Communications in Applied Geometry.
ISSN 2249-4286 Volume 2, Number 1 (2012) pp. 7-23

http://www.ripublication.com/cag.htm

A Conjecture of Ducci Sequences

Euich Miztani, Akihiro Nozaki, and Toru Sawatari

JEIN Institute for Fundamental Science(JIFS)
VBL Kyoto University, Yoshida honmachi,
sakyo-ku, Kyoto 600-8813, Japan

Professor Emeritus, Otsuma Women's University
2-7-1 Karakida, Tama-shi, Tokyo 206-8540, Japan

Microsoft Japan Co., Ltd
Shinagawa Grand Central Tower 2-16-3,
Konan Minato-ku, Tokyo 108-0075, Japan

Abstract

In this paper, we argue an arithmetic issue also known as Ducci's 4 or n-number game. First we set an arbitrary non-negative integer at each vertex of a polygon, and apply the following procedure: at the center of each edge, set the difference of two integers at both ends of the edge, and connect the centers of adjacent edges: in this way, we obtain a new polygon with non-negative integers at every vertex. We apply the same procedure recursively, until all numbers set around the polygon become zeros. In the case the number N of edges of the polygon is four, it has been claimed that the procedure will always terminate in finite steps, and it is proved by B. Freedman [1]. For an arbitrary N, we know the following facts.

1. If the number N of edges of the polygon is a power of two, the recursive procedure will always terminate in finite steps.
2. Otherwise, there exits some non-negative integers at vertices of the polygon, from which the recursive procedure will never terminate.

This time, we discuss the following fact: Let A, B, or C be positive integers on consecutive vertices on polygon A≤C≤B, A≥C≥B, B≤A≤C, or B≥A≥C, the recursive procedure by the binary operation which vertices N are four will always terminate by four steps. In this paper, we prove it and also discuss further general case.

Keywords: Ducci sequences, polygon subtraction, recursive procedure, binary operation.

2010 Mathematics Subject Classification: 11A99.

1. Introduction

Historically, the arithmetic issue is well known as Ducci's 4 or n-number game. It can be arranged as follows. As shown in Fig.1, The arranged game sets non-negative integers at every corner of a square, and put at the center of each edge the difference of two integers at both ends of the edge. By connecting the centers of adjacent edges, he obtains a new square with non-negative integers at every corner. The procedure will be repeated recursively, until all numbers at corners become zeros. He noted that the procedure will always terminate in finite steps.

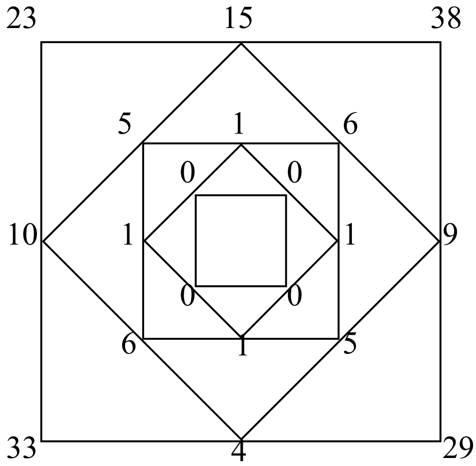

Figure 1

2. Notations and Basic Concepts

In what follows we give basic concepts and fundamental properties, in order to formulate our basic conjecture on the above mentioned problem.

Definition 2.1. Serializing any non-negative integers $a_1, a_2, a_3, \ldots, a_{n-1}, a_N$ for vertices of a polygon with N vertices, let $[a_{n-1}, a_n]$ and $[a_N, a_1]$ $(N \geq n)$ be absolute value of binary operation of the circular permutation by the recursive procedure noted above. Therefore, the equation of the recursive procedure is

$$[a_{n-1}, a_n] = |a_{n-1} - a_n|, \; [a_N, a_1] = |a_N - a_1| \qquad (1)$$

Then, repeat the recursive procedure until the terminate values are zeros.

Remark: a couple of members in square brackets are commutative because they are positive integers.

Lemma 2.1. Let A, B, and C be non-negative integers on consecutive vertices on polygon.
i). If $A<B<C$,
$[[A, B], [B, C]]=[C, [A, 2B]]$ (2)
ii). If $A>B>C$,
$[[A, B], [B, C]]=[A, [C, 2B]]$ (3)
iii). If $A\le C\le B$, $A\ge C\ge B$, $B\le A\le C$, or $B\ge A\ge C$,
$[[A, B], [B, C]]=[A, C]$ (4)

***Remark* :** Every combination of the six possible ordering of three positive integers A, B, and C is treated in one of the three cases i), ii) or iii).

***Proof*.**
i). If $A<B<C$,
$[[A, B], [B, C]]=|(B-A)-(C-B)|=|(-A+2B)-C|=[C, [A, 2B]]$
ii). If $A>B>C$,
$[[A, B], [B, C]]=|(A-B)-(B-C)|=|A-(2B-C)|=[A, [C, 2B]]$
iii).
If $A\le C\le B$,
$[[A, B], [B, C]]=(B-A)-(B-C)=C-A=[A, C]$
If $A\ge C\ge B$,
$[[A, B], [B, C]]=(A-B)-(C-B)=A-C=[A, C]$
If $B\le A\le C$,
$[[A, B], [B, C]]= (C-B)-(A-B)=C-A=[A, C]$
If $B\ge A\ge C$,
$[[A, B], [B, C]]=(B-C)-(B-A)=A-C=[A, C]$ □

Those case i), ii), and iii) are combined in the process of operations, but it is very difficult to research all the processes because of the complication. Therefore, let us check each of the cases separately.

Now, let us algorithmically verify positive integers set around all the vertices of square (; N=4) based on the lemma 2.1.

i). If $A<B<C$ for any consecutive a_n and the following integers after recursive procedures except for cases of a_{N-1}, a_N, and a_1 (; $a_1 < a_{N-1} < a_N$ by condition of the circular permutation, therefore it belongs to the case iii)) and a_N, a_1, and a_2 (; $a_1 < a_2 < a_N$),
Step1
$[a_1, a_2], [a_2, a_3], [a_3, a_4], [a_4, a_1]$
Step2
$[[a_1, a_2], [a_2, a_3]]=[a_3, [a_1, 2a_2]]$
$[[a_2, a_3], [a_3, a_4]]=[a_4, [a_2, 2a_3]]$
$[[a_3, a_4], [a_4, a_1]]=[a_3, a_1]$
$[[a_4, a_1], [a_1, a_2]]=[a_4, a_2]$
Step3
$[[a_3, [a_1, 2a_2]], [a_4, [a_2, 2a_3]]]$

$[[a_4, [a_2, 2a_3]], [a_3, a_1]]$
$[[a_3, a_1], [a_4, a_2]]$
$[[a_4, a_2], [a_3, [a_1, 2a_2]]]$
Step4
$[[[a_3, [a_1, 2a_2]], [a_4, [a_2, 2a_3]]], [[a_4, [a_2, 2a_3]], [a_3, a_1]]]$
$[[[a_4, [a_2, 2a_3]], [a_3, a_1]], [[a_3, a_1], [a_4, a_2]]]$
$[[[a_3, a_1], [a_4, a_2]], [[a_4, a_2], [a_3, [a_1, 2a_2]]]$
$[[[a_4, a_2], [a_3, [a_1, 2a_2]]], [[a_4, a_2]], [a_3, [a_1, 2a_2]]]$
Since the values in square brackets diverge, we stop the operation.

ii). If A>B>C for any consecutive a_n and the following integers after recursive procedures except for cases of a_{N-1}, a_N, a_1 and a_N, a_1, a_2,
then the results are the same as the case i) and we do not.

iii). If A≤C≤B, A≥C≥B, B≤A≤C, and B≥A≥C for any consecutive a_n,
Step1
$[a_1, a_2], [a_2, a_3], [a_3, a_4], [a_4, a_1]$
Step2
$[[a_1, a_2], [a_2, a_3]]=[a_1, a_3]$
$[[a_2, a_3], [a_3, a_4]]=[a_2, a_4]$
$[[a_3, a_4], [a_4, a_1]]=[a_3, a_1]$
$[[a_4, a_1], [a_1, a_2]]=[a_4, a_2]$
Step3
$[[a_1, a_3], [a_2, a_4]]$
$[[a_2, a_4], [a_3, a_1]]=[[a_1, a_3], [a_2, a_4]]$
$[[a_3, a_1], [a_4, a_2]]=[[a_1, a_3], [a_2, a_4]]$
$[[a_4, a_2], [a_1, a_3]]=[[a_1, a_3], [a_2, a_4]]$
Step4
0 allout.

