Optimal Management Policy for Heterogeneous Arrival M/E_k/1 Queueing System with Server Breakdowns and Multiple Vacations

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Abstract

This paper studies the optimal control of an M/E_k/1 queue with server breakdowns and vacations, where the arrival rate varies according to the server’s status with the length of the vacations determined by the number of arrivals during the vacation period. The server can be turned off and takes a vacation whenever the system is empty. If the number of units waiting in the system is less than N, the server will take another vacation. It is assumed that server breakdown according to a Poisson process and the repair time has an exponential distribution. Explicit expressions for the steady state distribution of the number of customers in the system are obtained and also derived various system measures. A cost model is developed to determine the optimal operating policy at minimum cost. Sensitivity analysis is also provided.

Keywords: Heterogeneous arrival, N-policy, server breakdowns, M/E_k/1 queueing system, probability generating function, multiple vacations.

1. Introduction:

In this paper it is assumed that the number of units arrived follow a Poisson process where the arrival rate varies according to the server status: idle, busy and breakdown. The management policy of our model is that as soon as the system becomes empty, the server is immediately turned off and takes a vacation of random length. The server takes repeated vacation of the same random length unless there are at least N units waiting in the system upon returning from the vacations. Once it starts the service, the server continues the service until all units are served. Whenever the server is working, it is assumed that the server may breakdown at any time and if the server fails, it is immediately sent for repair.
A practical problem related to a manufacturing system is presented for illustrative purpose. We consider the modeling of a production system in which the production times of the orders are made up of $k$ independent and identically distributed exponential random variables with mean $1/k\mu$ which yields an Erlang type $k$ distribution. The system operation starts (turned on) only when $N$ orders have accumulated and is shut down (turned off) when no orders are present. The production does not start until some specified number of units say $N$, are accumulated during the server off period. When the server is working, he may meet unpredictable breakdowns but it is immediately repaired. This production system can be modeled by an $M/E_k/1$ queue with server breakdowns and multiple vacations under $N$-policy.

The concept of the $N$ policy, was first introduced by Yadin and Naor [11]. Past work regarding queueing systems under the $N$ policy may be divided into two categories: (i) cases with server startup and (ii) cases with server’s breakdowns. For cases with server startup, Baker [2] first proposed the N-policy $M/M/1$ queueing system with exponential startup. Borthakur [3] extended Baker’s results to the general startup time. In case with server breakdowns, Wang [7] first proposed a management policy for Markovian queueing systems under the $N$ policy with server breakdowns. Wang [9] and Wang et al. [10] extended the model proposed by Wang [7] to $M/E_k/1$ and $M/H_2/1$ queueing systems, respectively. They developed the analytic closed-form solutions and provided a sensitivity analysis. Haridass and Arumuganathan [5] presented the operating characteristics of an $M^0/G/1$ queueing system with unreliable server and single vacation. Anantha Lakshmi et. al. [1] presented the optimal strategy of an N-policy bulk arrival queueing system with server startup and breakdowns. Vasantha kumar et. al. [6] studied two phase $M^4/E_k/1$ queueing system with server start up and breakdowns. They developed a cost model to determine the optimal operating policy at minimum cost through numerical illustrations. The purpose of this paper is threefold.

1. The steady state equations are established to get the steady state probability distribution and to show that it generalizes the previous results.
2. We formulate the system total expected cost in order to determine the optimal operating $N$-policy numerically at the minimum cost for various values of system parameters while maintaining the minimal service quantity.
3. We perform a sensitivity analysis.

2. Model description:
For the purpose of analytical investigation, we consider the model with the following assumptions:

1. The arrival is poisson process with various arrival rates $\lambda_i$, ($i=0, 1, 2,...$) where $\lambda_0, \lambda_1,$ and $\lambda_2$ denote the arrival rate during the server off, busy and breakdown respectively. The service times according to an Erlang distribution with mean $1/\mu$ and stage parameter $k$. The Erlang type $k$ distribution is made up of $k$ independent and identical exponential stages, each with mean $1/k\mu$. A customer goes into the first stage of the service then progresses through the remaining stages and must complete the last stage before the next customer enter the last stage. We assume
that customers arriving at the service station form a single waiting line and are served in the order of their arrivals.

