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Approximation Properties and Convergence Analysis of Gamma type Operators

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Abstract

The Gamma operators, constructed through the Gamma function, constitute an important class of positive linear operators with notable approximation properties. In this paper, we investigate the approximation behavior of these operators and establish several quantitative estimates for their convergence. In particular, error bounds are derived in terms of different moduli of continuity, providing a precise description of the rate of approximation. The obtained results contribute to the study of the direct approximation properties of Gamma-type operators and extend the existing convergence estimates in this framework.

Keywords: Gamma operators, Post-Widder operators, Rathore operators, linear positive operators, Modulus of continuity.

Mathematics Subject Classification: 41A25, 47B30

1. Introduction

The Gamma transform associated with a function f is given by

$$\Gamma_{\alpha,\beta}f = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty} \mathrm{e}^{-\beta u}u^{\alpha-1}f(u)\mathrm{d}u,$$

where $f \in L_{1,\,\mathrm{loc}}(0,\infty)$ such that $\Gamma_{\alpha,\beta}|f| < \infty$. $\Gamma_{\alpha,\beta}e_0 = 1, \Gamma_{\alpha,\beta}e_1 = \frac{\alpha}{\beta}$, $L_{1,\,\mathrm{loc}}(0,\infty)$

means the space of functions that are locally integrable on $(0,\infty)$.

In general $\Gamma_{\alpha,\beta}e_r = \frac{(\alpha)_r}{\beta^r}$, where $e_r(u) = u^r, r = 0,1,2,\ldots$ and the Pochhammer symbol $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1), (\alpha)_0 = 1$ .The operator $\Gamma_{\alpha,\beta}$ preserves constants, but it does not preserve the linear function $e_1(u) = u$, unless $\alpha = \beta$.

In order to preserve the linear functions, Miheşan in [9] considered $\beta = \alpha/x$ and the Gamma operators take the following form:

$$G_\alpha(f,x) = \left(\frac{\alpha}{x}\right)^\alpha \frac{1}{\Gamma(\alpha)} \int_0^\infty u^{\alpha-1} \mathrm{e}^{-\alpha u/x} f(u) \mathrm{d}u.$$

(1.1)

As some of the special cases, if

If $\alpha = nx$, it reduces to Rathore operators [12], defined as:

$$R_n(f,x) = \frac{n^{nx}}{\Gamma(nx)} \int_0^\infty u^{nx-1} \mathrm{e}^{-nu} f(u) \mathrm{d}u.$$

If $\alpha = n$, we immediately have Post-Widder operators [5], defined as:

$$P_n(f,x) = \left(\frac{n}{x}\right)^n \frac{1}{\Gamma(n)} \int_0^\infty u^{n-1} \mathrm{e}^{-nu/x} f(u) \mathrm{d}u.$$

The classical Lupaş and Baskakov operators can be obtained as compositions of the Rathore–Szász and Post–Widder–Szász operators, respectively. In particular, the relevant identities are typically written in the form:

$$L_n(f,x) := (R_n \circ S_n)f = \sum_{v=0}^{\infty} 2^{-nx} \frac{(nx)_v}{v!.\, 2^v} f\left(\frac{v}{n}\right)$$

and

$$B_n(f,x) := (P_n \circ S_n)f = \sum_{v=0}^{\infty} \binom{n+v-1}{v} \frac{x^v}{(1+x)^{n+v}} f\left(\frac{v}{n}\right)$$

where the Szász operators are given as:

$$S_n(f,x) = \sum_{v=0}^{\infty} \mathrm{e}^{-nx} \frac{(nx)^v}{v!} f\left(\frac{v}{n}\right).$$

In recent years, considerable attention has been devoted to the study of approximation properties and convergence behavior of various classes of linear positive operators. The author has also contributed to this area through several related studies; see [13, 14, 15]. A substantial body of literature has been developed on this topic. For further results concerning the convergence of linear positive operators, we refer the reader to [2, 7, 8, 11, 16, 17] and the references cited therein.