Therefore, we get the valid results for the case iii). As verified above, we assume that the terminal values based on the case iii) of lemma 2.1 will be zeros, only if the number of vertices is four. Then,

Conjecture. *Let A, B, and C be positive integers on consecutive vertices on polygon A≤C≤B, A≥C≥B, B≤A≤C, or B≥A≥C, the recursive procedure by the binary operation which the number of vertices is four will always terminate by four steps.*

Postscript.
We were originally supposed to offer the conjecture wider as follows,

Conjecture. *Let A, B, and C be positive integers on consecutive vertices on polygon A≤C≤B, A≥C≥B, B≤A≤C, or B≥A≥C, the recursive procedure by the binary operation which the number of vertices is a power of two will always terminate by the power of two steps.*

For example, under the special conditions, if N=2, 4, and 8, then the operation terminates by *2, 4, and 8 steps* as calculated above.

To verify it above, let A, B, and C be positive integers on consecutive vertices on polygon and A≤C≤B, A≥C≥B, B≤A≤C, or B≥A≥C, and we calculate the differences to see if the results by recursive procedures are terminally zeros.

First of all, if N=2,

Remark: a polygon with two vertices (and double sides).
Step1
$[a_1, a_2], [a_2, a_1]$
Step2
$[[a_1, a_2], [a_2, a_1]]=[a_1, a_1]=0$

N=3,
Step1
$[a_1, a_2], [a_2, a_3], [a_3, a_1]$
Step2
$[[a_1, a_2], [a_2, a_3]]=[a_1, a_3]$
$[[a_2, a_3], [a_3, a_1]]=[a_2, a_1]$
$[[a_3, a_1], [a_1, a_2]]=[a_3, a_2]$
Step3
$[[a_1, a_3], [a_2, a_1]]$
$[[a_2, a_1], [a_3, a_2]]$
$[[a_3, a_2], [a_1, a_3]]$
Step4 (*Remark:* any integer at vertices (, not only any initial integer before recursive procedure) follows the case iii) of lemma 2.1.)
$[[[a_1, a_3], [a_2, a_1]], [[a_2, a_1], [a_3, a_2]]]$(=[A', B']. [B', C'])
$=[[a_1, a_3], [a_3, a_2]]$(=[[A, B], [B, C]])$=[[a_1, a_2]$
$[[[a_2, a_1], [a_3, a_2]], [[a_3, a_2], [a_1, a_3]]]=[a_2, a_3]$
$[[[a_3, a_2], [a_1, a_3]], [[a_1, a_3], [a_2, a_1]]]=[a_3, a_1]$
Step5
Return to Step2; the values are never zeros.

N=4, it is as calculated above.

N=5,
Step1
$[a_1, a_2], [a_2, a_3], [a_3, a_4], [a_4, a_5], [a_5, a_1]$
Step2
$[[a_1, a_2], [a_2, a_3]]=[a_1, a_3]$
$[[a_2, a_3], [a_3, a_4]]=[a_2, a_4]$
$[[a_3, a_4], [a_4, a_5]]=[a_3, a_5]$
$[[a_4, a_5], [a_5, a_1]]=[a_4, a_1]$
$[[a_5, a_1], [a_1, a_2]]=[a_5, a_2]$
Step3
$[[a_1, a_3], [a_2, a_4]]$
$[[a_2, a_4], [a_3, a_5]]$

$[[a_3, a_5], [a_4, a_1]]$
$[[a_4, a_1], [a_5, a_2]]$
$[[a_5, a_2], [a_1, a_3]]$
Step4 (*Remark: any integer* at vertices follows the case iii) of lemma 2.1.)
$[[[a_1, a_3], [a_2, a_4]], [[a_2, a_4], [a_3, a_5]]]$(=[[A', B'], [B', C']]=[A', C'])
$=[[a_1, a_3], [a_3, a_5]]$(=[[A, B], [B, C]])$=[a_1, a_5]=[a_5, a_1]$
$[[[a_2, a_4], [a_3, a_5]], [[a_3, a_5], [a_4, a_1]]]=[[a_2, a_4], [a_4, a_1]]=[a_2, a_1]=[a_1, a_2]$
$[[[a_3, a_5], [a_4, a_1]], [[a_4, a_1], [a_5, a_2]]]=[[a_3, a_5], [a_5, a_2]]=[a_3, a_2]=[a_2, a_3]$
$[[[a_4, a_1], [a_5, a_2]], [[a_5, a_2], [a_1, a_3]]]=[[a_4, a_1], [a_1, a_3]]=[a_4, a_3]=[a_3, a_4]$
$[[[a_5, a_2], [a_1, a_3]], [[a_1, a_3], [a_2, a_4]]]=[[a_5, a_2], [a_2, a_4]]=[a_5, a_4]=[a_4, a_5]$
Step5
Return to Step2

N=6,
Step1
$[a_1, a_2], [a_2, a_3], [a_3, a_4], [a_4, a_5], [a_5, a_6], [a_6, a_1]$
Step2
$[[a_1, a_2], [a_2, a_3]]=[a_1, a_3]$
$[[a_2, a_3], [a_3, a_4]]=[a_2, a_4]$
$[[a_3, a_4], [a_4, a_5]]=[a_3, a_5]$
$[[a_4, a_5], [a_5, a_6]]=[a_4, a_6]$
$[[a_5, a_6], [a_6, a_1]]=[a_5, a_1]$
$[[a_6, a_1], [a_1, a_2]]=[a_6, a_2]$
Step3
$[[a_1, a_3], [a_2, a_4]]$
$[[a_2, a_4], [a_3, a_5]]$
$[[a_3, a_5], [a_4, a_6]]$
$[[a_4, a_6], [a_5, a_1]]$
$[[a_5, a_1], [a_6, a_2]]$
$[[a_6, a_2], [a_1, a_3]]$
Step4 (*Remark: any integer* at vertices follows the case iii) of lemma 2.1.)
$[[[a_1, a_3], [a_2, a_4]], [[a_2, a_4], [a_3, a_5]]]=[[a_1, a_3], [a_3, a_5]]=[a_1, a_5]=[a_5, a_1]$
$[[[a_2, a_4], [a_3, a_5]], [[a_3, a_5], [a_4, a_6]]]=[[a_2, a_4], [a_4, a_6]]=[a_2, a_6]=[a_6, a_2]$
$[[[a_3, a_5], [a_4, a_6]], [[a_4, a_6], [a_5, a_1]]]=[[a_3, a_5], [a_5, a_1]]=[a_3, a_1]=[a_1, a_3]$
$[[[a_4, a_6], [a_5, a_1]], [[a_5, a_1], [a_6, a_2]]]=[[a_4, a_6], [a_6, a_2]]=[a_4, a_2]=[a_2, a_4]$
$[[[a_5, a_1], [a_6, a_2]], [[a_6, a_2], [a_1, a_3]]]=[[a_5, a_1], [a_1, a_3]]=[a_5, a_3]=[a_3, a_5]$
$[[[a_6, a_2], [a_1, a_3]], [[a_1, a_3], [a_2, a_4]]]=[[a_6, a_2], [a_2, a_4]]=[a_6, a_4]=[a_4, a_6]$
Return to Step3.

N=7,
Step1
$[a_1, a_2], [a_2, a_3], [a_3, a_4], [a_4, a_5], [a_5, a_6], [a_6, a_7], [a_7, a_1]$
Step2
$[[a_1, a_2], [a_2, a_3]]=[a_1, a_3]$
$[[a_2, a_3], [a_3, a_4]]=[a_2, a_4]$

