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Plane-Point Method For Solving Solid Assignment Problems

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Abstract

A new method namely, plane-point method is proposed for determining an optimal solution of solid assignment problems. The proposed method is based on the concept of planes at a point, but not on Hungarian method and branch and bound procedure. The optimality of the solution is derived mathematically. Numerical example is presented to demonstrate the proposed method. The plane-point method helps the decision makers to take an appropriate decision while handling various types of three dimensional assignment problems in real life situations.

Keywords: Solid assignment problem, Optimal solution, Planes at a point, Plane-point method.

Introduction

A special case of the transportation problem in the operations research [9] is the assignment problem in which each supply is one and each demand is one. The assignment problem is viewed as one of the fundamental combinatorial optimization problems. The solid assignment (SA) problem is an extension of the two-dimensional assignment problem in which three items are involved. The SA problem is also, called a three-dimensional assignment problem and, it is NP-hard [6]. SA problems have wide applications in management science, engineering, technology, social science and medical science. In particular, SA problems have interesting applications in time tabling problems.

Pierskalla [12] was the first, who proposed a heuristic algorithm for solving SA problems. Most of the algorithms for solving SA problems split the current problem into two sub-problems. Vlach [15] developed a branch bound algorithm for planar three dimensional assignment problems by applying row and column reductions. Hansen and Kaufman [8] described a primal-dual method similar to the Hungarian method [9] for three-dimensional linear assignment problems. Frieze and Yadegar [7] presented an algorithm for solving SA problems with application to scheduling in a teaching practice. Balas [2] provided an algorithm for solving SA problems. Crama and Spieksma [4] developed approximation algorithms for solving SA problems based on 3-partite graph with triangle inequalities. Magos and Miliotis [11] is proposed a branch and bound procedure for obtaining an optimal solution of planar 3-index assignment problems.

Magos [10] developed a tabu search for solving planar three-index assignment problems. A new Lagrangean relaxation based algorithm for a class of multidimensional assignment problems was discussed by Poore and Robertson [13]. Burkand [3] surveyed recent developments of some assignment problems. Storms and Spieksma [14] proposed a solution procedure for geometric three-dimensional assignment problems. Federico Perea [5] discussed multi dimensional assignment problems and some of its applications. Anuradha and Pandian[1] proposed the reduction method for solving SA problems.

In the paper, we propose a new method namely, plane-point method to find an optimal solution of solid assignment problems based on planes at a point. We derive the optimality of the solution obtained by the proposed method mathematically. The plane-point method is not based on Hungarian method and also, we do not use branch and bound procedure. We provide a numerical example for demonstrating and showing the importance of the proposed method. When the decision makers are handling various types of three dimensional assignment problems, the plane-point method will help them for constructing an appropriate assignment plan.

Solid Assignment Problem

The SA problem states as : " In a factory, m workers, n jobs and l machines are available. Let c_{ijk} be the amount to be paid the ith worker when he/she is assigned to the jth job in the kth machine. In the factory, the policy "one job to one worker in one machine only" should be followed. Let x_{ijk} denote the jth job assigned to ith worker in the machine k, that is, $x_{ijk} = 1$ if jth job is assigned to ith worker in the kth machine and otherwise, $x_{ijk} = 0$. Determine an assignment planning schedule to minimize the total assignment cost satisfying workers, jobs and machines constraints".

Now, the linear programming model of the above SA problem is given below:

(P) Minimize $z = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} c_{ijk} x_{ijk}$

subject to

$$\sum_{j=1}^{n} \sum_{k=1}^{l} x_{ijk} = 1, \ i \in I \qquad (1)$$

$$\sum_{i=1}^{m} \sum_{k=1}^{l} x_{ijk} = 1, \ j \in J \qquad (2)$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} = 1, \; k \in K \tag{3}$$

$$x_{ijk} = 1 \text{ or } 0, \text{ for all i, j and k} \tag{4}$$

where $I = \{1,2,\ldots,m\}$, $J = \{1,2,\ldots,n\}$ and $K = \{1,2,\ldots,l\}$.

