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On the Existence of Triangles

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Abstract

We formulate criterions about the existence of triangles depending on their sidelengths.

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It is well-known that a genuine triangle with sides of lengths a, b and c exists if and only if three inequalities are fulfilled. They are called the 'triangle inequalities'.

Proposition 1. A triangle with sidelenghts a, b and c exists if and only if three inequalities hold.

$$a+b>c \tag{1}$$
$$a+c>b \tag{2}$$
$$b+c>a \tag{3}$$

These inequalities can be combined by the inequality

$$Minimum\{a+b-c, a+c-b, b+c-a\} > 0 \tag{4}$$

We show that this inequality can be replaced by another inequality with a product of three factors.

Proposition 2. A triangle with sidelengths a,b and c exists if and only if the following inequality holds.

$$(a+b-c)\cdot(a+c-b)\cdot(b+c-a)>0 \tag{5}$$

Proof. We show that out of the three factors, two cannot simultaneously be negative. Assume three real numbers x,y,z such that both $x+y-z$ and $z+x-y$ are negative. This means $x+y<z$ and $z+x<y$. This means $z+x<y<z-x$, hence $x<-x$. This is only possible with a negative x. This proves the proposition, since a,b and c are positive. □

The example $a=10$ and $b=c=1$ shows that all inequalities (1), (2) and (3) are necessary to generate a triangle.

We show that there are two further inequalities which are equivalent to the triangle inequalities of Proposition 1.

Let a,b,c be positive real numbers such that $b,c\leq a$.

Proposition 3. A triangle with sidelengths a,b and c exists if and only if the following inequality (6) holds for any natural number n or equivalently if and only if the next inequality (7) holds.

$$(a-c)^n<b^n \tag{6}$$

$$a-c<b<a+c \tag{7}$$

Proof. We show that (6) is equivalent to (3). Because of $0<b,c\leq a$ it follows that both $a-c$ and b are positive or zero. Hence we can extract the root. This means that (6) is equivalent to $a-c<b$. Since $0<b,c\leq a$ we have (1) and (2). Further, (7) is equivalent to (2), (3). □

Remark 1. Note that we get equivalent inequalities if we exchange the variables b and c in (6) and (7).

Furthermore we ask by given line segments with lengths a and b for a segment of length c such that a,b and c are sidelengths of a triangle.

Proposition 4. Let a and b are line segments which meet only once in a single point called C at the end. The endpoints of a are B and C, while the endpoints of b are A and C. The generated angle is $\angle(ACB)$. ($\angle(XYZ)$ means the angle from three points X,Y,Z such that Y is the apex). Let $\angle(ACB)<180°$.

A triangle with sidelengths a,b and c exists if and only if the following equation holds.

$$c=a\cdot\cos\angle(ABC)+b\cdot\cos\angle(BAC)=\sqrt{a^2+b^2-2\cdot a\cdot b\cdot\cos\angle(ACB)} \tag{8}$$

A and B have the distance c.

Proof. The equation is well-known. It is true due to the law of cosines. □

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