common fixed point in X.

Proof:

Let x_0 be an arbitrary point in X .

We define $x_{2n+1} = T_1T_2\ (x_{2n}), n = 0,1,2...$ and $x_{2n} = T_3T_4\ (x_{2n-1}), n = 1,2...$

Then we have

$$\left[d\left(x_{2n+1}, x_{2n}, a\right)\right]^2 = \left[d\left(T_1T_2(x_{2n}), T_3T_4(x_{2n-1}), a\right)\right]^2$$

$$\le \alpha_1\left[d\left(x_{2n}, x_{2n-1}, a\right)\right]^2$$

$$+\alpha_2\left[d\left(x_{2n}, T_1T_2(x_{2n}), a\right)d\left(x_{2n-1}, T_3T_4(x_{2n-1}), a\right)\right]$$

$$+\alpha_3\left[d\left(x_{2n}, T_1T_2(x_{2n}), a\right)d\left(x_{2n}, T_3T_4(x_{2n-1}), a\right)\right]$$

$$+\alpha_4\left[d\left(x_{2n}, T_1T_2(x_{2n}), a\right)\right]^2$$

$$+\alpha_5\left[d\left(x_{2n-1}, T_3T_4(x_{2n-1}), a\right)d\left(x_{2n}, T_3T_4(x_{2n-1}), a\right)\right]$$

$$+\alpha_6\left[d\left(x_{2n-1}, T_3T_4(x_{2n-1}), a\right)\right]^2$$

$$+\alpha_7\left[d\left(x_{2n}, T_3T_4(x_{2n-1}), a\right)d\left(x_{2n-1}, T_1T_2(x_{2n}), a\right)\right]$$

$$+\alpha_8\left[d\left(x_{2n}, x_{2n-1}, a\right)d\left(x_{2n}, T_1T_2(x_{2n}), a\right)\right]$$

$$+\alpha_9\left[d\left(x_{2n}, x_{2n-1}, a\right)d\left(x_{2n-1}, T_3T_4(x_{2n-1}), a\right)\right]$$

$$+\alpha_{10}\left[d\left(x_{2n}, x_{2n-1}, a\right)d\left(x_{2n}, T_3T_4(x_{2n-1}), a\right)\right]$$

$$= \alpha_1\left[d\left(x_{2n}, x_{2n-1}, a\right)\right]^2$$

$$+\alpha_2\left[d\left(x_{2n}, x_{2n+1}, a\right)d\left(x_{2n-1}, x_{2n}, a\right)\right]$$

$$+\alpha_3\left[d\left(x_{2n}, x_{2n+1}, a\right)d\left(x_{2n}, x_{2n}, a\right)\right]$$

$$+\alpha_4\left[d\left(x_{2n}, x_{2n+1}, a\right)\right]^2$$

$$+\alpha_5\left[d\left(x_{2n-1}, x_{2n}, a\right)d\left(x_{2n}, x_{2n}, a\right)\right]$$

$$+\ \alpha_6\left[d\left(x_{2n-1}, x_{2n}, a\right)\right]^2$$

$$+\alpha_7\left[d\left(x_{2n}, x_{2n}, a\right)d\left(x_{2n-1}, x_{2n+1}, a\right)\right]$$

$$+\alpha_8\left[d\left(x_{2n}, x_{2n-1}, a\right)d\left(x_{2n}, x_{2n+1}, a\right)\right]$$

$$+\alpha_9\left[d\left(x_{2n}, x_{2n-1}, a\right)d\left(x_{2n-1}, x_{2n}, a\right)\right]$$

$$+\alpha_{10}\left[d\left(x_{2n}, x_{2n-1}, a\right)d\left(x_{2n}, x_{2n}, a\right)\right]$$

$$= \alpha_1\left[d\left(x_{2n}, x_{2n-1}, a\right)\right]^2$$

$$+\alpha_2\left[d\left(x_{2n}, x_{2n+1}, a\right)d\left(x_{2n-1}, x_{2n}, a\right)\right]$$