2. Moment Estimations:

Lemma 2.1: The r-th order moment for $r \in \mathbb{N} \cup \{0\}$, is given by the following relation:

$$G_\alpha(e_r, x) = \frac{(\alpha)_r \cdot x^r}{\alpha^r}, \quad \text{where} \quad e_r(u) = u^r. \tag{2.1}$$

In particular,

$$G_\alpha(1, x) = 1, G_\alpha(e_1, x) = x$$

$$G_\alpha(e_2, x) = \left[1 + \frac{1}{\alpha}\right] x^2$$

$$G_\alpha(e_3, x) = \left[1 + \frac{3}{\alpha} + \frac{2}{\alpha^2}\right] x^3$$

$$G_\alpha(e_4, x) = \left[1 + \frac{6}{\alpha} + \frac{11}{\alpha^2} + \frac{6}{\alpha^3}\right] x^4$$

$$G_\alpha(e_5, x) = \left[1 + \frac{10}{\alpha} + \frac{35}{\alpha^2} + \frac{50}{\alpha^3} + \frac{24}{\alpha^4}\right] x^5$$

$$G_\alpha(e_6, x) = \left[1 + \frac{15}{\alpha} + \frac{85}{\alpha^2} + \frac{225}{\alpha^3} + \frac{274}{\alpha^4} + \frac{120}{\alpha^5}\right] x^6.$$

Furthermore, the moment-generating function associated with G_α is given by:

$$G_\alpha\left(\mathrm{e}^{\theta u}, x\right) = \left(1 - \frac{\theta x}{\alpha}\right)^{-\alpha}$$

$$1 + \theta x + \frac{\alpha(\alpha+1)}{\alpha^2}\frac{\theta^2 x^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\alpha^3}\frac{\theta^3 x^3}{3!} + \cdots$$

It follows from the above expansion that the k-th order moment $G_\alpha(e_k, x), k = 0,1,2, \ldots$ is given by the coefficient of $\theta^k/k!$. Consequently, using the linearity of G_α, the corresponding central moments $\mu_{\alpha,k}(x) = G_\alpha((u-x)^k, x)$, are given by:

$$\mu_{\alpha,0}(x) = 1, \mu_{\alpha,1}(x) = 0, \mu_{\alpha,2}(x) = \frac{x^2}{\alpha},$$

$$\mu_{\alpha,4}(x) = \left[\frac{3}{\alpha^2} + \frac{6}{\alpha^3}\right] x^4,$$

$$\mu_{\alpha,6}(x) = \left[\frac{15}{\alpha^3} + \frac{130}{\alpha^4} + \frac{120}{\alpha^5}\right] x^6.$$

In this paper, we investigate the convergence properties of the Gamma operators defined in (1.1). Several direct approximation results are established in terms of both the usual and weighted moduli of continuity. Furthermore, a direct estimate is obtained in terms of the modulus of continuity for functions with exponential growth.

3. Convergence Estimates

Let $C_B[0,\infty)$ denotes the space of all continuous and bounded functions on $[0,\infty)$, equipped with the norm $\|f\| = \sup\{|f(x)|: x \in [0,\infty)\}$.
Furthermore, we consider the following K-functional:

$$K(f,\delta) = \inf_{h \in C_B^2[0,\infty)} \{\|f-h\| + \delta\|h''\|\},$$

where $\delta > 0$ and $C_B^2[0,\infty) = \{h \in C_B[0,\infty): h', h'' \in C_B[0,\infty)\}$. By [3], the following inequality holds:

$$K(f,\delta) \leq C\omega_2(f,\sqrt{\delta}), \delta > 0. \tag{3.1}$$

Theorem 3.1: Let $f \in C_B[0,\infty)$, then we have:

