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Generalized Hyers-Ulam-Rassias Stability of a New Quadratic Functional Equation

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ABSTRACT

In this paper, the authors investigate the general solution of new quadratic functional equation.
$f(2x+y)+f(x-2y)+f(x+y)+f(x-y)=7f(x)+7f(y)$
and discuss its generalized Hyers- Ulam-Rassias stability.

Key Words: Quadratic functional equation and Hyers – Ulam-Rassias stability.

1. INTRODUCTION

In 1940, S.M.Ulam[19] raised the following question concerning the stability of group homomorphisms. Under what condition does there is an additive mapping near an approximately additive mapping between a Group and a Metric Group?

The case of approximately additive function was solved by D.H.Hyers[10] under the assumption that $\in > 0$, for $E_1 \,\&\, E_2$ are Banach space and $f: E_1 \to E_2$ such that $\|f(x+y)-f(x)-f(y)\| \le \in$ for all $x, y \in E_1$, then there exist an unique additive mapping $T: E_1 \to E_2$ such that $\|f(x)-T(x)\| \le \in$ for all $x \in E_1$

In 1978, a generalized version of the theorem of Hyres for approximately liner mapping was given by Th.M.Rassias[16]. He proved that for a mapping $f: E_1 \to E_2$ be such that $f(tx)$ is continuous in $t \in R$ and for each fixed $x \in E_1$ assume that there exist a constant $\in > 0$ and $p \in \ 0,1$ with

$$\|f(x+y)-f(x)-f(y)\| \le \in \left[\|x\|^p + \|y\|^p\right] \tag{1.1}$$

For all $x, y \in E_1$, then there exist a unique R linear mapping $T : E_1 \to E_2$ such that

$$\| f(x) - T(x) \| \le \frac{2 \in}{2 - 2^p} \|x\|^p \qquad (1.2)$$

a number of mathematicians were attracted by the result of Th.M.Rassias. The Stability concept that was introduced and investigated by Th.M.Rassias is called Hyers – Ulam- Rassias stability. During the last three decades, the stability problems of several functional equations have been extensively investigated by a number of authors ([1], [5], [7], [10], [11], [12], [15]). In 1982 – 1989, Th.M.Rassias ([16], [17], [18]) replaced the sum appeared in the right hand side equation (1.1) by the product of powers of norms. In fact, he proved the following theorem.

Theorem 1.1:
Let $f : E_1 \to E_2$ be a mapping from a norm vector space E_1 into vector space E_2 subject to the in equality

$$\| f(x+y) - f(x) - f(y) \| \le \in \left[\|x\|^p \|y\|^p \right] \qquad (1.3)$$

For all $x, y \in E_1$, where $\in$ and p are constants with $\in > 0$ and $0 \le p < \frac{1}{2}$. Then the limit

$$L(x) = \underset{n \to \infty}{Lim} \frac{f(2^n x)}{2^n} \qquad (1.4)$$

exist for all $x \in E_1$, and $L : E_1 \to E_2$ is the unique additive mapping which satisfies

$$\| f(x) - L(x) \| \le \frac{\in}{2 - 2^{2p}} \|x\|^{2p} \qquad (1.5)$$

for all $x \in E_1$. If $p > \frac{1}{2}$ the inequality (1.3) holds for all $x, y \in E_1$ and the limit

$$A(x) = \underset{n \to \infty}{Lim} 2^n f(x/2^n) \qquad (1.6)$$

Exist for all $x \in E_1$ and $A : E_1 \to E_2$ is a unique additive mapping

$$\| f(x) - A(x) \| \le \frac{\in}{2^{2p} - 2} \|x\|^{2p} \qquad (1.7)$$

for all $x \in E_1$.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \qquad 1.8)$$

Is called quadratic functional equation, because, the function $f(x) = kx^2$ satisfies the equation (1.8). Infact every solution of the quadratic functional equation is said to be quadratic mapping.