2. Whenever the system is empty the server goes on a sequence of vacations \( \{V_1, V_2, \ldots \} \). The duration of those vacations are identically and independently distributed random variables denoted by \( V \), having exponential distribution with mean \( 1/r \). When the server comes back from a vacation and find at least \( N \) units waiting in the system, it begins the service immediately until the system is empty again. It is assumed that the duration of each vacation is independent of the arrival process, the service process and breakdown times.

3. When the server is working, the server may breakdown at any time with a Poisson breakdown rate \( \alpha \).

4. When the server fails, it is immediately repaired at a repair rate \( \beta \), where the repair times are exponentially distributed.

### 3. Steady State Results

In steady state the following notations are used.

- \( P_{n,0,0} \) = Probability that there are no customers in the system when the server is idle
- \( P_{n,i,1} \) = Probability that there are \( n \) customers in the system and the customer in service is in stage \( i \) while the server is busy.
- \( P_{n,i,2} \) = Probability that there are \( n \) customers in the system and the customer in service is in stage \( i \) when the server is in operation but found to be broken down.

The steady state equations are given as follows:

\[
\begin{align*}
\lambda_0 P_{1,0,0} &= \lambda_0 P_{0,0,0} \\
\lambda_0 P_{n,k,0} &= \lambda_0 P_{n-1,k,0} \quad (2 \leq n \leq N-1) \\
\lambda_0 P_{0,0,0} &= k\mu P_{1,1,1} \\
(\lambda_0 + r)P_{n,k,o} &= \lambda_0 P_{n-1,k,0} \quad (n \geq N) \\
(\lambda_0 + k\mu + \alpha)P_{1,i,1} &= k\mu P_{i+1,1,1} + \beta P_{1,i+1,2} \quad (1 \leq i \leq k-1) \\
(\lambda_0 + k\mu + \alpha)P_{1,k,1} &= k\mu P_{2,1,1} + \beta P_{1,k,2} \\
(\lambda_0 + k\mu + \alpha)P_{n,i,1} &= \lambda_i P_{n-1,i,1} + k\mu P_{n+1,i,1} + \beta P_{n,i+1,2} \quad (2 \leq n \leq N-1, 1 \leq i \leq k-1) \\
(\lambda_0 + k\mu + \alpha)P_{n,k,1} &= \lambda_i P_{n-1,k,1} + k\mu P_{n+1,k,1} + \beta P_{n,k+1,2} \quad (2 \leq n \leq N-1) \\
(\lambda_0 + k\mu + \alpha)P_{n,k,1} &= \lambda_i P_{n-1,k,1} + k\mu P_{n+1,k,1} + \beta P_{n,k+1,2} + \beta P_{n,k,2} \quad (n \geq N) \\
(\lambda_0 + k\mu + \alpha)P_{n,i,1} &= \lambda_i P_{n-1,i,1} + k\mu P_{n+1,i,1} + \beta P_{n,i+1,2} \quad (n \geq N, 1 \leq i \leq k-1) \\
(\lambda_0 + \beta)P_{1,k,2} &= \alpha P_{1,k,1} \\
(\lambda_0 + \beta)P_{n,i,2} &= \alpha P_{n,i-1,2} + \lambda_2 P_{n-1,i-2} \quad (1 \leq i \leq k-1, n \geq 2) \\
(\lambda_0 + \beta)P_{1,i,2} &= \alpha P_{1,i-1,2} \quad (1 \leq i \leq k-1) \\
(\lambda_0 + \beta)P_{n,k,2} &= \lambda_2 P_{n-1,k,2} + \alpha P_{n,k,1} \quad (n \geq 2)
\end{align*}
\]

Solving equations (1), (2) and (4) recursively, we finally get...
\[ P_{n,k,0} = \begin{cases} P_{0,0,0}, & 1 \leq n \leq N-1 \\ R^{n-(N-1)} P_{0,0,0}, & n \geq N \end{cases} \]  

(15)

where \( R = \frac{\lambda_0}{\lambda_0 + r} \).

