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Analysis of a Fractional Order Prey-Predator Model (3-Species)

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Abstract

In this paper, we discuss the dynamical behavior of a fractional order prey predator model. The equilibrium points are computed and stability of the equilibrium points are analyzed. The phase portraits are obtained for different sets of parameter values. Numerical simulations are performed and it is shown that the system exhibits rich dynamical behaviors.

Keywords and phrases: Fractional order, stability analysis, Prey-Predator model.

I. INTRODUCTION

The concept of fractional-order calculus was proposed early 300 years ago. Fractional order equations are more suitable than integer order ones in modeling biological, economic and social systems where memory effects are important. Fractional order differential equations are generalizations of integer order differential equations. There are several definitions of the fractional derivative / integral. The Riemann-Liouville definition is [4]

$$ {}_aD_t^r f(t) = \frac{1}{\Gamma(n-r)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{r-n+1}} d\tau $$

for $n-1 < r < n$ and where $\Gamma(.)$ is the Gamma function.

The Caputo's definition of fractional derivatives can be expressed as

$$ {}_aD_t^r f(t) = \frac{1}{\Gamma(n-r)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{r-n+1}} d\tau $$

for $(n-1 < r < n)$.

Lemma 1. [2, 5] The following linear commensurate fractional-order autonomous system $D^{\alpha}x(t) = Ax(t), \quad x(0) = x_0$ is asymptotically stable if and only if $|\arg\lambda| > \alpha\frac{\pi}{2}$ is satisfied for all eigen values (λ) of matrix A. Also, this system is stable if and only if $|\arg\lambda| > \alpha\frac{\pi}{2}$ is satisfied for all eigen values (λ) of matrix A, and those critical eigen values which satisfy $|\arg\lambda| = \alpha\frac{\pi}{2}$ have geometric multiplicity one, where $0 < \alpha < 1,\ x \in R^n$ and $A \in R^{n\times n}$.

Lemma 2. [2, 5] Consider the following autonomous system for internal stability definition $D^{\alpha}x(t) = Ax(t), \quad x(0) = x_0$ with $\alpha = [\alpha_1, \alpha_2, ..., \alpha_n]^T$ and its n-dimensional representation:

$$\begin{aligned} D^{\alpha_1}x_1(t) &= a_{11}x_1(t) + a_{12}x_2(t) + ... + a_{1n}x_n(t) \\ D^{\alpha_2}x_2(t) &= a_{21}x_1(t) + a_{22}x_2(t) + ... + a_{2n}x_n(t) \\ &..... \\ D^{\alpha_n}x_n(t) &= a_{n1}x_1(t) + a_{n2}x_2(t) + ... + a_{nn}x_n(t) \end{aligned} \tag{1}$$

where all α_i's are rational numbers between *0* and *2*. Assume *m* to be the *LCM* of the denominators u_i's of α_i's, where $\alpha_i = \frac{u_i}{v_i}, u_i, v_i \in Z^+$ for $i = 1, 2, ..., n$ and we set $\gamma = \frac{1}{m}$.

Define:

$$\det\begin{bmatrix} \lambda^{m\alpha_1} - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda^{m\alpha_2} - a_{22} & \cdots & -a_{2n} \\ \vdots & & & \\ -a_{n1} & -a_{n2} & \cdots & \lambda^{m\alpha_n} - a_{nn} \end{bmatrix} = 0 \tag{2}$$

The characteristic equation can be transformed to integer-order polynomial equation if all α_i's are rational number. Then the zero solution of system is globally asymptotically stable if all roots λ_i's of the characteristic (polynomial) equation satisfy:

$$|\arg\lambda| > \gamma\frac{\pi}{2}, \ \forall i.$$

II. MODEL DESCRIPTION AND EQUILIBRIUM POINTS

Mathematical modeling in population dynamics has gained a lot of attention during the last few decades. The dynamical relationship between predators and their prey has been an important topic. Many researchers investigated predator-prey population models with system of first order differential equations. Recently the fractional order differential equations have gained a lot of attention in many field of applied mathematics including population dynamics due to their ability to provide better description of different non-linear phenomena [4]. The Lotka-Volterra equations are a pair of first order differential equations used to describe the dynamics of interactions of two species. In 1926, Volterra came up with a model to describe the evolution of predator and prey fish populations in the Adriatic Sea. They were proposed independently by Alfred J.Lotka in 1925 [1, 2]. Several authors formulated fractional order systems and analyzed the dynamical and qualitative behavior of the systems [3, 6, 7, 8]. The fractional Lotka-Volterra equations are obtained from the classical equations by replacing the first order time derivatives by fractional derivatives [3].