The problem (P) is said to be balanced if $m = n = l$, that is , number of works, jobs and machines are equal. Otherwise, it is called unbalanced.

Definition 2.1.
Any set of non-negative assignments to the problem (P) which satisfies the equations (1) to (4), is said to be a feasible solution to the problem (P).

Definition 2.2.
A feasible solution to the problem (P) which minimizes the total assignment cost, that is, $\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} c_{ijk} x_{ijk}$ is said to be an optimal solution to the problem (P).

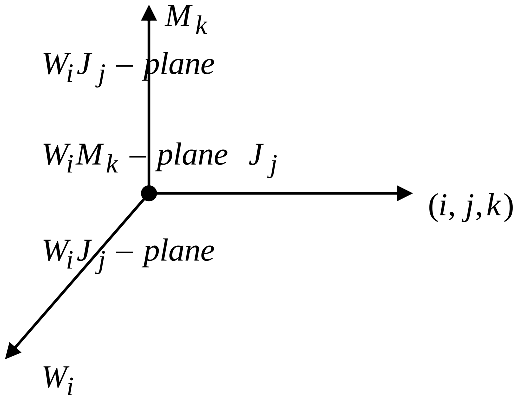
Result 2.1.
The balanced condition is the necessary and sufficient condition for the existence of an optimal solution of the problem (P).

Remark 2.1.
If l = 1, the number of machines is only one, the problem (P) reduces to the assignment problem, that is, the two-dimensional assignment problem or the plane assignment problem.

Now, the problem (P) can be represented as a table form, called SA table as shown below:

Jobs →		J_1			…	J_n		
↓ Workers	Machines →	M_1	…	M_l	…	M_1		M_l
W_1		c_{111}		c_{11l}		c_{1n1}		c_{1nl}
⋱		⋱	…	⋱	…	⋱	…	⋱
W_m		c_{n11}		c_{n1l}		c_{mn1}		c_{mnl}

Now, since three items are considered for each assignment in the problem (P), three planes at a point (i, j, k) namely, $W_i J_j$-plane at (i, j, k), $W_i M_k$-plane at (i, j, k) and $J_j M_k$-plane at (i, j, k) can be constructed.



Plane-Point Method

Now, we prove the following theorem which is used in the plane method for proving the optimality of a feasible solution of the SA problem obtained by the proposed method.

Theorem 3.1

Let $\{ x^{\circ}_{ijk},\ i \in I;\ j \in J$ and $k \in K \}$ be an optimal solution of the problem (Q) with optimum objective value zero where

(Q) Minimum $z^{*} = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} (c_{ijk} - u_i - v_j - w_k) x_{ijk}$

subject to (1) to (4) and

$c_{ijk} - u_i - v_j - w_k \geq 0$, for all i , j and k (5)

where u_i, v_j and w_k are some real values, then $\{ x^{\circ}_{ijk},\ i \in I;\ j \in J$ and $k \in K \}$ is an optimum solution of the problem (P) .

Proof:

Clearly, $\{x^{\circ}_{ijk},\ i \in I;\ j \in J$ and $k \in K\}$ is a feasible solution to the problem (P). We claim to prove that $\{x^{\circ}_{ijk},\ i \in I;\ j \in J$ and $k \in K\}$ is an optimal of (P).

Assume that $\{x^{\circ}_{ijk},\ i \in I;\ j \in J$ and $k \in K\}$ is not an optimal solution of (P).

Then, there exists a feasible solution $\{y_{ijk},\ i \in I;\ j \in J$ and $k \in K\}$ to the problem (P) such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} c_{ijk}\, y_{ijk} < \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} c_{ijk}\, x^{\circ}_{ijk}\,. \tag{6}$$

Now, since $\{ y_{ijk},\ i \in I;\ j \in J$ and $k \in K \}$ is feasible to (P), $\{ y_{ijk},\ i \in I;\ j \in J$ and $k \in K \}$ is feasible to the problem (Q).