4. Probability generating function

The technique of using the probability generating function may be applied in a recursive manner from equation (1) to (14) to obtain the analytic solution of \( P_{0,0,0} \) in neat closed form expression. Define the probability generating function of \( G_0(z) \), \( G_1(z) \) and \( G_2(z) \) respectively as follows:

\[ G_0(z) = P_{0,0,0} + \sum_{n=1}^{\infty} z^n P_{n,k,0} \quad H_i(z) = \sum_{n=1}^{\infty} z^n P_{n,i,1} \quad 1 \leq i \leq k - 1 \]

\[ G_i(z) = \sum_{i=1}^{K} H_i(z) \quad G_i(z) = \sum_{n=1}^{\infty} z^n P_{n,i,2} \quad (1 \leq i \leq k - 1) \quad G_2(z) = \sum_{i=1}^{K} G_i(z) \quad \text{where} \quad |z| \leq 1 \]

Applying algebraic manipulation technique to equations (2) and (3), we get the following

\[ G_0(z) = \left[ \frac{1 - z^N}{1 - z} + \frac{Rz^N}{1 - Rz} \right] P_{0,0,0} \]  

(16)

Multiplying equation (5) by \( z \) and (7) and (10) by \( nz \) and summary over \( n \) we get

\[ H_{i+1}(z) = (r_i + 1 - r_i z + s) H_i(z) - t G_i(z) \]  

(17)

Where \( r_i = \frac{\lambda_i}{k \mu} \quad s = \frac{\alpha}{k \mu} \quad t = \frac{\beta}{k \mu} \)

Multiplying equation (6) by \( z \) (8) and (9) by \( nz \) and summing over \( n \) we get

\[ (r_i + 1 + s)H_K(z) = rz H_K(z) + \frac{1}{z} H_i(z) + \lambda_0 \left[ \frac{rRz^N}{1 - Rz} - 1 \right] P_{0,0,0} + t G_K(z) \]  

(18)

Again multiplying (11) and (13), (12) and (14) respectively by appropriate powers of \( z \) and summary over \( n \) we find

\[ G_i(z) = \frac{\alpha}{\lambda_2 + \beta - \lambda_2 z} H_i(z) \]  

(19)

\[ G_K(z) = \frac{\alpha}{\lambda_2 + \beta - \lambda_2 z} H_K(z) \]  

(20)

Substituting \( G_i(z) \) in equation (23) we get

\[ H_{i+1}(z) = \left[ (r_i + 1 - r_i z + s) - t \frac{\alpha}{\lambda_2 + \beta - \lambda_2 z} \right] H_i(z) \]
Therefore $H_i(z) = \left( r_i + 1 - r_i z + s \right) - t \left( \frac{\alpha}{\lambda_2 + \beta - \lambda_2 z} \right)^{i-1} H_i(z)$ (21)

Solving (16) and (17) in $G_i(z) = \sum_{n=0}^{\infty} z^n P_{n,i,z}$ we obtain

$$H_i(z) = \frac{r_i z \left[ 1 - \frac{r R z}{\lambda_0 (1 - R z)} \right]}{1 - z \left[ (r_i + 1 - r_i z + s) - t \left( \frac{\alpha}{\lambda_2 + \beta - \lambda_2 z} \right) \right]}^{k} P_{0,0,0}$$ (22)

We solve equations (21) and (22) for $G_i(z)$ and $G_z(z)$ to obtain the following

$$G_i(z) = \frac{z \left[ \frac{r R z}{\lambda_0 (1 - R z)} - 1 \right]}{1 - z + s - t \left( \frac{\alpha}{\lambda_2 + \beta - \lambda_2 z} \right)}$$
$$G_z(z) = \frac{\alpha}{\lambda_2 + \beta - \lambda_2 z} G_i(z)$$ (24)

