

## A STRENGTHENED CARLEMAN'S INEQUALITY

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### Abstract

In this paper, it is proved that

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 - \sum_{k=1}^{\infty} b_k / (n+1)^k\right) a_n.$$

Where  $a_n \geq 0$ ,  $n = 1, 2, \dots$ ,  $0 < \sum_{n=1}^{\infty} a_n < \infty$  and  $b_0 = 1$ ,  $b_n = \frac{1}{n} \sum_{k=1}^n \frac{b_{n-k}}{k+1}$ ,  $n = 1, 2, \dots$ .

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## 1 Introduction

The following Carleman inequality is well known [1].

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n. \quad (1.1)$$

where  $e = \lim_{n \rightarrow \infty} (1 + 1/n)^n$ ,  $a_n \geq 0$ ,  $n = 1, 2, \dots$ , and  $0 < \sum_{n=1}^{\infty} a_n < \infty$ .

The constant  $e$  is sharp in the sense that it cannot be replaced by a smaller one.

Recently, the inequality (1) has also been improved by many authors, for example: Yang Bicheng and L. Debnath [2] with

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 - \frac{1}{2n+2}\right) a_n, \quad (1.2)$$

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in [3] by Yan Ping and Sun Guozheng with

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 + \frac{1}{n+1/5}\right)^{-1/2} a_n, \quad (1.3)$$

and in [4] by X. Yang with

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 - \frac{1}{2(n+1)} - \frac{1}{24(n+1)^2} - \frac{1}{48(n+1)^3}\right) a_n, \quad (1.4)$$

and

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 - \sum_{k=1}^6 \frac{b_k}{(n+1)^k}\right) a_n, \quad (1.5)$$

where  $b_1 = 1/2$ ,  $b_2 = 1/24$ ,  $b_3 = 1/48$ ,  $b_4 = 73/5760$ ,  $b_5 = 11/1280$ ,  $b_6 = 1945/580608$ . It is conjectured in [4] that if  $(1 + 1/x)^x = e \left[1 - \sum_{k=1}^{\infty} b_k/(x+1)^k\right]$ ,  $x > 0$ , then  $b_k > 0$ ,  $k = 1, 2, \dots$ . In this paper, we will prove the following Theorem and the conjecture is also proved to some extent.

**Theorem 1.1:** Let  $a_n \geq 0$ ,  $n = 1, 2, \dots$ , and  $0 < \sum_{n=1}^{\infty} a_n < \infty$ . Then we have

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 - \sum_{k=1}^{\infty} b_k/(n+1)^k\right) a_n. \quad (1.6)$$

Where  $b_n = \frac{1}{n} \sum_{k=1}^n \frac{b_{n-k}}{k+1}$ ,  $n = 1, 2, \dots$ .  $b_0 = 1$ .

## 2 Lemma

**Lemma 2.1:** Let  $f(t) = (1-t)^{1-1/t}$ , if  $0 < |t| < 1$  and  $f(0) = \lim_{t \rightarrow 0} f(t)$ . Then

$$f(t) = e(b_0 - b_1 t - b_2 t^2 - \dots). \quad (2.1)$$

Where  $b_0 = 1$ ,  $b_n = \frac{1}{n} \sum_{k=1}^n \frac{b_{n-k}}{k+1}$ ,  $n = 1, 2, \dots$ .

*Proof.* By  $f(t) = (1-t)^{1-1/t}$ , we have

$$\ln f(t) = \left(1 - \frac{1}{t}\right) \ln(1-t). \quad (2.2)$$

Using Maclaurin series, from (8), we obtain

$$\ln f(t) = 1 - \frac{t}{2} - \sum_{n=2}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) t^n. \quad (2.3)$$