Now, since the minimum value of $z^* = 0$, we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}(c_{ijk} - u_i - v_j - w_k)\, y_{ijk} \geq 0$$

That is,

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} c_{ijk}\, y_{ijk} - \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} u_i\, y_{ijk} - \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} v_j\, y_{ijk} - \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} w_k\, y_{ijk} \geq 0\,.$$

From (1) to (3), we conclude that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} c_{ijk}\, y_{ijk} \geq \sum_{i=1}^{m} u_i + \sum_{j=1}^{n} v_j + \sum_{k=1}^{l} w_k\,.$$

From (6), we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} c_{ijk}\, x^{\circ}_{ijk} > \sum_{i=1}^{m} u_i + \sum_{j=1}^{n} v_j + \sum_{k=1}^{l} w_k\,.$$

Using (1) to (3), we obtain the following:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} c_{ijk}\, x^{\circ}_{ijk} - \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} u_i\, x^{\circ}_{ijk} - \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} v_j\, x^{\circ}_{ijk} - \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} w_k\, x^{\circ}_{ijk} > 0\,.$$

That is, $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}(c_{ijk} - u_i - v_j - w_k)\, x^{\circ}_{ijk} > 0$

which contradicts the given hypothesis that $\{x^{\circ}_{ijk},\ i \in I;\ j \in J$ and $k \in K\}$ is an optimal solution of (Q). Therefore, $\{ x^{\circ}_{ijk},\ i \in I;\ j \in J$ and $k \in K \}$ is an optimal solution of the problem (P).

Hence, the theorem is proved.

Now, a new method namely, plane-point method is proposed for determining an optimal solution of the SA problem directly without splitting the problem..

The proposed method proceeds as follows.

Step 1. Construct the SA table for the given SA problem and then, convert it into a balanced one, if it is not.

Step 2. Subtract each worker's entries of the SA table from its minimum and then, subtract each job's entries of the resulting SA table from its minimum and also then, subtract each machine's entries of the resulting SA table from its minimum.

Step 3. Draw the minimum number of planes at the zero points to cover all the zeros of the reduced SA table. If the number of planes is equal to number of workers, jobs or machines, go to the Step 5. (Such table is known as an optimal assignment table). Otherwise, go to the Step 4..

Step 4: Develop the new revised reduced SA table as follows:

(a) Draw the minimum number of lines to cover all the zeros of the reduced SA table.

(b) Find the smallest entry of the reduced SA cost matrix not covered by any lines.

(c) Subtract all the uncovered entries in the reduced SA cost matrix from the smallest entry and add the same to all entries lying at the intersection of lines.

Then, move to the Step 3..

Step 5. Construct the set, S where S = $\{(w_a, j_b, m_c) : (w_a, j_b, m_c)$ is an originating point of a plane obtained in the Step 3.}. An element (w_a, j_b, m_c) in S provides an assignment " the *b*th job is assigned to *a*th worker in the machine *c* '' to the problem (P).

Step 6. The set of assignments of workers, jobs and machines obtained in the Step 5. is an optimal solution of the given SA problem by the Theorem 3.1.. and the sum of the costs of the assignments in the set of the assignments is the minimum total assignment cost of the given problem.

Now, we demonstrate the solution procedure of the proposed method, namely plane-point method for solving the solid assignment problem using the following numerical example.

