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Identities of generalized Carlitz's q-Euler polynomials under S_5

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Abstract

In this paper, we investigate some new identities of symmetry for the generalized Carlitz q-Euler polynomials under the symmetric group of degree five.

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1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic absolutely value $|\cdot|_p$ is normally defined by $|p|_p = \frac{1}{p}$ and the q-analogue of number x is denoted as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$. Let $f(x)$ be continuous function on $\mathbb{Z}_p$. Then the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty} \frac{[2]_q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{ (see [1]-[4]).} \quad (1.1)$$

For $d \in \mathbb{N}$ with $(d, p) = 1$ and $d \equiv 1(mod\ 2)$, we set

$$X = \lim_{\overleftarrow{N}} (\mathbb{Z}/dp^N\mathbb{Z}),\ \ X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p),$$

and

$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a(mod\ dp^N)\},$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$, $(N \in \mathbb{N})$.

Let χ be a Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1(mod\ 2)$. From (1.1), we have

$$\int_X f(x) d\mu_{-q}(x) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x), \tag{1.2}$$

and

$$q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l),\ \ \text{(see [5])}. \tag{1.3}$$

By (1.2), we easily get

$$\begin{aligned} \int_X \chi(y) e^{(x+y)t} d\mu_{-1}(y) &= \frac{2}{e^{dt}+1} \left(\sum_{a=0}^{d-1} (-1)^a \chi(a) e^{at} \right) e^{xt} \\ &= \sum_{n=0}^{\infty} E_{n,\chi}(x) \frac{t^n}{n!},\ \ \text{(see [3])}. \end{aligned} \tag{1.4}$$

where $E_{n,\chi}(x)$ are called the generalized Euler polynomials attached to χ.

In the viewpoint of (1.4), Kim considered the generalized Carlitz's type q-Euler polynomials attached to χ as follows:

$$\int_X \chi(y) e^{[x+y]_q t} d\mu_{-q}(y) = \sum_{n=0}^{\infty} \xi_{n,\chi,q}(x) \frac{t^n}{n!},\ \ \text{(see [5])}. \tag{1.5}$$

The purpose of this paper is to give some new identities of symmetry for the generalized Carlitz's type q-Euler polynomials attached to χ which are derived from the fermionic p-adic q-integral on $\mathbb{Z}_p$.

2. Identities for $\xi_{n,\chi,q}(x)$ under S_5

Let w_1, w_2, w_3, w_4, w_5 be natural numbers such that $w_1 \equiv w_2 \equiv w_3 \equiv w_4 \equiv w_5 \equiv 1(mod\ 2)$ and let χ be a Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1(mod\ 2)$.

Then, from (1.2), we have

$$\begin{aligned}
&\int_X \chi(y) e^{[w_1w_2w_3w_4y+w_1w_2w_3w_4w_5x+w_5w_4w_3w_2i+w_5w_4w_3w_1j+w_5w_4w_2w_1k+w_5w_3w_2w_1l]_q t} \\
&\quad d\mu_{-q^{w_1w_2w_3w_4}}(y) \\
&= \frac{[2]_{q^{w_1w_2w_3w_4}}}{2} \lim_{N\to\infty} \sum_{m=0}^{dw_5-1} \sum_{y=0}^{p^N-1} \chi(m)(-1)^{m+y} q^{w_1w_2w_3w_4(m+dw_5y)} \\
&\quad\times e^{[w_1w_2w_3w_4(m+dw_5y)+w_1w_2w_3w_4w_5x+w_5w_4w_3w_2i+w_5w_4w_3w_1j+w_5w_4w_2w_1k+w_5w_3w_2w_1l]_q t}.
\end{aligned} \tag{2.1}$$

From (2.1), we note that

$$\begin{aligned}
&\frac{2}{[2]_{q^{w_1w_2w_3w_4}}} \sum_{i=0}^{dw_1-1} \sum_{j=0}^{dw_2-1} \sum_{k=0}^{dw_3-1} \sum_{l=0}^{dw_4-1} (-1)^{i+j+k+l} \chi(i)\chi(j)\chi(k)\chi(l) \\
&\times q^{w_5w_4w_3w_2i+w_5w_4w_3w_1j+w_5w_4w_2w_1k+w_5w_3w_2w_1l} \\
&\times \int_X e^{[w_1w_2w_3w_4y+w_1w_2w_3w_4w_5x+w_5w_4w_3w_2i+w_5w_4w_3w_1j+w_5w_4w_1w_2k+w_5w_3w_1w_2l]_q t} \\
&\quad\times \chi(y) d\mu_{-q^{w_1w_2w_3w_4}}(y) \\
&= \lim_{N\to\infty} \sum_{i=0}^{dw_1-1} \sum_{j=0}^{dw_2-1} \sum_{k=0}^{dw_3-1} \sum_{l=0}^{dw_4-1} \sum_{m=0}^{dw_5-1} \chi(i)\chi(j)\chi(k)\chi(l)\chi(m)(-1)^{i+j+k+l+m+y} \\
&\times q^{w_5w_4w_3w_2i+w_5w_4w_3w_1j+w_5w_4w_2w_1k+w_5w_3w_2w_1l+w_1w_2w_3w_4(m+dw_5y)} \\
&\times e^{[w_1w_2w_3w_4(m+dw_5y)+w_1w_2w_3w_4w_5x+w_5w_4w_3w_2i+w_5w_4w_3w_1j+w_5w_4w_2w_1k+w_5w_3w_2w_1l]_q t}.
\end{aligned} \tag{2.2}$$