$[[a_3, a_4], [a_4, a_5]]=[a_3, a_5]$
$[[a_4, a_5], [a_5, a_6]]=[a_4, a_6]$
$[[a_5, a_6], [a_6, a_7]]=[a_5, a_7]$
$[[a_6, a_7], [a_7, a_1]]=[a_6, a_1]$
$[[a_7, a_1], [a_1, a_2]]=[a_7, a_2]$
Step3
$[[a_1, a_3], [a_2, a_4]]$
$[[a_2, a_4], [a_3, a_5]]$
$[[a_3, a_5], [a_4, a_6]]$
$[[a_4, a_6], [a_5, a_7]]$
$[[a_5, a_7], [a_6, a_1]]$
$[[a_6, a_1], [a_7, a_2]]$
$[[a_7, a_2], [a_1, a_3]]$
Step4 (*Remark: any integer* at vertices follows the case iii) of lemma 2.1.)
$[[[a_1, a_3], [a_2, a_4]], [[a_2, a_4], [a_3, a_5]]]=[[a_1, a_3], [a_3, a_5]]=[a_1, a_5]$
$[[[a_2, a_4], [a_3, a_5]], [[a_3, a_5], [a_4, a_6]]]=[[a_2, a_4], [a_4, a_6]]=[a_2, a_6]$
$[[[a_3, a_5], [a_4, a_6]], [[a_4, a_6], [a_5, a_7]]]=[[a_3, a_5], [a_5, a_7]]=[a_3, a_7]$
$[[[a_4, a_6], [a_5, a_7]], [[a_5, a_7], [a_6, a_1]]]=[[a_4, a_6], [a_6, a_1]]=[a_4, a_1]$
$[[[a_5, a_7], [a_6, a_1]], [[a_6, a_1], [a_7, a_2]]]=[[a_5, a_7], [a_7, a_2]]=[a_5, a_2]$
$[[[a_6, a_1], [a_7, a_2]], [[a_7, a_2], [a_1, a_3]]]=[[a_6, a_1], [a_1, a_3]]=[a_6, a_3]$
$[[[a_7, a_2], [a_1, a_3]], [[a_1, a_3], [a_2, a_4]]]=[[a_7, a_2], [a_2, a_4]]=[a_7, a_4]$
Step5
$[[a_1, a_5], [a_2, a_6]]$
$[[a_2, a_6], [a_3, a_7]]$
$[[a_3, a_7], [a_4, a_1]]$
$[[a_4, a_1], [a_5, a_2]]$
$[[a_5, a_2], [a_6, a_3]]$
$[[a_6, a_3], [a_7, a_4]]$
$[[a_7, a_4], [a_1, a_5]]$
Step6 (*Remark: any integer* at vertices follows the case iii) of lemma 2.1.)
$[[[a_1, a_5], [a_2, a_6]], [[a_2, a_6], [a_3, a_7]]]=[[a_1, a_5], [a_3, a_7]]$
$[[[a_2, a_6], [a_3, a_7]], [[a_3, a_7], [a_4, a_1]]]=[[a_2, a_6], [a_4, a_1]]$
$[[[a_3, a_7], [a_4, a_1]], [[a_4, a_1], [a_5, a_2]]]=[[a_3, a_7], [a_5, a_2]]$
$[[[a_4, a_1], [a_5, a_2]], [[a_5, a_2], [a_6, a_3]]]=[[a_4, a_1], [a_6, a_3]]$
$[[[a_5, a_2], [a_6, a_3]], [[a_6, a_3], [a_7, a_4]]]=[[a_5, a_2], [a_7, a_4]]$
$[[[a_6, a_3], [a_7, a_4]], [[a_7, a_4], [a_1, a_5]]]=[[a_6, a_3], [a_1, a_5]]$
$[[[a_7, a_4], [a_1, a_5]], [[a_1, a_5], [a_2, a_6]]]=[[a_7, a_4], [a_2, a_6]]$
Step7
$[[[a_1, a_5], [a_3, a_7]], [[a_2, a_6], [a_4, a_1]]]$
$[[[a_2, a_6], [a_4, a_1]], [[a_3, a_7], [a_5, a_2]]]$
$[[[a_3, a_7], [a_5, a_2]], [[a_4, a_1], [a_6, a_3]]]$
$[[[a_4, a_1], [a_6, a_3]], [[a_5, a_2], [a_7, a_4]]]$
$[[[a_5, a_2], [a_7, a_4]], [[a_6, a_3], [a_1, a_5]]]$
$[[[a_6, a_3], [a_1, a_5]], [[a_7, a_4], [a_2, a_6]]]$
$[[[a_7, a_4], [a_2, a_6]], [[a_1, a_5], [a_3, a_7]]]$

Step8 (*Remark: any integer* at vertices follows the case iii) of lemma 2.1.)
$[[[[a_1, a_5], [a_3, a_7]], [[a_2, a_6], [a_4, a_1]]], [[[a_2, a_6], [a_4, a_1]], [[a_3, a_7], [a_5, a_2]]]]=[a_1, a_2]$
$[[[[a_2, a_6], [a_4, a_1]], [[a_3, a_7], [a_5, a_2]]], [[[a_3, a_7], [a_5, a_2]], [[a_4, a_1], [a_6, a_3]]]]=[a_2, a_3]$
$[[[[a_3, a_7], [a_5, a_2]], [[a_4, a_1], [a_6, a_3]]], [[[a_4, a_1], [a_6, a_3]], [[a_5, a_2], [a_7, a_4]]]]=[a_3, a_4]$
$[[[[a_4, a_1], [a_6, a_3]], [[a_5, a_2], [a_7, a_4]]], [[[a_5, a_2], [a_7, a_4]], [[a_6, a_3], [a_1, a_5]]]]=[a_4, a_5]$
$[[[[a_5, a_2], [a_7, a_4]], [[a_6, a_3], [a_1, a_5]]], [[[a_6, a_3], [a_1, a_5]], [[a_7, a_4], [a_2, a_6]]]]=[a_5, a_6]$
$[[[[a_6, a_3], [a_1, a_5]], [[a_7, a_4], [a_2, a_6]]], [[[a_7, a_4], [a_2, a_6]], [[a_1, a_5], [a_3, a_7]]]]=[a_6, a_7]$
$[[[[a_7, a_4], [a_2, a_6]], [[a_1, a_5], [a_3, a_7]]], [[[a_1, a_5], [a_3, a_7]], [[a_2, a_6], [a_4, a_1]]]]=[a_7, a_1]$
Step9
Return to Step2.

N=8
Step1
$[a_1, a_2], [a_2, a_3], [a_3, a_4], [a_4, a_5], [a_5, a_6], [a_6, a_7], [a_7, a_8], [a_8, a_1]$
Step2
$[[a_1, a_2], [a_2, a_3]]=[a_1, a_3]$
$[[a_2, a_3], [a_3, a_4]]=[a_2, a_4]$
$[[a_3, a_4], [a_4, a_5]]=[a_3, a_5]$
$[[a_4, a_5], [a_5, a_6]]=[a_4, a_6]$
$[[a_5, a_6], [a_6, a_7]]=[a_5, a_7]$
$[[a_6, a_7], [a_7, a_8]]=[a_6, a_8]$
$[[a_7, a_8], [a_8, a_1]]=[a_7, a_1]$
$[[a_8, a_1], [a_1, a_2]]=[a_8, a_2]$
Step3
$[[a_1, a_3], [a_2, a_4]]$
$[[a_2, a_4], [a_3, a_5]]$
$[[a_3, a_5], [a_4, a_6]]$
$[[a_4, a_6], [a_5, a_7]]$
$[[a_5, a_7], [a_6, a_8]]$
$[[a_6, a_8], [a_7, a_1]]$
$[[a_7, a_1], [a_8, a_2]]$
$[[a_8, a_2], [a_1, a_3]]$
Step4 (*Remark: any integer* at vertices follows the case iii) of lemma 2.1.)
$[[[a_1, a_3], [a_2, a_4]], [[a_2, a_4], [a_3, a_5]]]=[[a_1, a_3], [a_3, a_5]]=[a_1, a_5]$
$[[[a_2, a_4], [a_3, a_5]], [[a_3, a_5], [a_4, a_6]]]=[[a_2, a_4], [a_4, a_6]]=[a_2, a_6]$
$[[[a_3, a_5], [a_4, a_6]], [[a_4, a_6], [a_5, a_7]]]=[[a_3, a_5], [a_5, a_7]]=[a_3, a_7]$
$[[[a_4, a_6], [a_5, a_7]], [[a_5, a_7], [a_6, a_8]]]=[[a_4, a_6], [a_6, a_8]]=[a_4, a_8]$
$[[[a_5, a_7], [a_6, a_8]], [[a_6, a_8], [a_7, a_1]]]=[[a_5, a_7], [a_7, a_1]]=[a_5, a_1]$
$[[[a_6, a_8], [a_7, a_1]], [[a_7, a_1], [a_8, a_2]]]=[[a_6, a_8], [a_8, a_2]]=[a_6, a_2]$
$[[[a_7, a_1], [a_8, a_2]], [[a_8, a_2], [a_1, a_3]]]=[[a_7, a_1], [a_1, a_3]]=[a_7, a_3]$
$[[[a_8, a_2], [a_1, a_3]], [[a_1, a_3], [a_2, a_4]]]=[[a_8, a_2], [a_2, a_4]]=[a_8, a_4]$
Step5
$[[a_1, a_5], [a_2, a_6]]$
$[[a_2, a_6], [a_3, a_7]]$
$[[a_3, a_7], [a_4, a_8]]$