A Hyers – Ulam stability problem for the equation was discussed by skof[15], Cholewa[4], and Czerwol [6], in different settings.

In this paper we introduced a new quadratic functional equation

$$f(2x+y) + f(x-2y) + f(x+y) + f(x-y) = 7f(x) + 7f(x) \qquad (1.9)$$

and in section 2 we discuss the general solution of equation (1.9), in section 3 we discuss, the generalized Hyers-Ulam-Rassias stability of (1.9).

2. General solution of the quadratic functional equation (1.9)

In this section we derive the general solution of the quadratic functional equation (1.9).

For this we use the following theorem. Throughout this section, let X *and* Y be a real vector spaces.

Theorem 2.1

A function $f : X \to Y$ satisfies the quadratic functional equation (1.9) if and only if $f : X \to Y$ satisfies the quadratic functional equation (1.8).

Proof:

Putting $x = 0 = y$ in (1.9), we get $f(0) = 0$. Let $y = 0$ in (1.9) , we obtain

$$f(2x) = 4f(x) \qquad (2.1)$$

for all $x \in X$. Substituting $y = 0$ in (1.9), we get

$$f(-x) = f(x) \qquad (2.2)$$

for all $x \in X$ .put $(x, y) = (x - y, x + y)$ in (1.9), we get

$$f(3x-y) + f(x+3y) + 4f(x) + 4f(y) = 7f(x-y) + 7f(x+y) \qquad (2.3)$$

for all $x \in X$. Put $y = -y$ (1.12) and using (1.11), we get

$$f(3x+y) + f(x-3y) + 4f(x) + 4f(y) = 7f(x-y) + 7f(x+y) \qquad (2.4)$$

for all $x \in X$. Adding (1.12) with (1.13), we get

$$f(3x-y) + f(3x+y) + f(x+3y) + f(x-3y) + 8f(x) + 8f(y) = 14f(x-y) + 14f(x+y) \qquad (2.5)$$

for all $x \in X$. Again replace $(x, y) = (x + y, 2y)$ in (1.9), we get

$$4f(x+2y) + f(x-3y) + f(x+3y) + f(x-y) = 7f(x+y) + 28f(y) \qquad (2.6)$$

for all $x \in X$. Replace $y = -y$ in (1.15) and using (1.11), we get

$$4f(x-2y) + f(x+3y) + f(x-3y) + f(x+y) = 7f(x-y) + 28f(y) \qquad (2.7)$$

for all $x \in X$. Replace $(x, y) = (y, x)$ in (1.15), and using evenness of f we get

$$4f(2x+y) + f(3x-y) + f(3x+y) + f(x-y) = 7f(x+y) + 28f(x) \qquad (2.8)$$

for all $x \in X$. Adding (1.16) with (1.17), we get

$$4[f(x-2y)+f(2x+y)]+f(x+3y)+f(x-3y)+f(3x-y)+f(3x+y) = 6f(x+y)+6f(x-y)+28f(x)+28f(y) \quad (2.9)$$

for all $x \in X$. Using (1.14) in (1.18), we get

$$4[f(x-2y)+f(2x+y)] = -8f(x+y)-8f(x-y)+36f(x)+36f(y) \quad (2.10)$$

for all $x \in X$. Dividing (1.19) by 4, we get

$$f(x-2y)+f(2x+y) = -2f(x+y)-2f(x-y)+9f(x)+9f(y) \quad (2.11)$$

for all $x \in X$. Using (1.20) in (1.9), we get

$$-f(x+y)-f(x-y) = -2f(x)-2f(y) \quad (2.12)$$

for all $x \in X$. Interchange the LHS and RHS of (1.21), we get

$$f(x+y)+f(x-y) = 2f(x)+2f(y) \quad (2.13)$$

Conversely assume f satisfies (1.22). And set $x = 0 = y$ in (1.22), we get $f(0) = 0$ and put $y = x$ in (1.22), it shows that $f(2x) = 4f(x)$ also set $x = 0$ in (1.22), it gives $f(-y) = f(y)$.