The expression of (2.2) is an invariant under S_5. Therefore, by (2.2), we obtain the following equation :

$$\begin{aligned}
&\frac{2}{[2]_{q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}}}} \sum_{i=0}^{dw_{\sigma(1)}-1} \sum_{j=0}^{dw_{\sigma(2)}-1} \sum_{k=0}^{dw_{\sigma(3)}-1} \sum_{l=0}^{dw_{\sigma(4)}-1} (-1)^{i+j+k+l} \\
&\quad\times \chi(i)\chi(j)\chi(k)\chi(l) q^{(w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(3)}w_{\sigma(2)}i+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(3)}w_{\sigma(1)}j+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(1)}k+w_{\sigma(5)}w_{\sigma(3)}w_{\sigma(2)}w_{\sigma(1)}l)} \\
&\times \int_X e^{[w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}y+w_1w_2w_3w_4w_5x+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(3)}w_{\sigma(2)}i+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(3)}w_{\sigma(1)}j+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(1)}k+w_{\sigma(5)}w_{\sigma(3)}w_{\sigma(2)}w_{\sigma(1)}l]_q t} \chi(y) d\mu_{-q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}}}(y)
\end{aligned} \tag{2.3}$$

are the same for any $\sigma \in S_5$.

Note that

$$\int_X \chi(y) e^{[w_1w_2w_3w_4y+w_1w_2w_3w_4w_5x+w_5w_4w_3w_2i+w_5w_4w_3w_1j+w_5w_4w_2w_1k+w_5w_3w_2w_1l]_q t} d\mu_{-q^{w_1w_2w_3w_4}}(y)$$
$$= \sum_{n=0}^{\infty} [w_1w_2w_3w_4]_q^n \int_X \chi(y) \left[y + w_5x + \frac{w_5}{w_1}i + \frac{w_5}{w_2}j + \frac{w_5}{w_3}k + \frac{w_5}{w_4}l\right]_{q^{w_1w_2w_3w_4}}^n d\mu_{-q^{w_1w_2w_3w_4}}(y)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} [w_1w_2w_3w_4]_q^n \xi_{n,\chi,q^{w_1w_2w_3w_4}}(w_5x + \frac{w_5}{w_1}i + \frac{w_5}{w_2}j + \frac{w_5}{w_3}k + \frac{w_5}{w_4}l)\frac{t^n}{n!}. \tag{2.4}$$

Thus, (2.4), we get

$$\int_X \chi(y)[w_1w_2w_3w_4y + w_1w_2w_3w_4w_5x + w_5w_4w_3w_2i + w_5w_4w_3w_1j + w_5w_4w_1w_2k + w_5w_3w_1w_2l]_q^n d\mu_{-q^{w_1w_2w_3w_4}}(y) \tag{2.5}$$
$$= \xi_{n,\chi,q^{w_1w_2w_3w_4}}(w_5x + \frac{w_5}{w_1}i + \frac{w_5}{w_2}j + \frac{w_5}{w_3}k + \frac{w_5}{w_4}l)[w_1w_2w_3w_4]_q^n, \ (n \geq 0).$$

From (2.3) and (2.5), we have

$$\frac{2[w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}]_q^n}{[2]_{q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}}}} \sum_{i=0}^{dw_{\sigma(1)}-1}\sum_{j=0}^{dw_{\sigma(2)}-1}\sum_{k=0}^{dw_{\sigma(3)}-1}\sum_{l=0}^{dw_{\sigma(4)}-1} (-1)^{i+j+k+l}$$
$$\times \chi(i)\chi(j)\chi(k)\chi(l) q^{(w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(3)}w_{\sigma(2)}i + w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(3)}w_{\sigma(1)}j + w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(1)}k + w_{\sigma(5)}w_{\sigma(3)}w_{\sigma(2)}w_{\sigma(1)}l)} \tag{2.6}$$
$$\times \xi_{n,\chi,q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}}}(w_{\sigma(5)}x + \frac{w_{\sigma(5)}}{w_{\sigma(1)}}i + \frac{w_{\sigma(5)}}{w_{\sigma(2)}}j + \frac{w_{\sigma(5)}}{w_{\sigma(3)}}k + \frac{w_{\sigma(5)}}{w_{\sigma(4)}}l)$$

are the same for any $\sigma \in S_5$.