$[[a_4, a_8], [a_5, a_1]]$
$[[a_5, a_1], [a_6, a_2]]$
$[[a_6, a_2], [a_7, a_3]]$
$[[a_7, a_3], [a_8, a_4]]$
$[[a_8, a_4], [a_1, a_5]]$
Step6 (*Remark: any integer* at vertices follows the case iii) of lemma 2.1.)
$[[[a_1, a_5], [a_2, a_6]], [[a_2, a_6], [a_3, a_7]]]=[[a_1, a_5], [a_3, a_7]]$
$[[[a_2, a_6], [a_3, a_7]], [[a_3, a_7], [a_4, a_8]]]=[[a_2, a_6], [a_4, a_8]]$
$[[[a_3, a_7], [a_4, a_8]], [[a_4, a_8], [a_5, a_1]]]=[[a_3, a_7], [a_5, a_1]]$
$[[[a_4, a_8], [a_5, a_1]], [[a_5, a_1], [a_6, a_2]]]=[[a_4, a_8], [a_6, a_2]]$
$[[[a_5, a_1], [a_6, a_2]], [[a_6, a_2], [a_7, a_3]]]=[[a_5, a_1], [a_7, a_3]]$
$[[[a_6, a_2], [a_7, a_3]], [[a_7, a_3], [a_8, a_4]]]=[[a_6, a_2], [a_8, a_4]]$
$[[[a_7, a_3], [a_8, a_4]], [[a_8, a_4], [a_1, a_5]]]=[[a_7, a_3], [a_1, a_5]]$
$[[[a_8, a_4], [a_1, a_5]], [[a_1, a_5], [a_2, a_6]]]=[[a_8, a_4], [a_2, a_6]]$
Step7
$[[[a_1, a_5], [a_3, a_7]], [[a_2, a_6], [a_4, a_8]]]$
$[[[a_2, a_6], [a_4, a_8]], [[a_3, a_7], [a_5, a_1]]]=[[[a_1, a_5], [a_3, a_7]], [[a_2, a_6], [a_4, a_8]]]$
$[[[a_3, a_7], [a_5, a_1]], [[a_4, a_8], [a_6, a_2]]]=[[[a_1, a_5], [a_3, a_7]], [[a_2, a_6], [a_4, a_8]]]$
$[[[a_4, a_8], [a_6, a_2]], [[a_5, a_1], [a_7, a_3]]]=[[[a_1, a_5], [a_3, a_7]], [[a_2, a_6], [a_4, a_8]]]$
$[[[a_5, a_1], [a_7, a_3]], [[a_6, a_2], [a_8, a_4]]]=[[[a_1, a_5], [a_3, a_7]], [[a_2, a_6], [a_4, a_8]]]$
$[[[a_6, a_2], [a_8, a_4]], [[a_7, a_3], [a_1, a_5]]]=[[[a_1, a_5], [a_3, a_7]], [[a_2, a_6], [a_4, a_8]]]$
$[[[a_7, a_3], [a_1, a_5]], [[a_8, a_4], [a_2, a_6]]]=[[[a_1, a_5], [a_3, a_7]], [[a_2, a_6], [a_4, a_8]]]$
$[[[a_8, a_4], [a_2, a_6]], [[a_1, a_5], [a_3, a_7]]]=[[[a_1, a_5], [a_3, a_7]], [[a_2, a_6], [a_4, a_8]]]$
Step8
0 allout

Eliminating any exceptional integer according to the four inequalities in the process of recursive operations, it logically makes sense thus. However, after this paper was accepted, we detected a counter example of our conjecture to the case of N=8 after the vast data verification. It is as follows.

	1	2	3	4	5	6	7	8
S	0	0	0	0	1	1	3	3
1	0	0	0	1	0	2	0	3
2	0	0	1	1	2	2	3	3
3	0	1	0	1	0	1	0	3
4	1	1	1	1	1	1	3	3
5	0	0	0	0	0	2	0	2
6	0	0	0	0	2	2	2	2
7	0	0	0	2	0	0	0	2
8	0	0	2	2	0	0	2	2
9	0	2	0	2	0	2	0	2
10	2	2	2	2	2	2	2	2
11	0	0	0	0	0	0	0	0

In this case, the recursive operation takes 11 steps. According to further computational checks when N=8 except for A=B=C at the first step, the conjecture is valid with base three numbers (0,1,2) (, but it is not by congruent expression modulo 3 on the recursive operation.) On the other hand, we detects counter examples in the operations with more than base three numbers like the case mentioned above (; 0,1,2,3 or more.) So that, we cannot say the conjecture will be true more than the case of N=4 at this time.

If the conjecture is discussed only with binary numbers (0, 1), it needs none of any four inequalities as the conditions to the conjecture. However, it is easily possible to prove such a case with binary numbers so that it will be no longer conjecture. On the other hand, the data suggests the conjecture will be valid also for the case with the three numbers (0, 1, 2). Thus, the general conjecture still seems to make sense in such a narrow domain. At the same time, we have no idea why it does until the case with base three numbers and does not more than the case with three base numbers against the recursive operations we logically expand. Additionally to say, if consecutive integers A, B, C, D, E, and F are A=B<C=D<E=F, such any sequences will not follow the conjecture: reasoning aside, Eliminating this condition from the conjecture seems to be a solution to exclude the counter example. Or, we may not notice an extra condition necessary for the conjecture. We would therefore like to know the preferable solution and offer these issues of our conjecture.

References

[1] B. Freedman, The Four Number Game, Scripta Math., 14: 35-47, (1948)
[2] M. Chamberland and D. M. Thomas, The N-Number Ducci Game(in the section Open Problems and Conjectures edited by G. Ladas), J. of Difference Equations and Applications, Vol. 10, No. 3, 339–342, (Mar. 2004). Retrieved 2009-01-26.

Supplementary Data (, N=16 in the case iii).)
N=16,
Step1
$[a_1, a_2]$, $[a_2, a_3]$, $[a_3, a_4]$, $[a_4, a_5]$, $[a_5, a_6]$, $[a_6, a_7]$, $[a_7, a_8]$, $[a_8, a_9]$, $[a_9, a_{10}]$, $[a_{10}, a_{11}]$, $[a_{11}, a_{12}]$, $[a_{12}, a_{13}]$, $[a_{13}, a_{14}]$, $[a_{14}, a_{15}]$, $[a_{15}, a_{16}]$, $[a_{16}, a_1]$
Step2
$[[a_1, a_2], [a_2, a_3]]=[a_1, a_3]$
$[[a_2, a_3], [a_3, a_4]]=[a_2, a_4]$
$[[a_3, a_4], [a_4, a_5]]=[a_3, a_5]$
$[[a_4, a_5], [a_5, a_6]]=[a_4, a_6]$
$[[a_5, a_6], [a_6, a_7]]=[a_5, a_7]$
$[[a_6, a_7], [a_7, a_8]]=[a_6, a_8]$
$[[a_7, a_8], [a_8, a_9]]=[a_7, a_9]$
$[[a_8, a_9], [a_9, a_{10}]]=[a_8, a_{10}]$
$[[a_9, a_{10}], [a_{10}, a_{11}]]=[a_9, a_{11}]$