$$|G_\alpha(f,x) - f(x)| \leq C\omega_2\left(f,\frac{x}{\sqrt{\alpha}}\right).$$

Proof: For $h \in C_B^2[0,\infty)$ and $x,u \in [0,\infty)$, Taylor's expansion yields:

$$h(u) = h(x) + (u-x)h'(x) + \int_x^u (u-t)h''(t)dt.$$

Then using Lemma 2.1, we have:

$$|G_\alpha(h,x) - h(x)| = \left|G_\alpha\left(\int_x^u (u-t)h''(t)dt, x\right)\right|$$

$$\leq \tfrac{1}{2}\mu_{\alpha,2}(x)\|h''\| = \tfrac{x^2}{2\alpha}\|h''\|. \tag{3.2}$$

We have:

$$|G_\alpha(f,x)| \leq \|f\|. \tag{3.3}$$

Hence, by (3.2) and (3.3), we have:

$$|G_\alpha(f,x) - f(x)| \leq |G_\alpha(f-h,x) - (f-h)(x)| + |G_\alpha(h,x) - h(x)|$$
$$\leq 2\|f-h\| + \tfrac{x^2}{2\alpha}\|h''\|. \tag{3.4}$$

Taking infimum over all $h \in C_B^2[0,\infty)$, and using (3.1), we get the desired result.

Corollary 3.2: As special cases, and following Theorem 3.1, we have
(1) For $\alpha = n$, the corresponding operators reduce to the Post–Widder operators. Consequently

$$|P_n(f,x) - f(x)| \le C\omega_2\left(f, \frac{x}{\sqrt{n}}\right).$$

(2) For the particular choice $\alpha = nx$, we obtain the Rathore operators, which satisfy
(2)

$$|R_n(f,x) - f(x)| \le C\omega_2\left(f, \sqrt{\frac{x}{n}}\right).$$

Following [10], we define the weighted modulus $\omega_\varphi(f;g)$, as

$$\omega_\varphi(f;g) = \sup_{\substack{|x-y|\le g\varphi\left(\frac{x+y}{2}\right) \\ x\ge 0, y\ge 0, g\ge 0}} |f(x) - f(y)|,$$

where $\varphi(x) = \frac{\sqrt{x}}{1+x^k}, x \in [0,\infty), k \in \mathbb{N}, k \ge 2$. Aral et al. [1] established a Voronovskaja-type asymptotic formula associated with this weighted modulus of continuity. In the present work, we consider the class of functions satisfying the following property:

$$\lim_{h\to 0} \omega_\varphi(f;g) = 0,$$

whenever $f \circ e_2$ is uniformly continuous on [0, 1] and $f \circ e_r, r = \frac{2}{2k+1}$ is uniformly continuous on $[1,\infty)$, $e_r(x) = x^r, x \ge 0$. We denote by $W_\varphi[0,\infty)$, the subspace of all real-valued functions defined on $[0,\infty)$, satisfying the above conditions. Further, let E denotes the subspace of $C[0,\infty)$, such that $C_m[0,\infty) \subset E$, where $m = \max\{k + r + 1, 2r + 2, 2k\}$ and
$C_m[0,\infty) = \{f \in C[0,\infty), \text{there exists } Q > 0: |f(x)| \le Q(1 + x^m), \forall x \ge 0\}, m \in \mathbb{N}.$

In [19], the following quantitative asymptotic formula was obtained.
Theorem 3.3: [19] Let $L_n: E \to C[0,\infty), C_m[0,\infty) \subset E, m = \max\{k + 3, 6, 2k\}$ be a sequence of linear positive operators, reproducing the linear functions. If $f \in C^2[0,\infty) \cap E$ and $f'' \in W_\varphi[0,\infty)$, then we have for $x \in (0,\infty)$ that:

$$\left|L_n(f,x) - f(x) - \frac{1}{2}f''(x)\mu_{n,2}^L(x)\right|$$
$$\le \frac{1}{2}\left[\mu_{n,2}^L(x) + \sqrt{2}\sqrt{L_n\left(\left[1 + \left(x + \frac{|u-x|}{2}\right)^k\right]^2; x\right)}\right]\omega_\varphi\left(f''; \left(\frac{\mu_{n,6}^L}{x}\right)^{1/2}\right)$$
$$\mu_{n,k}^L(x) = L_n((u-x)^k, x)$$
.
As an application of Theorem 3.3, we present the following exact estimate for the Gamma operators.