In this paper, we propose a system of fractional order prey-predator model. The stability of equilibrium points is studied. Numerical solutions and simulations of this model are provided. We consider the fractional order model as follows:

$$\begin{aligned} D^{\alpha_1}x(t) &= ax(t) - bx^2(t) - x(t)y(t) - x(t)z(t) \\ D^{\alpha_2}y(t) &= (1-c)y(t) + dx(t)y(t) \qquad (3) \\ D^{\alpha_3}z(t) &= (1-e)z(t) + fx(t)z(t) + gy(t)z(t) \end{aligned}$$

where the parameters a, b, c, d, e, f, g are positive and $\alpha_1, \alpha_2, \alpha_3$ are fractional orders. To evaluate the equilibrium points, let us consider

$D^{\alpha_1}x(t) = 0;\ D^{\alpha_2}y(t) = 0;\ D^{\alpha_3}z(t) = 0$.

The fractional order system has five equilibria

$E_0 = (0,0,0)$ (trivial), $E_1 = \left(\frac{a}{b}, 0, 0\right)$ (axial), $E_2 = \left(\frac{c-1}{d}, \frac{ad - b(c-1)}{d}, 0\right)$ (axial), $E_3 = \left(\frac{e-1}{f}, 0, \frac{af - b(e-1)}{f}\right)$ (axial) and

$E_4 = \left(\frac{c-1}{d}, \frac{d(e-1) - f(c-1)}{gd}, \frac{d(1+ag-e) + (f-bg)(c-1)}{gd}\right)$ (coexistence).

To accommodate biological meaning, the existence conditions for the equilibria require that they are nonnegative. It obvious E_0 and E_1 always exist, E_2 exist when $c > 1$ and

$ad > b(c-1)$, while E_3 exist when $e > 1$ and $af > b(e-1)$. The interior equilibrium E_4 exist when $d(e-1) > f(c-1)$ and $d > \frac{(bg-f)(c-1)}{(1+ag-e)}$.

III. STABILITY OF EQUILIBRIA

Based on (3), to investigate the local stability of each equilibrium point (x^*, y^*, z^*), we provide the Jacobian matrix $J(x^*, y^*, z^*)$.

$$J=\begin{bmatrix} a-2bx^{*}-y^{*}-z^{*} & -x^{*} & -x^{*} \\ dy^{*} & 1-c+dx^{*} & 0 \\ fz^{*} & gz^{*} & 1-e+fx^{*}+gy^{*} \end{bmatrix}. \quad (4)$$

For E_0, we have

$$J\left(E_0\right)=\begin{bmatrix} a & 0 & 0 \\ 0 & 1-c & 0 \\ 0 & 0 & 1-e \end{bmatrix}.$$

The eigen values are $\lambda_1=a,\ \lambda_2=1-c$ and $\lambda_3=1-e$. It is clear that E_0 is a saddle point, while for E_1 - we have

$$J\left(E_1\right)=\begin{bmatrix} -a & -\frac{a}{b} & -\frac{a}{b} \\ 0 & 1-c+\frac{ad}{b} & 0 \\ 0 & 0 & 1-e+\frac{af}{b} \end{bmatrix}.$$

The eigen values are $\lambda_1=-a,\ \lambda_2=1-c+\frac{ad}{b}$ and $\lambda_3=1-e+\frac{af}{b}$. Hence E_1 is asymptotically stable when $ad<bc$ and $af<be$. Jacobian of E_2 is

$$J\left(E_2\right)=\begin{bmatrix} \frac{b}{d}(1-c) & \frac{1-c}{d} & \frac{1-c}{d} \\ ad+b(1-c) & 0 & 0 \\ 0 & 0 & 1-e-\frac{f}{d}(1-c)+\frac{g}{d}[ad+b(1-c)] \end{bmatrix}.$$

which has the following eigen values:

$\lambda_1=1-e-\frac{f}{d}(1-c)+\frac{g}{d}[ad+b(1-c)]$ and

$$\lambda_{2,3}=\frac{b(1-c)\pm\sqrt{b^2(1-c)^2+4d(1-c)[ad+b(1-c)]}}{2d}.$$

Since both of λ_2 and λ_3 are negative, local stability of E_2 is determined by λ_1. Hence E_2 is stable note when $g<\frac{de+f(1-c)}{ad+b(1-c)}$ and $d>\frac{b(1-c)(2-c)}{1-a(1-c)}$.