for all $y \in X$. Put $(x, y) = (2x+y, x-2y)$ in (1.22), we get

$$2f(2x+y)+2f(x-2y) = f(3x-y)+f(x+3y) \quad (2.14)$$

for all $x \in X$. Again replace $x = 2x$ *and* $y = x-y$ in (1.22), we derive

$$f(3x-y) = 8f(x)+2f(x-y)-f(x+y) \quad (2.15)$$

for all $x \in X$. Change $y = -y$ in (1.24), we get

$$f(3x+y) = 8f(x)+2f(x+y)-f(x-y) \quad (2.16)$$

for all $x \in X$. Replace $(x, y) = (y, x)$ in (1.25) and using evenness of f , we get

$$f(x+3y) = 8f(y)+2f(x+y)-f(x-y) \quad (2.17)$$

for all $x \in X$. Adding (1.24) with (1.26), we get

$$f(3x-y)+f(x+3y) = 8f(x)+8f(y)+f(x+y)+f(x-y) \quad (2.18)$$

for all $x \in X$. Substitute (1.27) in (1.23), we get

$$2f(2x+y)+2f(x-2y) = 8f(x)+8f(y)+f(x+y)+f(x-y) \quad (2.19)$$

for all $x \in X$. Use (1.22) in (1.28) and dividing the resultant equation by 2

$$f(2x+y)+f(x-2y) = 5f(x)+5f(y) \quad (2.20)$$

for all $x \in X$. Now adding (1.9) with (1.22), we get

$$f(2x+y)+f(x-2y)+f(x+y)+f(x-y) = 7f(x)+7f(x) \quad (2.21)$$

for all $x \in X$. Hence proved **Theorem 2.1.**

Section 3:
In this section we discuss the **Generalized Hyers-Ulam-Rassias Stability** of the functional equation (1.9)

Throughout this section consider X be a Normed Linear Space and Y be a Banach Space, we present the Hyers- Ulam- Rassias Stability of the Functional Equation (1.9) involving sum of powers of norms

Theorem:3.1
Let X Be a Normed Linear Space and let Y be a Banach Space. Let $\phi : X \to [0,\infty)$ be a function such that $\sum_{k=0}^{\infty} \frac{\phi\left(2^k x\right),}{4^k}$ converges for all $x \in X$ and

$$\lim_{n\to\infty} \frac{\phi(2^n x, 2^n y)}{4^n} = 0 \quad for\, all\, x, y \in X . \tag{3.1}$$

If a function $f : X \to Y$ satisfies the inequality

$$\| f(2x+y) + f(x-2y) + f(x+y) + f(x-y) - 7f(x) - 7f(y) \| \le \phi(x,y) \quad \text{for all } x, y \in X \tag{3.2}$$

Then there exist a unique function $Q : X \to Y$ such that Q satisfies (1.9) and

$$\| f(x) - Q(x) \| \le \frac{1}{4} \sum_{k=0}^{n-1} \frac{\phi(2^k x, 0)}{4^k} \qquad for\, all\, x \in X \tag{3.3}$$

Where

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n} \quad for\, all\, x \in X . \tag{3.4}$$

Proof:
We will first prove the case when the condition (3.1) holds. Letting $(x, y) = (x, 0)$ in (3.2) we get

$$\| f(2x) - 4f(x) \| \le \phi(x,0) \qquad for\, all\, x \in X \tag{3.5}$$

Now divide throughout by 4 we get,

$$\left\| \frac{f(2x)}{4} - f(x) \right\| \le \frac{1}{4} \phi(x,0) \quad for\, all\, x \in X \tag{3.6}$$