5. The probability of Empty system
We evaluate the probability $P_{0,0,0}$ using normalizing condition. For this purpose we evaluate $G_0(l), G_i(l)$ and $G_z(l)$ from equations (16), (23) and (24) respectively as

$$G_0(l) = \left( N + \frac{\lambda_0}{r} \right) P_{0,0,0}$$
$$G_i(l) = \left( N + \frac{\lambda_0}{r} \right) \left[ 1 - \rho \left[ 1 + \frac{\alpha}{\beta} \right] \right] P_{0,0,0}$$
$$G_z(l) = \frac{\alpha}{\beta} G_i(l)$$

Now using the normalizing condition given by $G(l) = G_0(l) + G_i(l) + G_z(l) = 1$, we obtain the value of probability

$$P_{0,0,0} = \frac{1 - \rho \left[ 1 + \frac{\alpha}{\beta} \right]}{N + \frac{\lambda_0}{r}}$$ (25)

6. Expected number of units in the system
Using the probability generating functions expected number of customers in the system at different states are presented below.

Let $L_i, L_0$ and $L_d$ denotes the expected number of units in the system when the
server is in vacation, working and breakdown respectively.

To find \( L_v \), we compute \( G_0'(1) \) in equation (16) by applying L’Hospital rule twice to obtain.

\[
L_v = \left\{ \frac{N(N-1)}{2} + \frac{\lambda_0}{r} \right\} \left[ 1 - \rho \left( \frac{1}{\beta} \right) \right]
\]

(26)

Similarly, we compute \( G_1'(1) \) and \( G_2'(1) \) in equations (23) and (24) respectively, by applying L’hospital rule twice to obtain

\[
L_v = -\frac{1}{2\left( 1 + \frac{a}{\beta} \right)} \left[ \frac{2s^2 + \lambda_0^2}{\beta} \left( \frac{1}{\beta} \right) \rho + \rho \left( \frac{N(N-1)}{2} + 2\frac{\lambda_0}{r} \right) \left[ \left( 1 + \frac{a}{\beta} \right) \right] \right]
\]

(27)

\[
L_d = \frac{\alpha}{\beta} G_1'(1) + G_1(1) \left[ -\frac{\alpha \lambda_2}{\beta^2} \right]
\]

(28)

The expected number of units in the system is given by

\[
L_v = G_0'(1) + G_1'(1) + G_2'(1)
\]

(29)

7. Special Cases

In this section we present some existing results in the literature which are special cases of our model.

Case (i): If \( \lambda_1 = \lambda_2 = \lambda_3, r=\infty, \alpha=0 \) and \( \beta=\infty \), expressions (26), (27) and (29) reduces to a special cases of \( L_{\text{off}}, L_{\text{on}} \) and \( L_{\text{N}} \) respectively of Wang and Huang [8](p. 1019)

Case (ii): If \( N=1, \lambda_1 = \lambda_2 = \lambda_3, r=\infty, \alpha=0 \) and \( \beta=\infty \), the expression (25) reduces to a special case of expression (2. 8) of Gross and Harris [4]

8. Other System Characteristic

Using the concept of grand vacation process, we have the expected length of idle period

\[
E(I) = \frac{N}{\lambda_0}
\]

(30)

If E(B), E(D) and E(C) denote the expected busy period, breakdown period and busy cycle respectively, we have \( E(C) = E(I) + E(B)+E(D) \)

The long run fraction of time for the server is idle, busy and breakdown respectively is given by

\[
\frac{E(I)}{E(C)} = G_0(l) = 1 - \rho \left( \frac{1}{\beta} \right), \quad \frac{E(B)}{E(C)} = G_1(l) = \rho, \quad \frac{E(D)}{E(C)} = G_2(l) = \frac{\alpha}{\beta} \rho
\]
Thus we have the number of cycles per unit time
\[ E(C) = \frac{N}{\lambda_0 \left[ 1 - \rho \left( 1 + \frac{\alpha}{\beta} \right) \right]} \]