$$+\alpha_4\left[d\left(x_{2n}, x_{2n+1}, a\right)\right]^2$$

$$+\alpha_6\left[d\left(x_{2n-1},x_{2n},a\right)\right]^2$$
$$+\alpha_8\left[d\left(x_{2n},x_{2n-1},a\right)d\left(x_{2n},x_{2n+1},a\right)\right]$$
$$+\alpha_9\left[d\left(x_{2n},x_{2n-1},a\right)\right]^2$$
$$\leq(\alpha_1+\alpha_6+\alpha_9)\left[d\left(x_{2n},x_{2n-1},a\right)\right]^2+\frac{\alpha_2}{2}\left\{\left[d\left(x_{2n},x_{2n+1},a\right)\right]^2+\left[d\left(x_{2n-1},x_{2n},a\right)\right]^2\right\}$$
$$+\alpha_4\left[d\left(x_{2n},x_{2n+1},a\right)\right]^2$$
$$+\frac{\alpha_8}{2}\left\{\left[d\left(x_{2n},x_{2n-1},a\right)\right]^2+\left[d\left(x_{2n},x_{2n+1},a\right)\right]^2\right\}$$
$$=\left(\alpha_1+\alpha_6+\alpha_9+\frac{\alpha_2}{2}+\frac{\alpha_8}{2}\right)\left[d\left(x_{2n},x_{2n-1},a\right)\right]^2$$
$$+\left(\frac{\alpha_2}{2}+\frac{\alpha_8}{2}+\alpha_4\right)\left[d\left(x_{2n},x_{2n+1},a\right)\right]^2$$
$$\Rightarrow\left[1-\left(\frac{\alpha_2}{2}+\frac{\alpha_8}{2}+\alpha_4\right)\right]\left[d\left(x_{2n},x_{2n+1},a\right)\right]^2\leq\left(\alpha_1+\alpha_6+\alpha_9+\frac{\alpha_2}{2}+\frac{\alpha_8}{2}\right)\left[d\left(x_{2n},x_{2n-1},a\right)\right]^2$$
$$\Rightarrow\left[d\left(x_{2n},x_{2n+1},a\right)\right]^2\leq\frac{\left(\alpha_1+\alpha_6+\alpha_9+\frac{\alpha_2}{2}+\frac{\alpha_8}{2}\right)\left[d\left(x_{2n},x_{2n-1},a\right)\right]^2}{\left[1-\left(\frac{\alpha_2}{2}+\frac{\alpha_8}{2}+\alpha_4\right)\right]}$$
$$\Rightarrow\left[d\left(x_{2n+1},x_{2n},a\right)\right]^2\leq k^2\left[d\left(x_{2n},x_{2n-1},a\right)\right]^2$$

Where

$$k^2=\frac{\left(\alpha_1+\alpha_6+\alpha_9+\frac{\alpha_2}{2}+\frac{\alpha_8}{2}\right)}{\left[1-\left(\frac{\alpha_2}{2}+\frac{\alpha_8}{2}+\alpha_4\right)\right]}$$

$$\Rightarrow\ d\left(x_{2n+1},x_{2n},a\right)\leq k\ d\left(x_{2n},x_{2n-1},a\right)$$

Similarly, $d\left(x_{2n},x_{2n-1},a\right)\ \leq k\ \ d\left(x_{2n-1},x_{2n-2},a\right)$

Then we get $d\left(x_{2n+1},x_{2n},a\right)\leq k^2 d\left(x_{2n-1},x_{2n-2},a\right)$

Continuing in this process, we get $d\left(x_{2n+1},x_{2n},a\right)\leq k^{2n}d\left(x_1,x_0,a\right)\ \rightarrow$ (1)

Since $k^2=\dfrac{\left(\alpha_1+\alpha_6+\alpha_9+\frac{\alpha_2}{2}+\frac{\alpha_8}{2}\right)}{\left[1-\left(\frac{\alpha_2}{2}+\frac{\alpha_8}{2}+\alpha_4\right)\right]}\leq\dfrac{\left(\alpha_1+\alpha_3+\alpha_6+\alpha_7+\alpha_9+\alpha_{10}+\frac{\alpha_2}{2}+\frac{\alpha_8}{2}\right)}{\left[1-\left(\frac{\alpha_2}{2}+\frac{\alpha_8}{2}+\alpha_4\right)\right]}<1$

We have $k^{2n}\rightarrow 0$ as $n\rightarrow\infty$

Hence from (1), it follows that $\{x_n\}$ is a Cauchy Sequence in X. Since X is a complete 2-metric space, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

Now we prove that z is a common fixed point of T_1T_2 and T_3T_4.