Replace $x = 2x$ in (3.6) and divide it by 4 we get,

$$\left\| \frac{f(2^2 x)}{4^2} - \frac{f(2x)}{4} \right\| \le \frac{1}{4^2} \phi(2x,0) \quad for\, all\, x \in X \tag{3.7}$$

From (3.6) and (3.7)

$$\left\| \frac{f(2^2 x)}{4^2} - f(x) \right\| \le \frac{1}{4}\left[\phi(x,0) + \frac{\phi(2x,0)}{4} \right] \ for\, all\, x \in X \qquad (3.8)$$

Generalizing (3.8) we get,

$$\left\| \frac{f(2^n x)}{4^n} - f(x) \right\| \le \frac{1}{4}\sum_{k=0}^{n-1} \frac{\phi(2^k x,0)}{4^k} \qquad (3.9)$$

for every positive integer n and $for\, all\, x \in X$.

Now to show the sequence $\left\{ \frac{f(2^n x)}{4^n} \right\}$ converges.

Replace $x\, by\, 2^m x$ in (3.9) and divide it by 4^m we get,

$$\left\| \frac{f(2^n 2^m x)}{4^n 4^m} - \frac{f(2^m x)}{4^m} \right\| \le \frac{1}{4}\sum_{k=0}^{n-1} \frac{\phi(2^{k+m} x,0)}{4^{k+m}}$$

By condition (3.1) the Right Hand side approaches $0\, as\, n \to \infty$.

Thus the sequence $\left\{ \frac{f(2^n x)}{4^n} \right\}$ converges for every positive integer n $for\, all\, x \in X$.

Since a Banch Space is complete, we let $Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$ $for\, all\, x \in X$.

By definition of $Q(x)$, we can see that the inequality (3.3) holds, in this case which the condition (3.1)holds.

Now to show $Q(x)$ satisfies the equation (1.9)

For this we set $(x,y) = (2^n x, 2^n y)$ in (3.2) and dividing it by 4^n, we get

$$\frac{1}{4n}\left\| f\, 2^n(2x+y + f\, 2^n(x-2y + f\, 2^n(x+y + f\, 2^n(x-y - 7f\, 2^n(x) - 7f\, 2^n(y) \right\| \le \frac{\phi(2^n x, 2^n y)}{4^n} \qquad (3.10)$$

allow $n \to \infty\, and\, apply\, the\, def.of$ $Q(x)$ we get,

$$\| Q(2x+y) + Q(x-2y) + Q(x+y) + Q(x-y) - 7Q(x) - 7Q(y) \| \le 0 \quad for\, all\, x \in X .$$

Therefore $Q(x)$ satisfies the equation (1.9).

Uniqueness of $Q(x)$:

Suppose that there exist another Quadratic function $S(x): X \to Y$ such that $S(x)$ satisfies equation (1.9) and $\| S(x) - f(x) \| \le \frac{1}{4}\sum_{k=0}^{\infty} \frac{\phi(2^k x,0)}{4^k}$ $for\, all\, x \in X$.

By theorem (2.1) every solution of (2.1) is also the solution of $Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$.

Thus every solution has Quadratic property,

That is , $Q(nx) = n^2 Q(x)$ for every positive integer n and $for\, all\, x \in X$.

Therefore $\|S(x)-Q(x)\|=\frac{1}{4n}\|S(2^n x)-Q(2^n x)\|$

$\leq \frac{1}{2}\sum_{k=0}^{\infty}\frac{\phi(2^k 2^n x,0)}{4^{k+n}}$

Right hand side tends to $0\, as\, n\to\infty$.

Thus $S(x)=Q(x)$. Hence $Q(x)$ is unique.