Example 3.1. When there are three workers denoted by W1, W2 and W3 , three machines denoted by M1, M2 and M3, and three jobs denoted by J1, J2 and J3 . It is obvious that c_{ijk} is the cost of the assigning jth job to be performed by the ith worker in the kth machine. Besides, three workers, three machines and three jobs can be associated with only one of the others. The assignment costs $c_{ijk}'s$ are given below in the following table

Workers	Jobs	J1			J2			J3		
	Machines	M1	M2	M3	M1	M2	M3	M1	M2	M3
W1		10	8	12	9	10	27	15	10	13
W2		8	6	7	9	6	12	7	11	12
W3		9	7	6	10	7	12	8	6	8

Our objective is to minimize total cost assignment plan.

Here, the problem is balanced.

Now, using the Step 2., the following reduced SA table is obtained:

Workers	Jobs	J1			J2			J3		
	Machines	M1	M2	M3	M1	M2	M3	M1	M2	M3
W1		1	0	4	0	2	19	6	2	5
W2		1	0	1	2	0	6	0	5	6
W3		2	1	0	3	1	6	1	0	2

Now, applying the Step 3., all zeroes in the reduced SA table can be covered by W_3M_2 – plane at (3,3,2) and W_1M_1 – plane at (1,2,1) as shown below:

Workers	Jobs	J1			J2			J3		
	Machines	M1	M2	M3	M1	M2	M3	M1	M2	M3
W1		1	0	4	0	2	19	6	2	5
W2		1	0	1	2	0	6	0	5	6
W3		2	1	0	3	1	6	1	0	2

Since the number of planes covered all zeros is 2 < 3, the reduced SA table is not an optimal assignment table.

Now, using M_1 – line, M_2 – line and W_3 – line, we cover all zeros as shown below:

Workers	Jobs	J1			J2			J3		
	Machines	M1	M2	M3	M1	M2	M3	M1	M2	M3
W1		1	0	4	0	2	19	6	2	5
W2		1	0	1	2	0	6	0	5	6
W3		2	1	0	3	1	6	1	0	2

Now, applying the Step 4., we obtain the following reduced SA table.

Workers	Jobs	J1			J2			J3		
	Machines	M1	M2	M3	M1	M2	M3	M1	M2	M3
W1		1	0	3	0	2	18	6	2	4
W2		1	0	0	2	0	5	0	5	5
W3		3	2	0	4	2	6	2	1	2

Now, the following optimal assignment table for the given problem by using the Step 3. and the Step 4. of the Plane-Point method is obtained since all zeroes can be covered by W_1M_1 – plane at (1,2,1), W_2M_3 – plane at (2,1,3) and W_3M_2 – plane at (3,3,2) as shown below.

Workers	Jobs	J1			J2			J3		
	Machines	M1	M2	M3	M1	M2	M3	M1	M2	M3
W1		2	0	2	**0**	1	16	5	0	1
W2		3	1	**0**	3	0	4	0	4	3
W3		5	3	0	5	2	5	2	**0**	0

Now, by the Steps 5. and 6., the optimal solution to the given SA problem is given below:

$W_1 \xrightarrow{M_1} J_2$; $W_2 \xrightarrow{M_3} J_1$ and $W_3 \xrightarrow{M_2} J_3$

and the total minimum assignment cost is 22.

Remark 3.1:

In [1], Anuraha and Pandian showed that the optimal solution of the Example 3.1. is $W_1 \xrightarrow{M_1} J_2$; $W_2 \xrightarrow{M_3} J_3$ and $W_3 \xrightarrow{M_3} J_1$ with the total minimum assignment cost 23 which is not optimal, but the proposed method is given the optimal solution to the given problem.

Conclusion

In this paper, we consider SA problems that deal with real life linear assignment problems. A new method namely, plane-point method for finding an optimal solution of solid assignment problems, is developed in which the problem is not decomposed. Mathematically, the optimality of the solution obtained by the plane-point method is derived. In the proposed method, Hungarian method and branch and bound procedure are not used. For understanding the efficiency and importance of the proposed method, a numerical example is presented. The plane-point method can serve as an appropriate tool for decision makers to take a suitable decision on the assignment plan when they are working assignment problems of real life situations involving three parameters.

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