Local stability of $E_3=\left(\frac{e-1}{f},0,\frac{af-b(e-1)}{f}\right)$ is determined by investigating the eigen values of

$$J\left(E_3\right)=\begin{bmatrix} \frac{b}{f}(1-e) & \frac{1-e}{f} & \frac{1-e}{f} \\ 0 & 1-c-\frac{d}{f}(1-e) & 0 \\ af+b(1-e) & \frac{g}{f}[af+b(1-e)] & 0 \end{bmatrix}.$$

namely $\lambda_1=1-c-\frac{d}{f}(1-e)$ and $\lambda_{2,3}=\frac{b(1-e)\pm\sqrt{b^2(1-e)^2+4f(1-e)[af+b(1-e)]}}{2f}$.

Obviously E_3 is stable when $d<\frac{cf}{e-1}$ and $f>\frac{b(1-e)(2-e)}{1-a(1-e)}$. Finally the local stability of the interior equilibrium point is investigated by considering the Jacobian matrix

$$J\left(E_4\right)=\begin{bmatrix} A & B & B \\ C & 0 & 0 \\ D & E & 0 \end{bmatrix}.$$

$A=\frac{b}{d}(1-c)$, $B=\frac{1-c}{d}$, $C=\frac{d(e-1)-f(c-1)}{g}$, $D=\frac{df(1+ag-e)+(c-1)f(f-bg)}{gd}$

and $E=(1+ag-e)+\frac{(c-1)(f-bg)}{d}$.The characteristic polynomial $P(\lambda)$ for $J\left(E_4\right)$ is $P(\lambda)=\lambda^3+a_1\lambda^2+a_2\lambda+a_3$ where $a_1=-A;\ a_2=-B(C+D);\ a_3=-BCE$. It is obvious that $a_1>0$ and $a_3>0$. If $a_1a_2>a_3$, then Routh Hurwitz criterion implies that all roots of $P(\lambda)$ have negative real parts, or in other words, E_4 is a stable point. It can be shown that equation $a_1a_2-a_3=B\left[A(C+D)+CE\right]$ is positive if $f>\frac{bd(c-1)[w+b(c-1)]}{w(b+d)+bd(c-1)}$, where $w=d(1+ag-e)+(f-bg)(c-1)$. This condition is in contrast to the existence condition of E_4. It means that E_4 is unstable. We shall complete this section by summarizing the existence and stability condition of all the equilibrium points in the following table:

Equilibrium Point	**Existence condition**	**Stability Condition**
E_0	-	Always saddle
E_1	-	$ad<bc$ and $af<be$
E_2	$c>1$ and $ad>b(c-1)$	$ad+b(1-c)<\frac{de+f(1-c)}{g}$ and $1-a(1-c)>\frac{b(1-c)(2-c)}{d}$

E_3	$e>1$ and $af>b(e-1)$	$d(e-1)<cf$ and $1-a(1-e)>\dfrac{b(1-e)(2-e)}{f}$.
E_4	$d(e-1)>f(c-1)$ and $d>\dfrac{(c-1)(bg-f)}{1+ag-e}$	$f>\dfrac{bd(c-1)[w+b(c-1)]}{w(b+d)+bd(c-1)}$

IV. DYNAMIC BEHAVIOR WITH NUMERICAL SOLUTIONS

Numerical solution of the fractional-order Prey-Predator system is given as follows [2]:

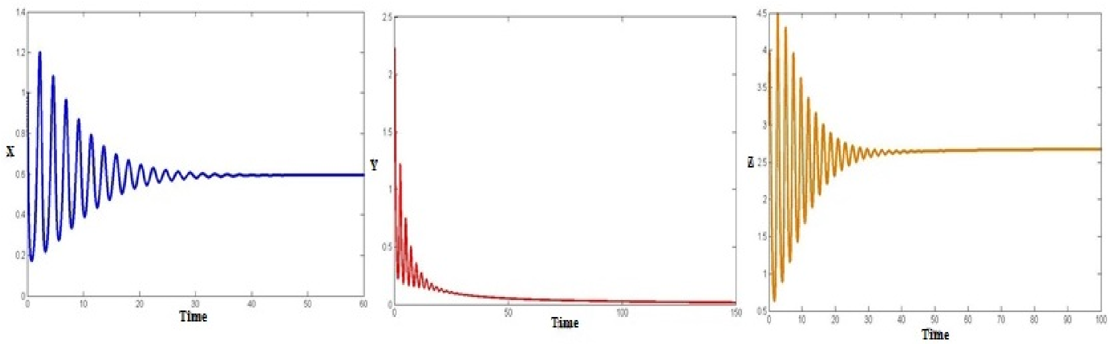
$$x(t_k)=\left(ax(t_{k-1})-bx^2(t_{k-1})-x(t_{k-1})y(t_{k-1})-x(t_{k-1})z(t_{k-1})\right)h^{\alpha_1}-\sum_{j=v}^{k}c_j^{(\alpha_1)}x(t_{k-j})$$