$[[a_{10}, a_{11}], [a_{11}, a_{12}]]=[a_{10}, a_{12}]$
$[[a_{11}, a_{12}], [a_{12}, a_{13}]]=[a_{11}, a_{13}]$
$[[a_{12}, a_{13}], [a_{13}, a_{14}]]=[a_{12}, a_{14}]$
$[[a_{13}, a_{14}], [a_{14}, a_{15}]]=[a_{13}, a_{15}]$
$[[a_{14}, a_{15}], [a_{15}, a_{16}]]=[a_{14}, a_{16}]$
$[[a_{15}, a_{16}], [a_{16}, a_{1}]]=[a_{15}, a_{1}]$
$[[a_{16}, a_{1}], [a_{1}, a_{2}]]=[a_{16}, a_{2}]$
Step3
$[[a_{1}, a_{3}], [a_{2}, a_{4}]]$
$[[a_{2}, a_{4}], [a_{3}, a_{5}]]$
$[[a_{3}, a_{5}], [a_{4}, a_{6}]]$
$[[a_{4}, a_{6}], [a_{5}, a_{7}]]$
$[[a_{5}, a_{7}], [a_{6}, a_{8}]]$
$[[a_{6}, a_{8}], [a_{7}, a_{9}]]$
$[[a_{7}, a_{9}], [a_{8}, a_{10}]]$
$[[a_{8}, a_{10}], [a_{9}, a_{11}]]$
$[[a_{9}, a_{11}], [a_{10}, a_{12}]]$
$[[a_{10}, a_{12}], [a_{11}, a_{13}]]$
$[[a_{11}, a_{13}], [a_{12}, a_{14}]]$
$[[a_{12}, a_{14}], [a_{13}, a_{15}]]$
$[[a_{13}, a_{15}], [a_{14}, a_{16}]]$
$[[a_{14}, a_{16}], [a_{15}, a_{1}]]$
$[[a_{15}, a_{1}], [a_{16}, a_{2}]]$
$[[a_{16}, a_{2}], [a_{1}, a_{3}]]$
Step4 (*Remark: any integer* at vertices follows the case iii) of lemma 2.1.)
$[[[a_{1}, a_{3}], [a_{2}, a_{4}]], [[a_{2}, a_{4}], [a_{3}, a_{5}]]]=[[a_{1}, a_{3}], [a_{3}, a_{5}]]=[a_{1}, a_{5}]$
$[[[a_{2}, a_{4}], [a_{3}, a_{5}]], [[a_{3}, a_{5}], [a_{4}, a_{6}]]]=[[a_{2}, a_{4}], [a_{4}, a_{6}]]=[a_{2}, a_{6}]$
$[[[a_{3}, a_{5}], [a_{4}, a_{6}]], [[a_{4}, a_{6}], [a_{5}, a_{7}]]]=[[a_{3}, a_{5}], [a_{5}, a_{7}]]=[a_{3}, a_{7}]$
$[[[a_{4}, a_{6}], [a_{5}, a_{7}]], [[a_{5}, a_{7}], [a_{6}, a_{8}]]]=[[a_{4}, a_{6}], [a_{6}, a_{8}]]=[a_{4}, a_{8}]$
$[[[a_{5}, a_{7}], [a_{6}, a_{8}]], [[a_{6}, a_{8}], [a_{7}, a_{9}]]]=[[a_{5}, a_{7}], [a_{7}, a_{9}]]=[a_{5}, a_{9}]$
$[[[a_{6}, a_{8}], [a_{7}, a_{9}]], [[a_{7}, a_{9}], [a_{8}, a_{10}]]]=[[a_{6}, a_{8}], [a_{8}, a_{10}]]=[a_{6}, a_{10}]$
$[[[a_{7}, a_{9}], [a_{8}, a_{10}]], [[a_{8}, a_{10}], [a_{9}, a_{11}]]]=[[a_{7}, a_{9}], [a_{9}, a_{11}]]=[a_{7}, a_{11}]$
$[[[a_{8}, a_{10}], [a_{9}, a_{11}]], [[a_{9}, a_{11}], [a_{10}, a_{12}]]]=[[a_{8}, a_{10}], [a_{10}, a_{12}]]=[a_{8}, a_{12}]$
$[[[a_{9}, a_{11}], [a_{10}, a_{12}]], [[a_{10}, a_{12}], [a_{11}, a_{13}]]]=[[a_{9}, a_{11}], [a_{11}, a_{13}]]=[a_{9}, a_{13}]$
$[[[a_{10}, a_{12}], [a_{11}, a_{13}]], [[a_{11}, a_{13}], [a_{12}, a_{14}]]]=[[a_{10}, a_{12}], [a_{12}, a_{14}]]=[a_{10}, a_{14}]$
$[[[a_{11}, a_{13}], [a_{12}, a_{14}]], [[a_{12}, a_{14}], [a_{13}, a_{15}]]]=[[a_{11}, a_{13}], [a_{13}, a_{15}]]=[a_{11}, a_{15}]$
$[[[a_{12}, a_{14}], [a_{13}, a_{15}]], [[a_{13}, a_{15}], [a_{14}, a_{16}]]]=[[a_{12}, a_{14}], [a_{14}, a_{16}]]=[a_{12}, a_{16}]$
$[[[a_{13}, a_{15}], [a_{14}, a_{16}], [[a_{14}, a_{16}], [a_{15}, a_{1}]]]=[[a_{13}, a_{15}], [a_{15}, a_{1}]]=[a_{13}, a_{1}]$
$[[[a_{14}, a_{16}], [a_{15}, a_{1}]], [[a_{15}, a_{1}], [a_{16}, a_{2}]]]=[[a_{14}, a_{16}], [a_{16}, a_{2}]]=[a_{14}, a_{2}]$
$[[[a_{15}, a_{1}], [a_{16}, a_{2}]], [[a_{16}, a_{2}], [a_{1}, a_{3}]]]=[[a_{15}, a_{1}], [a_{1}, a_{3}]]=[a_{15}, a_{3}]$
$[[[a_{16}, a_{2}], [a_{1}, a_{3}]], [[a_{1}, a_{3}], [a_{2}, a_{4}]]]=[[a_{16}, a_{2}], [a_{2}, a_{4}]]=[a_{16}, a_{4}]$
Step5
$[[a_{1}, a_{5}], [a_{2}, a_{6}]]$
$[[a_{2}, a_{6}], [a_{3}, a_{7}]]$
$[[a_{3}, a_{7}], [a_{4}, a_{8}]]$

$[[a_4, a_8], [a_5, a_9]]$
$[[a_5, a_1], [a_6, a_{10}]]$
$[[a_6, a_2], [a_7, a_{11}]]$
$[[a_7, a_3], [a_8, a_{12}]]$
$[[a_8, a_4], [a_9, a_{13}]]$
$[[a_9, a_5], [a_{10}, a_{14}]]$
$[[a_{10}, a_6], [a_{11}, a_{15}]]$
$[[a_{11}, a_7], [a_{12}, a_{16}]]$
$[[a_{12}, a_8], [a_{13}, a_1]]$
$[[a_{13}, a_1], [a_{14}, a_2]]$
$[[a_{14}, a_2], [a_{15}, a_3]]$
$[[a_{15}, a_3], [a_{16}, a_4]]$
$[[a_{16}, a_4], [a_1, a_5]]$

Step6 (*Remark: any integer* at vertices follows the case iii) of lemma 2.1.)

$[[[a_1, a_5], [a_2, a_6]], [[a_2, a_6], [a_3, a_7]]]=[[a_1, a_5], [a_3, a_7]]$
$[[[a_2, a_6], [a_3, a_7]], [[a_3, a_7], [a_4, a_8]]]=[[a_2, a_6], [a_4, a_8]]$
$[[[a_3, a_7], [a_4, a_8]], [[a_4, a_8], [a_5, a_9]]]=[[a_3, a_7], [a_5, a_9]]$
$[[[a_4, a_8], [a_5, a_9]], [[a_5, a_9], [a_6, a_{10}]]]=[[a_4, a_8], [a_6, a_{10}]]$
$[[[a_5, a_9], [a_6, a_{10}]], [[a_6, a_{10}], [a_7, a_{11}]]]=[[a_5, a_9], [a_7, a_{11}]]$
$[[[a_6, a_{10}], [a_7, a_{11}]], [[a_7, a_{11}], [a_8, a_{12}]]]=[[a_6, a_{10}], [a_8, a_{12}]]$
$[[[a_7, a_{11}], [a_8, a_{12}]], [[a_8, a_{12}], [a_9, a_{13}]]]=[[a_7, a_{11}], [a_9, a_{13}]]$
$[[[a_8, a_{12}], [a_9, a_{13}]], [[a_9, a_{13}], [a_{10}, a_{14}]]]=[[a_8, a_{12}], [a_{10}, a_{14}]]$
$[[[a_1, a_{13}], [a_{10}, a_{14}]], [[a_{10}, a_{14}], [a_{11}, a_{15}]]]=[[a_9, a_{13}], [a_{11}, a_{15}]]$
$[[[a_2, a_{14}], [a_{11}, a_{15}]], [[a_{11}, a_{15}], [a_{12}, a_{16}]]]=[[a_{10}, a_{14}], [a_{12}, a_{16}]]$
$[[[a_3, a_{15}], [a_{12}, a_{16}]], [[a_{12}, a_{16}], [a_{13}, a_1]]]=[[a_{11}, a_{15}], [a_{13}, a_1]]$
$[[[a_4, a_{16}], [a_{13}, a_1]], [[a_{13}, a_1], [a_{14}, a_2]]]=[[a_{12}, a_{16}], [a_{14}, a_2]]$
$[[[a_5, a_1], [a_{14}, a_2]], [[a_{14}, a_2], [a_{15}, a_3]]]=[[a_{13}, a_1], [a_{15}, a_3]]$
$[[[a_6, a_2], [a_{15}, a_3]], [[a_{15}, a_3], [a_{16}, a_4]]]=[[a_{14}, a_2], [a_{16}, a_4]]$
$[[[a_7, a_3], [a_{16}, a_4]], [[a_{16}, a_4], [a_1, a_5]]]=[[a_{15}, a_3], [a_1, a_5]]$
$[[[a_8, a_4], [a_1, a_5]], [[a_1, a_5], [a_2, a_6]]]=[[a_{16}, a_4], [a_2, a_6]]$