Now

$$\left[d\left(T_1T_2(z), x_{2n}, a\right)\right]^2 = \left[d\left(T_1T_2(z), T_3T_4(x_{2n-1}), a\right)\right]^2$$
$$\leq \alpha_1 \left[d\left(z, (x_{2n-1}), a\right)\right]^2$$
$$+\ \alpha_2\left[d\left(z, T_1T_2(z), a\right) d\left(x_{2n-1}, T_3T_4(x_{2n-1}), a\right)\right]$$
$$+\ \alpha_3\left[d\left(z, T_1T_2(z), a\right) d\left(z, T_3T_4(x_{2n-1}), a\right)\right]$$
$$+\ \alpha_4\left[d\left(z, T_1T_2(z), a\right)\right]^2$$
$$+\alpha_5\left[d\left(x_{2n-1}, T_3T_4(x_{2n-1}), a\right) d\left(z, T_3T_4(x_{2n-1}), a\right)\right]$$
$$+\ \alpha_6\left[d\left(x_{2n-1}, T_3T_4(x_{2n-1}), a\right)\right]^2$$
$$+\ \alpha_7\left[d\left(z, T_3T_4(x_{2n-1}), a\right) d\left(x_{2n-1}, T_1T_2(z), a\right)\right]$$
$$+\ \alpha_8\left[d\left(z, x_{2n-1}, a\right) d\left(z, T_1T_2(z), a\right)\right]$$
$$+\ \alpha_9\left[d\left(z, x_{2n-1}, a\right) d\left(z, T_3T_4(x_{2n-1}), a\right)\right]$$
$$+\alpha_{10}\left[d\left(z, x_{2n-1}, a\right) d\left(z, T_3T_4(x_{2n-1}), a\right)\right]$$
$$= \alpha_1\left[d\left(z, x_{2n-1}, a\right)\right]^2 + \alpha_2\left[d\left(z, T_1T_2(z), a\right) d\left(x_{2n-1}, x_{2n}, a\right)\right]$$
$$+\alpha_3\left[d\left(z, T_1T_2(z), a\right) d\left(z, x_{2n}, a\right)\right]$$
$$+\alpha_4\left[d\left(z, T_1T_2(z), a\right)\right]^2$$
$$+\alpha_5\left[d\left(x_{2n-1}, x_{2n}, a\right) d\left(z, x_{2n}, a\right)\right]$$
$$+\alpha_6\left[d\left(x_{2n-1}, x_{2n}, a\right)\right]^2$$
$$+\alpha_7\left[d\left(z, x_{2n}, a\right) d\left(x_{2n-1}, T_1T_2(z), a\right)\right]$$
$$+\alpha_8\left[d\left(z, x_{2n-1}, a\right) d\left(z, T_1T_2(z), a\right)\right]$$
$$+\alpha_9\left[d\left(z, x_{2n-1}, a\right) d\left(x_{2n-1}, x_{2n}, a\right)\right]$$
$$+\alpha_{10}\left[d\left(z, x_{2n-1}, a\right) d\left(z, x_{2n}, a\right)\right]$$

Letting $n \to \infty$, we get $\left[d\left(T_1T_2z, z, a\right)\right]^2 \leq \alpha_4\left[d\left(T_1T_2z, z, a\right)\right]^2$

$\Rightarrow \left[d\left(T_1T_2z, z, a\right)\right]^2 = 0$ for every a in X $\Rightarrow$ $T_1T_2z = z$ Similarly $\Rightarrow$ $T_3T_4z = z$.

Hence z is a common fixed point of T_1T_2 and T_3T_4.

Now we prove that z is a fixed point of T_1.