$$y(t_k)=\left((1-c)y(t_{k-1})+dx(t_k)y(t_{k-1})\right)h^{\alpha_2}-\sum_{j=v}^{k}c_j^{(\alpha_2)}y(t_{k-j})$$

$$z(t_k)=\left((1-e)z(t_{k-1})+fx(t_k)z(t_{k-1})+gy(t_k)z(t_{k-1})\right)h^{\alpha_3}-\sum_{j=v}^{k}c_j^{(\alpha_3)}z(t_{k-j})$$

where T_{sim} is the simulation time, $k=1,2,\ldots,N$ for $N=[T_{sim}/h]$, and $(x(0),y(0),z(0))$ is the start point (initial conditions).

Example 1. Let us consider the parameters with values $a=3; b=0.5; c=4;$ $d=5; e=4;$ $f=5;$ $g=0.4$ and the derivative order $\alpha_1=\alpha_2=\alpha_3=0.99$. For these parameters, the corresponding eigen values are $\lambda_1=0$ and $\lambda_{2,3}=-0.1500\pm i8.9987$ for E_4, which satisfy $|\arg\lambda|>\alpha\dfrac{\pi}{2}$. It means the system (3) is stable, see fig-1. Also the characteristic equation of the linearized system (3) at the equilibrium point E_4 is $\lambda^{297}+0.3\lambda^{198}+81\lambda^{99}=0$.



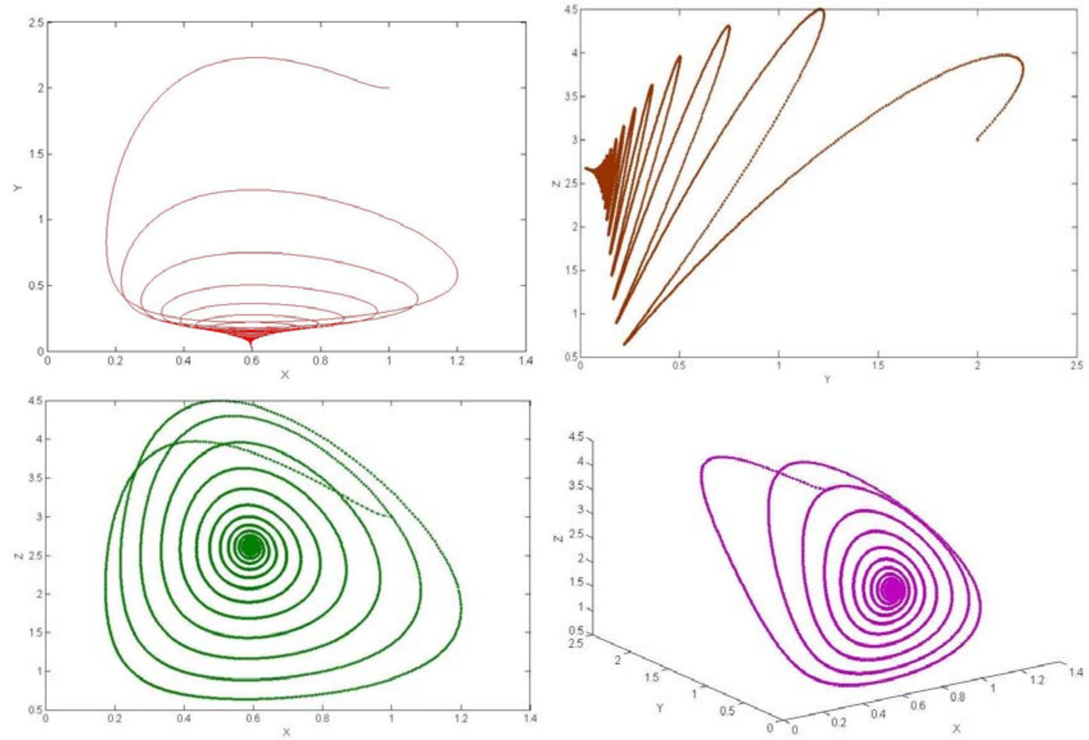
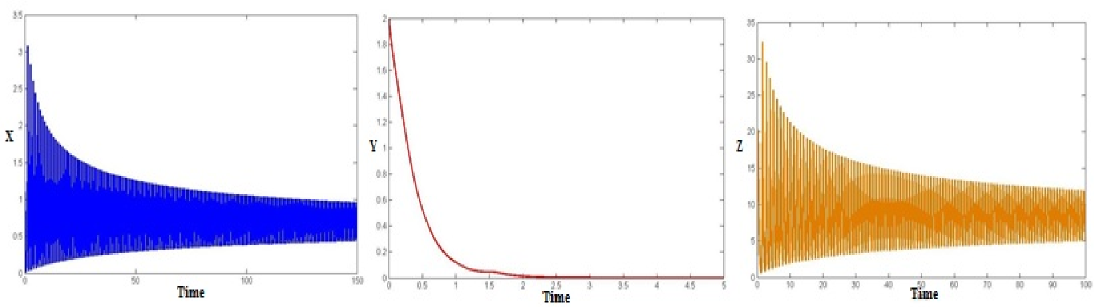