9. Optimal Control policy
We develop a steady state total expected cost function per unit time for the heterogeneous arrival of an N-policy M/\(E_k/1\) Queueing system with server vacations and breakdowns in which N is a decision variable. With the cost structure being constructed, our objective is to determine the optimal management policy to minimize this cost function, while maintaining the minimal service quality to the customers. Let
\[ C_h = \text{holding cost per unit time for each customer present in the system.} \]
\[ C_o = \text{Cost per unit time for the operating service station.} \]
\[ C_s = \text{Set up cost per cycle.} \]
\[ C_d = \text{breakdown cost per unit time} \]
\[ C_{\gamma} = \text{reward per unit time for the server being on vacation.} \]

Using the definition of each cost elements and its corresponding characteristics, the total expected cost function per unit time is given by
\[ T(N) = C_h L_N + \frac{C_s}{E(C)} E(B) + C_o \frac{E(D)}{E(C)} + C_d \frac{E(I)}{E(C)} - C_{\gamma} \frac{E(L)}{E(C)} \]  
(31)

We obtain the optimal value \(N^*\), which minimizes cost function by differentiating it with respect to N and setting the result to be zero. i.e., \(\frac{\partial}{\partial N} (T(N)) = 0\). The solution \(N\) to (31) may not be an integer and the optimal positive integer value of \(N\) is one of the integers surrounding \(N^*\) which gives a smaller cost \(T\). Here we should be pointed out explicitly that the solution really gives the minimum value and \(\frac{\partial^2}{\partial N^2} (T(N)) \) at \(N=N^*\) is greater than zero when the values of system parameters satisfy suitable conditions. However, it is quite tedious to present the explicit expression. Therefore we will perform the numerical experiments to demonstrate that the function is really convex and the solution gives a minimum.

10. Sensitivity Analysis
In the course of analysis, sensitivity analysis has been carried out to find the optimum value of \(N\) (ie., \(N^*\)), expected system length and minimum cost based on changes in the system parameters.

In order to arrive at the conclusions, the following arbitrary values of the system parameters are considered.
The optimal value $N^*$ and the corresponding minimum expected cost $T(N^*)$ are summarized in Table (1) for the parameters $(\lambda_0, \lambda_1, \lambda_2, \mu, \beta) = (0.6, 0.8, 0.6, 2.5, 4.0)$ and various values of $(r, \alpha)$. From table (1) we observe that i) the value of $N^*$ and expected cost $T(N^*)$ decreases as $r$ increases, and ii) $N^*$ remains unchanged when $\alpha$ increases from 0.05 to 0.5. Thus $N^*$ is insensitive to the changes in $\alpha$ when the parameter values are considered in some interval. The optimal value $N^*$ and the corresponding minimum expected cost $T(N^*)$ are summarized in Table (2) for the parameters $(\lambda_0, \lambda_1, \lambda_2, r, \alpha) = (0.4, 0.6, 0.4, 2.0, 0.05)$ and various values of $(\mu, \beta)$. From table (2) we observe that i) the value of $N^*$ increases and expected cost $T(N^*)$ decreases as $\mu$ increases. ii) $N^*$ remains unchanged when $\beta$ changes from 2.0 to 6.0. Thus $N^*$ is insensitive to the changes in $\beta$ when the parameter values are considered in some interval. iii) the value of $T(N^*)$ decreases when $\beta$ changes from 2.0 to 6.0.