Step7

$[[[a_1, a_5], [a_3, a_7]], [[a_2, a_6], [a_4, a_8]]]$
$[[[a_2, a_6], [a_4, a_8]], [[a_3, a_7], [a_5, a_9]]]$
$[[[a_3, a_7], [a_5, a_9]], [[a_4, a_8], [a_6, a_{10}]]]$
$[[[a_4, a_8], [a_6, a_{10}]], [[a_5, a_9], [a_7, a_{11}]]]$
$[[[a_5, a_9], [a_7, a_{11}]], [[a_6, a_{10}], [a_8, a_{12}]]]$
$[[[a_6, a_{10}], [a_8, a_{12}]], [[a_7, a_{11}], [a_9, a_{13}]]]$
$[[[a_7, a_{11}], [a_9, a_{13}]], [[a_8, a_{12}], [a_{10}, a_{14}]]]$
$[[[a_8, a_{12}], [a_{10}, a_{14}]], [[a_9, a_{13}], [a_{11}, a_{15}]]]$
$[[[a_9, a_{13}], [a_{11}, a_{15}]], [[a_{10}, a_{14}], [a_{12}, a_{16}]]]$
$[[[a_{10}, a_{14}], [a_{12}, a_{15}]], [[a_{11}, a_{15}], [a_{13}, a_1]]]$
$[[[a_{11}, a_{15}], [a_{13}, a_1]], [[a_{12}, a_{16}], [a_{14}, a_2]]]$
$[[[a_{12}, a_{16}], [a_{14}, a_2]], [[a_{13}, a_1], [a_{15}, a_3]]]$
$[[[a_{13}, a_1], [a_{15}, a_3]], [[a_{14}, a_2], [a_{15}, a_4]]]$
$[[[a_{14}, a_2], [a_{16}, a_4]], [[a_{15}, a_3], [a_1, a_5]]]$

$[[[a_{15}, a_3], [a_1, a_5]], [[a_{16}, a_4], [a_2, a_6]]]$

$[[[a_{16}, a_4], [a_2, a_6]], [[a_1, a_5], [a_3, a_7]]]$

Step8 (*Remark: any integer* at vertices follows the case iii) of lemma 2.1.)

$[[[[a_1, a_5], [a_3, a_7]], [[a_2, a_6], [a_4, a_8]]], [[[a_2, a_6], [a_4, a_8]], [[a_3, a_7], [a_5, a_9]]]]=[a_1, a_9]$

$[[[[a_2, a_6], [a_4, a_8]], [[a_3, a_7], [a_5, a_9]]], [[[a_3, a_7], [a_5, a_9]], [[a_4, a_8], [a_6, a_{10}]]]]=[a_2, a_{10}]$

$[[[[a_3, a_7], [a_5, a_9]], [[a_4, a_8], [a_6, a_{10}]]], [[[a_4, a_8], [a_6, a_{10}]], [[a_5, a_9], [a_7, a_{11}]]]]=[a_3, a_{11}]$

$[[[[a_4, a_8], [a_6, a_{10}]], [[a_5, a_9], [a_7, a_{11}]]], [[[a_5, a_9], [a_7, a_{11}]], [[a_6, a_{10}], [a_8, a_{12}]]]]=[a_4, a_{12}]$

$[[[[a_5, a_9], [a_7, a_{11}]], [[a_6, a_{10}], [a_8, a_{12}]]], [[[a_6, a_{10}], [a_8, a_{12}]], [[a_7, a_{11}], [a_9, a_{13}]]]]=[a_5, a_{13}]$

$[[[[a_6, a_{10}], [a_8, a_{12}]], [[a_7, a_{11}], [a_9, a_{13}]]], [[[a_7, a_{11}], [a_9, a_{13}]], [[a_8, a_{12}], [a_{10}, a_{14}]]]]=[a_6, a_{14}]$

$[[[[a_7, a_{11}], [a_9, a_{13}]], [[a_8, a_{12}], [a_{10}, a_{14}]]], [[[a_8, a_{12}], [a_{10}, a_{14}]], [[a_9, a_{13}], [a_{11}, a_{15}]]]]=[a_7, a_{15}]$

$[[[[a_8, a_{12}], [a_{10}, a_{14}]], [[a_9, a_{13}], [a_{11}, a_{15}]]], [[[a_9, a_{13}], [a_{11}, a_{15}]], [[a_{10}, a_{14}], [a_{12}, a_{16}]]]]=[a_8, a_{16}]$

$[[[[a_9, a_{13}], [a_{11}, a_{15}]], [[a_{10}, a_{14}], [a_{12}, a_{16}]]], [[[a_{10}, a_{14}], [a_{12}, a_{15}]], [[a_{11}, a_{15}], [a_{13}, a_1]]]]=[a_9, a_1]$

$[[[[a_{10}, a_{14}], [a_{12}, a_{15}]], [[a_{11}, a_{15}], [a_{13}, a_1]]], [[[a_{11}, a_{15}], [a_{13}, a_1]], [[a_{12}, a_{16}], [a_{14}, a_2]]]]=[a_{10}, a_2]$

$[[[[a_{11}, a_{15}], [a_{13}, a_1]], [[a_{12}, a_{16}], [a_{14}, a_2]]], [[[a_{12}, a_{16}], [a_{14}, a_2]], [[a_{13}, a_1], [a_{15}, a_3]]]]=[a_{11}, a_3]$

$[[[[a_{12}, a_{16}], [a_{14}, a_2]], [[a_{13}, a_1], [a_{15}, a_3]]], [[[a_{13}, a_1], [a_{15}, a_3]], [[a_{14}, a_2], [a_{15}, a_4]]]]=[a_{12}, a_4]$

$[[[[a_{13}, a_1], [a_{15}, a_3]], [[a_{14}, a_2], [a_{15}, a_4]]], [[[a_{14}, a_2], [a_{16}, a_4]], [[a_{15}, a_3], [a_1, a_5]]]]=[a_{13}, a_5]$

$[[[[a_{14}, a_2], [a_{16}, a_4]], [[a_{15}, a_3], [a_1, a_5]]], [[[a_{15}, a_3], [a_1, a_5]], [[a_{16}, a_4], [a_2, a_6]]]]=[a_{14}, a_6]$

$[[[[a_{15}, a_3], [a_1, a_5]], [[a_{16}, a_4], [a_2, a_6]]], [[[a_{16}, a_4], [a_2, a_6]], [[a_1, a_5], [a_3, a_7]]]]=[a_{15}, a_7]$

$[[[[a_{16}, a_4], [a_2, a_6]], [[a_1, a_5], [a_3, a_7]]], [[[a_1, a_5], [a_3, a_7]], [[a_2, a_6], [a_4, a_8]]]]=[a_{16}, a_8]$