Figure 1: Time Series and Phase diagram of Fixed Point E_4 with Stability.

Example 2. Let us consider the parameters values $a = 8; b = 0.05; c = 4; d = 1; e = 7; f = 9; g = 4$ and the derivative order $\alpha_1 = \alpha_2 = \alpha_3 = 0.99$. For these parameters, the corresponding eigen values are $\lambda_1 = 1.6163$ and $\lambda_{2,3} = -0.8832 \pm i\,22.5793$ for E_4, which satisfy $|\arg \lambda| > \alpha \frac{\pi}{2}$. It means the system (3) is unstable, see fig-2. Also the characteristic equation of the linearized system (3) at the equilibrium point E_4 is
$\lambda^{297} + 0.15\lambda^{198} + 508.05\lambda^{99} + 825.3 = 0$.



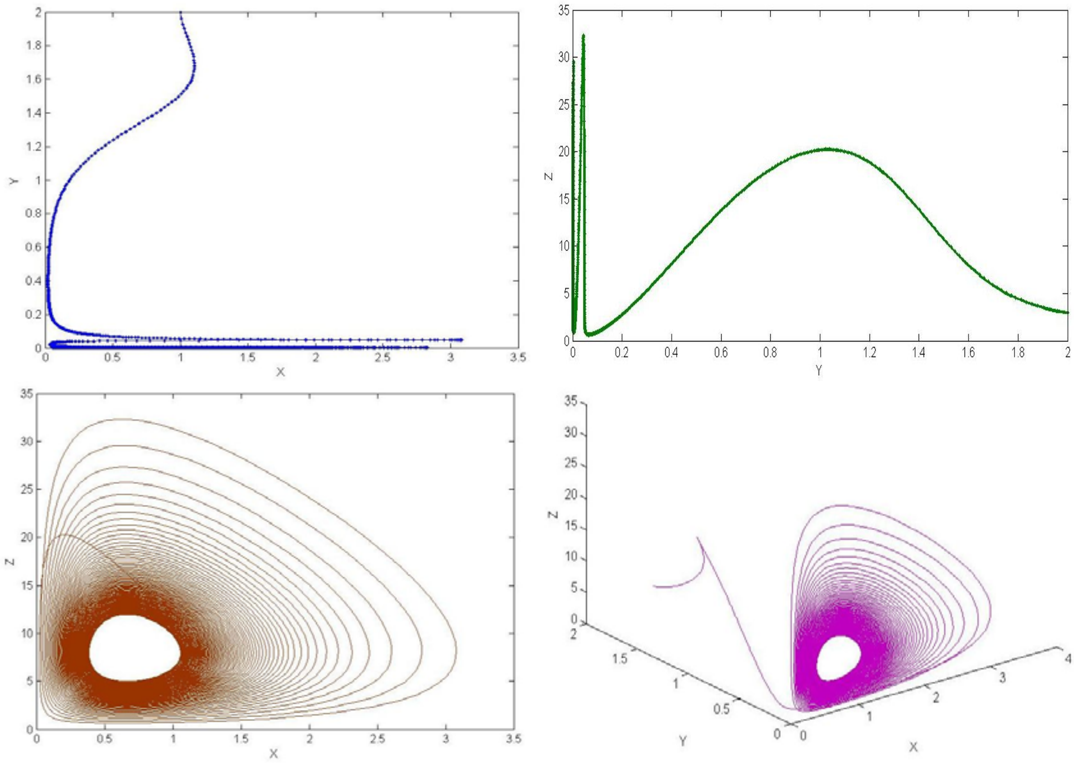


Figure 2: Time Series and Phase diagram of Fixed Point E_4 with Unstability.

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