It is easy to show that

$$\left[y + w_5x + \frac{w_5}{w_1}i + \frac{w_5}{w_2}j + \frac{w_5}{w_3}k + \frac{w_5}{w_4}l\right]_{q^{w_1w_2w_3w_4}}^n$$
$$= \sum_{m=0}^{n} \binom{n}{m} \left(\frac{[w_5]_q}{[w_1w_2w_3w_4]_q}\right)^{n-m} \tag{2.7}$$
$$\times [w_4w_3w_2i + w_4w_3w_1j + w_4w_1w_2k + w_3w_1w_2l]_{q^{w_5}}^{n-m}$$
$$\times q^{m(w_5w_4w_3w_2i + w_5w_4w_3w_1j + w_5w_4w_1w_2k + w_5w_3w_1w_2l)}[y + w_5x]_{q^{w_1w_2w_3w_4}}^m.$$

Thus, by (2.7), we get

$$\begin{aligned}
&\frac{2[w_1w_2w_3w_4]_q^n}{[2]_{q^{w_1w_2w_3w_4}}}\sum_{i=0}^{dw_1-1}\sum_{j=0}^{dw_2-1}\sum_{k=0}^{dw_3-1}\sum_{l=0}^{dw_4-1}(-1)^{i+j+k+l}\chi(i)\chi(j)\chi(k)\chi(l)\\
&\times q^{w_5w_4w_3w_2i+w_5w_4w_3w_1j+w_5w_4w_2w_1k+w_5w_3w_2w_1l}\\
&\times\int_{\mathbb{Z}_p}\left[y+w_5x+\frac{w_5}{w_1}i+\frac{w_5}{w_2}j+\frac{w_5}{w_3}k+\frac{w_5}{w_4}l\right]_{q^{w_1w_2w_3w_4}}^n\chi(y)d\mu_{-q^{w_1w_2w_3w_4}}(y)\\
&=\sum_{m=0}^{n}\binom{n}{m}\frac{2[w_1w_2w_3w_4]_q^m}{[2]_{q^{w_1w_2w_3w_4}}}[w_5]_q^{n-m}\xi_{m,\chi,q^{w_1w_2w_3w_4}}(w_5x)\\
&\times\sum_{i=0}^{dw_1-1}\sum_{j=0}^{dw_2-1}\sum_{k=0}^{dw_3-1}\sum_{l=0}^{dw_4-1}(-1)^{i+j+k+l}\chi(i)\chi(j)\chi(k)\chi(l)\\
&\times q^{(m+1)(w_5w_4w_3w_2i+w_5w_4w_3w_1j+w_5w_4w_1w_2k+w_5w_3w_1w_2l)}\\
&\times[w_4w_3w_2i+w_4w_1w_3j+w_4w_1w_2k+w_3w_1w_2l]_{q^{w_5}}^{n-m}\\
&=\sum_{m=0}^{n}\binom{n}{m}\frac{2[w_1w_2w_3w_4]_q^m}{[2]_{q^{w_1w_2w_3w_4}}}[w_5]_q^{n-m}\widehat{S}_{n,\chi,q^{w_5}}(w_1,w_2,w_3,w_4\mid m)\\
&\times\xi_{m,\chi,q^{w_1w_2w_3w_4}}(w_5x),
\end{aligned}\tag{2.8}$$

where

$$\begin{aligned}
\widehat{S}_{n,\chi,q}(w_1,w_2,w_3,w_4\mid m)&=\sum_{i=0}^{dw_1-1}\sum_{j=0}^{dw_2-1}\sum_{k=0}^{dw_3-1}\sum_{l=0}^{dw_4-1}(-1)^{i+j+k+l}\chi(i)\chi(j)\chi(k)\chi(l)\\
&\times q^{(m+1)(w_4w_3w_2i+w_4w_3w_1j+w_4w_1w_2k+w_3w_1w_2l)}\\
&\times[w_4w_3w_2i+w_4w_3w_1j+w_4w_1w_2k+w_3w_1w_2l]_q^{n-m}.
\end{aligned}\tag{2.9}$$

We note that the expression of equation (2.8) is also an invariant under S_5. Therefore, by (2.8), we obtain the following equation:

$$\begin{aligned}
&\sum_{m=0}^{n}\binom{n}{m}\frac{2[w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}]_q^m}{[2]_{q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}}}}[w_{\sigma(5)}]_q^{n-m}\\
&\times\xi_{m,\chi,q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}}}(w_{\sigma(5)}x)\widehat{S}_{n,\chi,q^{w_{\sigma(5)}}}(w_{\sigma(1)},w_{\sigma(2)},w_{\sigma(3)},w_{\sigma(4)}\mid m)
\end{aligned}$$

are the same for any $\sigma\in S_5$.

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