Step9

$[[a_1, a_9], [a_2, a_{10}]]$

$[[a_2, a_{10}], [a_3, a_{11}]]$

$[[a_3, a_{11}], [a_4, a_{12}]]$

$[[a_4, a_{12}], [a_5, a_{13}]]$

$[[a_5, a_{13}], [a_6, a_{14}]]$

$[[a_6, a_{14}], [a_7, a_{15}]]$

$[[a_7, a_{15}], [a_8, a_{16}]]$

$[[a_8, a_{16}], [a_9, a_1]]$

$[[a_9, a_1], [a_{10}, a_2]]$

$[[a_{10}, a_2], [a_{11}, a_3]]$

$[[a_{11}, a_3], [a_{12}, a_4]]$

$[[a_{12}, a_4], [a_{13}, a_5]]$

$[[a_{13}, a_5], [a_{14}, a_6]]$

$[[a_{14}, a_6], [a_{15}, a_7]]$

$[[a_{15}, a_7], [a_{16}, a_8]]$
$[[a_{16}, a_8], [a_1, a_9]]$
Step10 (*Remark: any integer* at vertices follows the case iii) of lemma 2.1.)
$[[[a_1, a_9], [a_2, a_{10}]], [[a_2, a_{10}], [a_3, a_{11}]]]=[[a_1, a_9], [a_3, a_{11}]]$
$[[[a_2, a_{10}], [a_3, a_{11}]], [[a_3, a_{11}], [a_4, a_{12}]]]=[[a_2, a_{10}], [a_4, a_{12}]]$
$[[[a_3, a_{11}], [a_4, a_{12}]], [[a_4, a_{12}], [a_5, a_{13}]]]=[[a_3, a_{11}], [a_5, a_{13}]]$
$[[[a_4, a_{12}], [a_5, a_{13}]], [[a_5, a_{13}], [a_6, a_{14}]]]=[[a_4, a_{12}], [a_6, a_{14}]]$
$[[[a_5, a_{13}], [a_6, a_{14}]], [[a_6, a_{14}], [a_7, a_{15}]]]=[[a_5, a_{13}], [a_7, a_{15}]]$
$[[[a_6, a_{14}], [a_7, a_{15}]], [[a_7, a_{15}], [a_8, a_{16}]]]=[[a_6, a_{14}], [a_8, a_{16}]]$
$[[[a_7, a_{15}], [a_8, a_{16}]], [[a_8, a_{16}], [a_9, a_1]]]=[[a_7, a_{15}], [a_9, a_1]]$
$[[[a_8, a_{16}], [a_9, a_1]], [[a_9, a_1], [a_{10}, a_2]]]=[[a_8, a_{16}], [a_{10}, a_2]]$
$[[[a_9, a_1], [a_{10}, a_2]], [[a_{10}, a_2], [a_{11}, a_3]]]=[[a_9, a_1], [a_{11}, a_3]]$
$[[[a_{10}, a_2], [a_{11}, a_3]], [[a_{11}, a_3], [a_{12}, a_4]]]=[[a_{10}, a_2], [a_{12}, a_4]]$
$[[[a_{11}, a_3], [a_{12}, a_4]], [[a_{12}, a_4], [a_{13}, a_5]]]=[[a_{11}, a_3], [a_{13}, a_5]]$
$[[[a_{12}, a_4], [a_{13}, a_5]], [[a_{13}, a_5], [a_{14}, a_6]]]=[[a_{12}, a_4], [a_{14}, a_6]]$
$[[[a_{13}, a_5], [a_{14}, a_6]], [[a_{14}, a_6], [a_{15}, a_7]]]=[[a_{13}, a_5], [a_{15}, a_7]]$
$[[[a_{14}, a_6], [a_{15}, a_7]], [[a_{15}, a_7], [a_{16}, a_8]]]=[[a_{14}, a_6], [a_{16}, a_8]]$
$[[[a_{15}, a_7], [a_{16}, a_8]], [[a_{16}, a_8], [a_1, a_9]]]=[[a_{15}, a_7], [a_1, a_9]]$
$[[[a_{16}, a_8], [a_1, a_9]], [[a_1, a_9], [a_2, a_{10}]]]=[[a_{16}, a_8], [a_2, a_{10}]]$
Step11
$[[[a_1, a_9], [a_3, a_{11}]], [[a_2, a_{10}], [a_4, a_{12}]]]$
$[[[a_2, a_{10}], [a_4, a_{12}]], [[a_3, a_{11}], [a_5, a_{13}]]]$
$[[[a_3, a_{11}], [a_5, a_{13}]], [[a_4, a_{12}], [a_6, a_{14}]]]$
$[[[a_4, a_{12}], [a_6, a_{14}]], [[a_5, a_{13}], [a_7, a_{15}]]]$
$[[[a_5, a_{13}], [a_7, a_{15}]], [[a_6, a_{14}], [a_8, a_{16}]]]$
$[[[a_6, a_{14}], [a_8, a_{16}]], [[a_7, a_{15}], [a_9, a_1]]]$
$[[[a_7, a_{15}], [a_9, a_1]], [[a_8, a_{16}], [a_{10}, a_2]]]$
$[[[a_8, a_{16}], [a_{10}, a_2]], [[a_9, a_1], [a_{11}, a_3]]]$
$[[[a_9, a_1], [a_{11}, a_3]], [[a_{10}, a_2], [a_{12}, a_4]]]$
$[[[a_{10}, a_2], [a_{12}, a_4]], [[a_{11}, a_3], [a_{13}, a_5]]]$
$[[[a_{11}, a_3], [a_{13}, a_5]], [[a_{12}, a_4], [a_{14}, a_6]]]$
$[[[a_{12}, a_4], [a_{14}, a_6]], [[a_{13}, a_5], [a_{15}, a_7]]]$
$[[[a_{13}, a_5], [a_{15}, a_7]], [[a_{14}, a_6], [a_{16}, a_8]]]$
$[[[a_{14}, a_6], [a_{16}, a_8]], [[a_{15}, a_7], [a_1, a_9]]]$
$[[[a_{15}, a_7], [a_1, a_9]], [[a_{16}, a_8], [a_2, a_{10}]]]$
$[[[a_{16}, a_8], [a_2, a_{10}]], [[a_1, a_9], [a_3, a_{11}]]]$
Step12 (*Remark: any integer* at vertices follows the case iii) of lemma 2.1.)
$[[[[a_1, a_9], [a_3, a_{11}]], [[a_2, a_{10}], [a_4, a_{12}]]], [[[a_2, a_{10}], [a_4, a_{12}]], [[a_3, a_{11}], [a_5, a_{13}]]]]=[[a_1, a_9], [a_5, a_{13}]]$
$[[[[a_2, a_{10}], [a_4, a_{12}]], [[a_3, a_{11}], [a_5, a_{13}]]], [[[a_3, a_{11}], [a_5, a_{13}]], [[a_4, a_{12}], [a_6, a_{14}]]]]=[[a_2, a_{10}], [a_6, a_{14}]]$
$[[[[a_3, a_{11}], [a_5, a_{13}]], [[a_4, a_{12}], [a_6, a_{14}]]], [[[a_4, a_{12}], [a_6, a_{14}]], [[a_5, a_{13}], [a_7, a_{15}]]]]=[[a_3, a_{11}], [a_7, a_{15}]]$
$[[[[a_4, a_{12}], [a_6, a_{14}]], [[a_5, a_{13}], [a_7, a_{15}]]], [[[a_5, a_{13}], [a_7, a_{15}]], [[a_6, a_{14}], [a_8, a_{16}]]]]=[[a_4, a_{12}], [a_8, a_{16}]]$

$[[[[a_5, a_{13}], [a_7, a_{15}]], [[a_6, a_{14}], [a_8, a_{16}]]], [[[a_6, a_{14}], [a_8, a_{16}]], [[a_7, a_{15}], [a_9, a_1]]]]=[[a_5, a_{13}], [a_9, a_1]]=[[a_1, a_9], [a_5, a_{13}]]$

$[[[[a_6, a_{14}], [a_8, a_{16}]], [[a_7, a_{15}], [a_9, a_1]]], [[[a_7, a_{15}], [a_9, a_1]], [[a_8, a_{16}], [a_{10}, a_2]]]]=[[a_6, a_{14}], [a_{10}, a_2]]=[[a_2, a_{10}], [a_6, a_{14}]]$

$[[[[a_7, a_{15}], [a_9, a_1]], [[a_8, a_{16}], [a_{10}, a_2]]], [[[a_8, a_{16}], [a_{10}, a_2]], [[a_9, a_1], [a_{11}, a_3]]]]=[[a_7, a_{15}], [a_{11}, a_3]]=[[a_3, a_{11}], [a_7, a_{15}]]$

$[[[[a_8, a_{16}], [a_{10}, a_2]], [[a_9, a_1], [a_{11}, a_3]]], [[[a_9, a_1], [a_{11}, a_3]], [[a_{10}, a_2], [a_{12}, a_4]]]]=[[a_8, a_{16}], [a_{12}, a_4]]=[[a_4, a_{12}], [a_8, a_{16}]]$

$[[[[a_9, a_1], [a_{11}, a_3]], [[a_{10}, a_2], [a_{12}, a_4]]], [[[a_{10}, a_2], [a_{12}, a_4]], [[a_{11}, a_3], [a_{13}, a_5]]]]=[[a_9, a_1], [a_{13}, a_5]]=[[a_1, a_9], [a_5, a_{13}]]$

$[[[[a_{10}, a_2], [a_{12}, a_4]], [[a_{11}, a_3], [a_{13}, a_5]]], [[[a_{11}, a_3], [a_{13}, a_5]], [[a_{12}, a_4], [a_{14}, a_6]]]]=[[a_{10}, a_2], [a_{14}, a_6]]=[[a_2, a_{10}], [a_6, a_{14}]]$

$[[[[a_{11}, a_3], [a_{13}, a_5]], [[a_{12}, a_4], [a_{14}, a_6]]], [[[a_{12}, a_4], [a_{14}, a_6]], [[a_{13}, a_5], [a_{15}, a_7]]]]=[[a_{11}, a_3], [a_{15}, a_7]]=[[a_3, a_{11}], [a_7, a_{15}]]$

$[[[[a_{12}, a_4], [a_{14}, a_6]], [[a_{13}, a_5], [a_{15}, a_7]]], [[[a_{13}, a_5], [a_{15}, a_7]], [[a_{14}, a_6], [a_{16}, a_8]]]]=[[a_{12}, a_4], [a_{16}, a_8]]=[[a_4, a_{12}], [a_8, a_{16}]]$

$[[[[a_{13}, a_5], [a_{15}, a_7]], [[a_{14}, a_6], [a_{16}, a_8]]], [[[a_{14}, a_6], [a_{16}, a_8]], [[a_{15}, a_7], [a_1, a_9]]]]=[[a_{13}, a_5], [a_1, a_9]]=[[a_1, a_9], [a_5, a_{13}]]$

$[[[[a_{14}, a_6], [a_{16}, a_8]], [[a_{15}, a_7], [a_1, a_9]]], [[[a_{15}, a_7], [a_1, a_9]], [[a_{16}, a_8], [a_2, a_{10}]]]]=[[a_{14}, a_6], [a_2, a_{10}]]=[[a_2, a_{10}], [a_6, a_{14}]]$