Theorem 3.2:

Let X Be a Normed Linear Space and let Y be a Banach Space. Let $\phi: X\to[0,\infty)$ be a function such that $\sum_{k=1}^{\infty}4^k\phi(x/2^n, y/2^n)$ converges for all $x\in X$ and $\lim_{n\to\infty}4^n\phi(x/2^n, y/2^n)=0 \; for\, all\, x,y\in X$

If a function $f: X\to Y$ satisfies the inequality

$\|f(2x+y)+f(x-2y)+f(x+y)+f(x-y)-7f(x)-7f(y)\|\leq\phi(x,y)$ for all $x,y\in X$ (3.11) then there exist a unique function $Q: X\to Y$ such that Q satisfies (1.9) and

$$\|f(x)-Q(x)\|\leq\frac{1}{4}\sum_{k=1}^{\infty}4^k\phi(x/2^k,0) \quad for\, all\, x\in X \tag{3.12}$$

Where

$$Q(x)=\lim_{n\to\infty}4^n f(x/2^n) \quad for\, all\, x\in X .$$

Proof:

For proving this theorem we substitute $(x,y)=(x/2,0)$ in (3.11) we get,

$$\|f(x)-4f(x/2)\|\leq\phi(x/2,0) \; for\, all\, x\in X$$

Generalizing this we get,

$\|f(x)-4^n f(x/2^n)\|\leq\frac{1}{4}\sum_{k=1}^{n}4^k\phi(x/2^k,0)$ for every positive integer n and $for\, all\, x\in X$. Now we can show that the sequence $4^n f(x/2^n)$ converges $for\, all\, x\in X$ and let $Q(x)=\lim_{n\to\infty}4^n f(x/2^n)$ and then follow the proof of theorem (3.1) we can obtain theorem (3.2).

corollary 3.1:

Let X Be a Normed Linear Space and let Y be a Banach Space. Let $f: X\to Y$ be a function satisfies the inequality

$$\|f(2x+y)+f(x-2y)+f(x+y)+f(x-y)-7f(x)-7f(y)\|\leq\in \tag{3.13}$$

For some real number $\in>0$, then there exist a unique function $Q: X\to Y$ such that Q satisfies (1.9) and

$$\|f(x)-Q(x)\| \le \frac{\in}{3} \quad for\ all\ x \in X \tag{3.14}$$

Proof:

We can choose $\phi(x,y) = \in$ for all $x, y \in X$ and also by theorem (3.1) follows that

$$\|f(x)-Q(x)\| \le \frac{1}{4}\sum_{k=0}^{\infty} \in/4^k$$

$$\|f(x)-Q(x)\| \le \frac{\in}{3} \ for\ all\ x \in X$$ as desired.

corollary 3.2:

Let X Be a Normed Vector Space and let Y be a Banach Space. Given positive real number $0 < p < 2 \in with\ p \ne 2$. If a function $f : X \to Y$ satisfies the inequality

$$\|f(2x+y)+f(x-2y)+f(x+y)+f(x-y)-7f(x)-7f(y)\| \le \in\left[\|x\|^p + \|y\|^p\right] \tag{3.15}$$

for all $x, y \in X$.

Then there exist a unique function $Q : X \to Y$ such that Q satisfies (1.9) and

$$\|f(x)-Q(x)\| \le \frac{\in}{\|4-2^p\|}\|x\|^p \ for\ all\ x \in X \tag{3.16}$$

Proof:

We can choose $\phi(x,y) = \in\left[\|x\|^p + \|y\|^p\right]$ for all $x, y \in X$. If $0 < p < 2$, then the condition (3.1) of Theorem(3.1) satisfies and consequently

$$\|f(x)-Q(x)\| \le \frac{1}{4}\sum_{k=0}^{\infty}\frac{\in 2^{kp}\|x\|^p}{4^k}$$

$$\le \frac{1}{4} \in \|x\|^p \sum_{k=0}^{\infty}\frac{2^{kp}}{2^{2k}}$$

$$\le \in \|x\|^p\left[\frac{1}{4-2^p}\right], p \ne 2 \ for\ all\ x \in X$$

Hence corollary (3.2).

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