We have

$$\left[d\left(T_1z,z,a\right)\right]^2=\left[d\left(T_1\left(T_1T_2(z)\right),T_3T_4(z),a\right)\right]^2$$
$$=\left[d\left(T_1\left(T_2T_1(z)\right),T_3T_4(z),a\right)\right]^2$$
$$=\left[d\left(T_1T_2\left(T_1(z)\right),T_3T_4(z),a\right)\right]^2$$
$$\leq\alpha_1\left[d\left(T_1(z),z,a\right)\right]^2+\alpha_2\left[d\left(T_1(z),T_1T_2\left(T_1(z)\right),a\right)d\left(z,T_3T_4(z),a\right)\right]$$
$$+\alpha_3\left[d\left(T_1(z),T_1T_2\left(T_1(z)\right),a\right)d\left(T_1(z),T_3T_4(z),a\right)\right]$$
$$+\ \alpha_4\left[d\left(T_1(z),T_1T_2\left(T_1(z)\right),a\right)\right]^2$$
$$+\ \alpha_5\left[d\left(z,T_3T_4(z),a\right)d\left(T_1(z),T_3T_4(z),a\right)\right]$$
$$+\ \alpha_6\left[d\left(z,T_3T_4(z),a\right)\right]^2$$
$$+\alpha_7\left[d\left(T_1(z),T_3T_4(z),a\right)d\left(z,T_1T_2\left(T_1(z)\right),a\right)\right]$$
$$+\ \alpha_8\left[d\left(T_1(z),z,a\right)d\left(T_1(z),T_1T_2\left(T_1(z)\right),a\right)\right]$$
$$+\ \alpha_9\left[d\left(T_1(z),z,a\right)d\left(z,T_3T_4(z),a\right)\right]$$
$$+\alpha_{10}\left[d\left(T_1(z),z,a\right)d\left(T_1(z),T_3T_4(z),a\right)\right]$$
$$=\alpha_1\left[d\left(T_1(z),z,a\right)\right]^2+\alpha_2\left[d\left(T_1(z),T_1(z),a\right)d\left(z,z,a\right)\right]$$
$$+\alpha_3\left[d\left(T_1(z),T_1(z),a\right)d\left(T_1(z),z,a\right)\right]$$
$$+\alpha_4\left[d\left(T_1(z),T_1(z),a\right)\right]^2$$
$$+\alpha_5\left[d\left(z,z,a\right)d\left(T_1(z),z,a\right)\right]$$
$$+\alpha_6\left[d\left(z,z,a\right)\right]^2$$
$$+\alpha_7\left[d\left(T_1(z),z,a\right)d\left(z,T_1(z),a\right)\right]$$
$$+\alpha_8\left[d\left(T_1(z),z,a\right)d\left(T_1(z),T_1(z),a\right)\right]$$
$$+\alpha_9\left[d\left(T_1(z),z,a\right)d\left(z,z,a\right)\right]$$
$$+\alpha_{10}\left[d\left(T_1(z),z,a\right)d\left(T_1(z),z,a\right)\right]$$
$$=\left(\alpha_1+\alpha_7+\alpha_{10}\right)\left[d\left(T_1(z),z,a\right)\right]^2$$

$\Rightarrow\ \left(1-\alpha_1-\alpha_7-\alpha_{10}\right)\left[d\left(T_1(z),z,a\right)\right]^2\leq 0$ for every a in X .

$\Rightarrow\ \left[d\left(T_1(z),z,a\right)\right]^2=0$ for every a in X .

$\Rightarrow\ T_1(z)=z$

Hence z is a fixed point of T_1.

Now $z=\left(T_1T_2\right)(z)=T_2T_1(z)=T_2\left(T_1(z)\right)=T_2(z)$

Hence z is a common fixed point of T_1 and T_2.

Similarly z is a common fixed point of T_3 and T_4.

Let w be another common fixed point of T_1T_2 and T_3T_4.

Now it follows that $T_3T_4(w) = T_4T_3(w) = w$.

Now $\left[d\left(z,w,a\right)\right]^2 = \left[d\left(T_1T_2(z),T_3T_4(w),a\right)\right]^2 \le \left(\alpha_1 + \alpha_7 + \alpha_{10}\right)\left[d\left(z,w,a\right)\right]^2$

$\Rightarrow \ \left(1-\alpha_1-\alpha_7-\alpha_{10}\right)\left[d\left(z,w,a\right)\right]^2 \le 0$

$\Rightarrow \ d\left(z,w,a\right)=0$ for every a in $X \ \Rightarrow \ z=w$

Hence z is a unique common fixed point of T_1, T_2, T_3 and T_4.

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