Table 1: $(\lambda_0, \lambda_1, \lambda_2, \mu, \beta) = (0.6, 0.8, 0.6, 2.5, 4.0)$

<table>
<thead>
<tr>
<th>$(r, \alpha)$</th>
<th>(2.0, 0.05)</th>
<th>(5.0, 0.05)</th>
<th>(10.0, 0.05)</th>
<th>(2.5, 0.05)</th>
<th>(2.5, 0.2)</th>
<th>(2.5, 0.5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N^*$</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$T(N^*)$</td>
<td>83.4945</td>
<td>82.8881</td>
<td>82.6868</td>
<td>23.2918</td>
<td>33.4923</td>
<td>35.0531</td>
</tr>
</tbody>
</table>

Table 2: $(\lambda_0, \lambda_1, \lambda_2, r, \alpha) = (0.4, 0.6, 0.4, 2.0, 0.05)$

<table>
<thead>
<tr>
<th>$(\mu, \beta)$</th>
<th>(0.8, 3.0)</th>
<th>(1.0, 3.0)</th>
<th>(1.2, 2.0)</th>
<th>(1.0, 2.0)</th>
<th>(1.0, 4.0)</th>
<th>(1.0, 6.0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N^*$</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$T(N^*)$</td>
<td>154.2810</td>
<td>105.3469</td>
<td>57.3954</td>
<td>143.2689</td>
<td>138.1420</td>
<td>134.0857</td>
</tr>
</tbody>
</table>

From table (3) and (4) we observe that i) $N^*$ increases as $\lambda_1$ or $\lambda_0$ increases. ii) $N^*$ remains unchanged when $\lambda_2$ changes from 0.4 to 0.8. In general $N^*$ is insensitive to the changes in $\lambda_2$ when the parameter values values are considered in some interval. iii) $T(N^*)$ decreases as $\lambda_1$ increases from 0.4 to 0.8. From our numerical investigation, it appears that i) $\alpha$, $\beta$ and $\lambda_2$ do not affect $N^*$. ii) $\lambda_0$ and $\lambda_1$ rarely affect $N^*$. but iii) $r$ and $\mu$ significantly affect $N^*$.

Table 3: $(\lambda_2, r, \alpha, \mu, \beta) = (0.5, 2.0, 0.05, 2.0, 5.0)$

<table>
<thead>
<tr>
<th>$(\lambda_0, \lambda_1)$</th>
<th>(0.4, 0.4)</th>
<th>(0.4, 0.6)</th>
<th>(0.4, 0.8)</th>
<th>(0.3, 0.7)</th>
<th>(0.5, 0.7)</th>
<th>(0.7, 0.7)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N^*$</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>$T(N^*)$</td>
<td>1006.543</td>
<td>587.095</td>
<td>378.2367</td>
<td>584.8592</td>
<td>589.0813</td>
<td>592.5658</td>
</tr>
</tbody>
</table>

Table 4: $(\lambda_1, r, \alpha, \mu, \beta) = (0.8, 1.0, 0.05, 1.0, 5.0)$

<table>
<thead>
<tr>
<th>$(\lambda_0, \lambda_2)$</th>
<th>(0.4, 0.4)</th>
<th>(0.4, 0.6)</th>
<th>(0.4, 0.8)</th>
<th>(0.4, 0.5)</th>
<th>(0.6, 0.5)</th>
<th>(0.8, 0.5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N^*$</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$T(N^*)$</td>
<td>70.3373</td>
<td>70.3831</td>
<td>70.4482</td>
<td>70.3577</td>
<td>72.3183</td>
<td>74.0572</td>
</tr>
</tbody>
</table>
11. Conclusion
Optimal control of an N-policy heterogeneous arrival M/E_k/1 queueing system with server breakdowns and multiple vacations is studied. Some of the system performance measures are derived. A cost function is formulated to determine the optimal value of N. Sensitivity analysis is carried out through numerical illustrations. These numerical values will be useful in analyzing practical queueing system and make decision to improve the grade of service by selecting appropriate system descriptors. The model investigated in this paper is more realistic than those existing once since it takes the behavior of the arriving units as well as the servers into consideration. The present study can be extended by working vacation.

References