Differentiating with respect to  $t$ , we get

$$\frac{f'(t)}{f(t)} = -\frac{1}{2} - \sum_{n=2}^{\infty} \left(1 - \frac{n}{n+1}\right) t^{n-1}, \quad (2.4)$$

and hence  $f'(t) = f(t)\varphi(t)$ , where  $\varphi(t) = -\frac{1}{2} - \sum_{n=2}^{\infty} (1 - \frac{n}{n+1}) t^{n-1}$ . Applying Leibnitz rule leads to

$$f^{(n)}(t) = \sum_{k=0}^{n-1} C_{n-1}^k f^{(n-k-1)}(t)\varphi^k(t), \quad n = 1, 2, \dots \tag{2.5}$$

Easy calculating reveals that

$$\varphi^{(n)}(0) = -\frac{n!}{n+2}, \quad n = 0, 1, 2, \dots \tag{2.6}$$

Hence

$$f^{(n)}(0) = -\sum_{k=0}^{n-1} \frac{k!}{k+2} C_{n-1}^k f^{(n-k-1)}(0). \tag{2.7}$$

Therefore

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n = f(0) - \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{k+2} \frac{f^{(n-k-1)}(0)}{(n-k-1)!} \right) t^n. \tag{2.8}$$

Let

$$b_0 = \frac{f(0)}{e}, b_n = \frac{f^{(n)}(0)}{n!e}, \quad n = 1, 2, \dots,$$

we conclude

$$f(t) = e(b_0 - b_1t - b_2t^2 - \dots).$$

The proof of Lemma 2.1 is complete. ■

**Lemma 2.2:** For all natural number  $m$ , then

$$\left(1 + \frac{1}{m}\right)^m = e \left( b_0 - \frac{b_1}{m+1} - \frac{b_2}{(m+1)^2} - \dots \right). \tag{2.9}$$

Where  $b_0 = 1, b_n = \frac{1}{n} \sum_{k=1}^n \frac{b_{n-k}}{k+1}, n = 1, 2, \dots$ .

*Proof.* Apply lemma 2.1 with  $t = 1/m + 1$ . ■

### 3 Proof of Theorem

Let  $c_i > 0(i = 1, 2, \dots)$ . By geometric and arithmetic means inequality, we have

$$(c_1a_1c_2a_2 \dots c_na_n)^{1/n} \leq \frac{1}{n} \sum_{m=1}^n c_m a_m. \tag{3.1}$$

Thus

$$\begin{aligned}
 & \sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} \\
 &= \sum_{n=1}^{\infty} \left( \frac{c_1 a_1 c_2 a_2 \cdots c_n a_n}{a_1 a_2 \cdots a_n} \right)^{1/n} \\
 &= \sum_{n=1}^{\infty} (c_1 c_2 \cdots c_n)^{-1/n} (c_1 a_1 c_2 a_2 \cdots c_n a_n)^{1/n} \\
 &\leq \sum_{n=1}^{\infty} (c_1 c_2 \cdots c_n)^{1/n} \frac{1}{n} \sum_{m=1}^n c_m a_m \\
 &= \sum_{m=1}^{\infty} c_m a_m \sum_{n=m}^{\infty} \frac{1}{n} (c_1 c_2 \cdots c_n)^{-1/n}.
 \end{aligned}$$

Set  $c_m = (m+1)^m / m^{m-1}$ , ( $m = 1, 2, \dots$ ), then

$$c_1 c_2 \cdots c_n = (n+1)^n,$$

and

$$\sum_{n=m}^{\infty} \frac{1}{n} (c_1 c_2 \cdots c_n)^{-1/n} = \sum_{n=m}^{\infty} \frac{1}{n(n+1)} = \frac{1}{m}.$$

Hence

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} \leq \sum_{m=1}^{\infty} \frac{c_m}{m} a_m = \sum_{m=1}^{\infty} \left(1 + \frac{1}{m}\right)^m a_m.$$

By Lemma 2.2, we get

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} \leq e \sum_{m=1}^{\infty} \left(1 - \sum_{k=1}^{\infty} \frac{b_k}{(m+1)^k}\right) a_m. \quad (3.2)$$

Because of  $0 < \sum_{n=1}^{\infty} a_n < \infty$ , the sign of equality in (17) not holds. The proof of Theorem 2.2 is complete.  $\blacksquare$

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