$[[[[a_{15}, a_7], [a_1, a_9]], [[a_{16}, a_8], [a_2, a_{10}]]], [[[a_{16}, a_8], [a_2, a_{10}]], [[a_1, a_9], [a_3, a_{11}]]]]=[[a_{15}, a_7], [a_3, a_{11}]]=[[a_3, a_{11}], [a_7, a_{15}]]$

$[[[[a_{16}, a_8], [a_2, a_{10}]], [[a_1, a_9], [a_3, a_{11}]]], [[[a_1, a_9], [a_3, a_{11}]], [[a_2, a_{10}], [a_4, a_{12}]]]]=[[a_{16}, a_8], [a_4, a_{12}]]=[[a_4, a_{12}], [a_8, a_{16}]]$

Step13

$[[[a_1, a_9], [a_5, a_{13}]], [[a_2, a_{10}], [a_6, a_{14}]]]$

$[[[a_2, a_{10}], [a_6, a_{14}]], [[a_3, a_{11}], [a_7, a_{15}]]]$

$[[[a_3, a_{11}], [a_7, a_{15}]], [[a_4, a_{12}], [a_8, a_{16}]]]$

$[[[a_4, a_{12}], [a_8, a_{16}]], [[a_1, a_9], [a_5, a_{13}]]]$

$[[[a_1, a_9], [a_5, a_{13}]], [[a_2, a_{10}], [a_6, a_{14}]]]$

$[[[a_2, a_{10}], [a_6, a_{14}]], [[a_3, a_{11}], [a_7, a_{15}]]]$

$[[[a_3, a_{11}], [a_7, a_{15}]], [[a_4, a_{12}], [a_8, a_{16}]]]$

$[[[a_4, a_{12}], [a_8, a_{16}]], [[a_1, a_9], [a_5, a_{13}]]]$

$[[[a_1, a_9], [a_5, a_{13}]], [[a_2, a_{10}], [a_6, a_{14}]]]$

$[[[a_2, a_{10}], [a_6, a_{14}]], [[a_3, a_{11}], [a_7, a_{15}]]]$

$[[[a_3, a_{11}], [a_7, a_{15}]], [[a_4, a_{12}], [a_8, a_{16}]]]$

$[[[a_4, a_{12}], [a_8, a_{16}]]. [[a_1, a_9], [a_5, a_{13}]]]$

$[[[a_1, a_9], [a_5, a_{13}]], [[a_2, a_{10}], [a_6, a_{14}]]]$

$[[[a_2, a_{10}], [a_6, a_{14}]], [[a_3, a_{11}], [a_7, a_{15}]]]$

$[[[a_3, a_{11}], [a_7, a_{15}]], [[a_4, a_{12}], [a_8, a_{16}]]]$

$[[[a_4, a_{12}], [a_8, a_{16}]], [[a_1, a_9], [a_5, a_{13}]]]$

Step14 (*Remark: any integer* at vertices follows the case iii) of lemma 2.1.)

$[[[[a_1, a_9], [a_5, a_{13}]], [[a_2, a_{10}], [a_6, a_{14}]]], [[[a_2, a_{10}], [a_6, a_{14}]], [[a_3, a_{11}], [a_7, a_{15}]]]]= [[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]]$

$[[[[a_2, a_{10}], [a_6, a_{14}]], [[a_3, a_{11}], [a_7, a_{15}]]], [[[a_3, a_{11}], [a_7, a_{15}]], [[a_4, a_{12}], [a_8,$

$a_{16}]]]]=[[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]]$
$[[[[a_3, a_{11}], [a_7, a_{15}]], [[a_4, a_{12}], [a_8, a_{16}]]], [[[a_4, a_{12}], [a_8, a_{16}]], [[a_1, a_9], [a_5, a_{13}]]]]=$
$[[[a_3, a_{11}], [a_7, a_{15}]], [[a_1, a_9], [a_5, a_{13}]]]$
$=[[[a_1, a_3], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]]$
$[[[[a_4, a_{12}], [a_8, a_{16}]], [[a_1, a_9], [a_5, a_{13}]]], [[[a_1, a_9], [a_5, a_{13}]], [[a_2, a_{10}], [a_6, a_{14}]]]]=$
$[[[a_4, a_{12}], [a_8, a_{16}]], [[a_2, a_{10}], [a_6, a_{14}]]]$
$=[[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]]$
$[[[[a_1, a_9], [a_5, a_{13}]], [[a_2, a_{10}], [a_6, a_{14}]]], [[[a_2, a_{10}], [a_6, a_{14}]], [[a_3, a_{11}], [a_7, a_{15}]]]]=[[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]]$
$[[[[a_2, a_{10}], [a_6, a_{14}]], [[a_3, a_{11}], [a_7, a_{15}]]], [[[a_3, a_{11}], [a_7, a_{15}]], [[a_4, a_{12}], [a_8, a_{16}]]]]=[[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]]$
$[[[[a_3, a_{11}], [a_7, a_{15}]], [[a_4, a_{12}], [a_8, a_{16}]]], [[[a_4, a_{12}], [a_8, a_{16}]], [[a_1, a_9], [a_5, a_{13}]]]]=[[[[a_1, a_3], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]]$
$[[[[a_4, a_{12}], [a_8, a_{16}]], [[a_1, a_9], [a_5, a_{13}]]], [[[a_1, a_9], [a_5, a_{13}]], [[a_2, a_{10}], [a_6, a_{14}]]]]=[[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]]$
$[[[[a_1, a_9], [a_5, a_{13}]], [[a_2, a_{10}], [a_6, a_{14}]]], [[[a_2, a_{10}], [a_6, a_{14}]], [[a_3, a_{11}], [a_7, a_{15}]]]]=[[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]]$
$[[[[a_2, a_{10}], [a_6, a_{14}]], [[a_3, a_{11}], [a_7, a_{15}]]], [[[a_3, a_{11}], [a_7, a_{15}]], [[a_4, a_{12}], [a_8, a_{16}]]]]=[[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]]$
$[[[[a_3, a_{11}], [a_7, a_{15}]], [[a_4, a_{12}], [a_8, a_{16}]]], [[[a_4, a_{12}], [a_8, a_{16}]]. [[a_1, a_9], [a_5, a_{13}]]]]=[[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]]$
$[[[[a_4, a_{12}], [a_8, a_{16}]]. [[a_1, a_9], [a_5, a_{13}]]], [[[a_1, a_9], [a_5, a_{13}]], [[a_2, a_{10}], [a_6, a_{14}]]]]=[[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]]$
$[[[[a_1, a_9], [a_5, a_{13}]], [[a_2, a_{10}], [a_6, a_{14}]]], [[[a_2, a_{10}], [a_6, a_{14}]], [[a_3, a_{11}], [a_7, a_{15}]]]]=[[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]]$
$[[[[a_2, a_{10}], [a_6, a_{14}]], [[a_3, a_{11}], [a_7, a_{15}]]], [[[a_3, a_{11}], [a_7, a_{15}]], [[a_4, a_{12}], [a_8, a_{16}]]]]=[[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]]$
$[[[[a_3, a_{11}], [a_7, a_{15}]], [[a_4, a_{12}], [a_8, a_{16}]]], [[[a_4, a_{12}], [a_8, a_{16}]], [[a_1, a_9], [a_5, a_{13}]]]]=[[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]]$
$[[[[a_4, a_{12}], [a_8, a_{16}]], [[a_1, a_9], [a_5, a_{13}]]], [[[a_1, a_9], [a_5, a_{13}]], [[a_2, a_{10}], [a_6, a_{14}]]]]=[[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]]$

Step15

$[[[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]], [[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]]]$
$[[[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]], [[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]]]$
$=[[[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]], [[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]]]$
$[[[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]], [[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]]]$
$[[[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]], [[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]]]$
$=[[[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]], [[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]]]$
$[[[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]], [[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]]]$
$[[[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]], [[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]]]$
$=[[[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]], [[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]]]$
$[[[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]], [[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]]]$
$[[[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]], [[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]]]$
$=[[[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]], [[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]]]$
$[[[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]], [[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]]]$

$[[[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]], [[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]]]$
$=[[[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]], [[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]]]$
$[[[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]], [[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]]]$
$[[[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]], [[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]]]$
$=[[[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]], [[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]]]$
$[[[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]], [[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]]]$
$[[[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]], [[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]]]$
$=[[[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]], [[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]]]$
$[[[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]], [[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]]]$
$[[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]], [[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]]]$
$=[[[[a_1, a_9], [a_5, a_{13}]], [[a_3, a_{11}], [a_7, a_{15}]]], [[[a_2, a_{10}], [a_6, a_{14}]], [[a_4, a_{12}], [a_8, a_{16}]]]]$
Step16
0 allout