Theorem 3.4: Assuming the conditions of Theorem 3.3, we obtain the following result for the family G_α of linear positive operators.

$$\left|G_\alpha(f,x)-f(x)-\frac{x^2}{2\alpha}f''(x)\right|$$

$$\leq \frac{1}{2}\left[\frac{x^2}{\alpha}+\sqrt{2}\sqrt{G_\alpha\left(\left[1+\left(x+\frac{|u-x|}{2}\right)^k\right]^2;x\right)}\right]$$

$$\omega_\varphi\left(f'';\left(\left[\frac{15}{\alpha^3}+\frac{130}{\alpha^4}+\frac{120}{\alpha^5}\right]x^5\right)^{1/2}\right).$$

Corollary 3.5: In view of the assumptions of Theorem 3.4, we have:
(1) In the special case $\alpha=n$, the proposed operators coincide with the Post–Widder operators, and hence

$$\left|P_n(f,x)-f(x)-\frac{x^2}{2n}f''(x)\right|$$

$$\leq \frac{1}{2}\left[\frac{x^2}{n}+\sqrt{2}\sqrt{P_n\left(\left[1+\left(x+\frac{|u-x|}{2}\right)^k\right]^2;x\right)}\right]$$

$$\omega_\varphi\left(f'';\left(\left[\frac{15}{n^3}+\frac{130}{n^4}+\frac{120}{n^5}\right]x^5\right)^{1/2}\right).$$

(2) When $\alpha=nx$, we obtain the Rathore operators, for which the following relation holds:

$$\left|R_n(f,x)-f(x)-\frac{x}{2n}f''(x)\right|$$

$$\leq \frac{1}{2}\left[\frac{x}{n}+\sqrt{2}\sqrt{R_n\left(\left[1+\left(x+\frac{|u-x|}{2}\right)^k\right]^2;x\right)}\right]$$

$$\omega_\varphi\left(f'';\left(\left[\frac{15x^2}{n^3}+\frac{130x}{n^4}+\frac{120}{n^5}\right]\right)^{\frac{1}{2}}\right).$$

Theorem 3.6: [19] Let $L_n: E \to C[0,\infty), C_m[0,\infty) \subset E, m = \max\{k+3,4\}$ be sequence of linear positive operators, retaining the linear functions. If $f \in C^2[0,\infty) \cap E$ and $f'' \in W_\varphi[0,\infty)$, then we have for $x \in (0,\infty)$ that:

$$\left|L_n(f,x) - f(x) - \frac{1}{2}f''(x)\mu_{n,2}^L(x)\right|$$

$$\leq \frac{1}{2}\left[\mu_{n,2}^L(x) + \frac{\sqrt{2}}{\sqrt{x}}\mu_{n,2}^L(x) \cdot C_{n,2,k}(x)\right]\omega_\varphi\left(f''; \sqrt{\frac{\mu_{n,4}^L(x)}{\mu_{n,2}^L(x)}}\right),$$

where

$$C_{n,2,k}(x) = 1 + \frac{1}{Q_{n,3}^L(x)} \cdot \sum_{m=0}^{k} \binom{k}{m} x^{k-m} \frac{Q_{n,m+3}^L(x)}{2^m}.$$

We suppose for the operators L_n that
$\frac{Q_{n,m}^L}{Q_{n,3}^L}, 4 \leq m \leq k,$ is a bounded ratio for fixed x and k, when $n \to \infty$ and $Q_{n,r}^L = L_n(|u-x|^r, x)$.
The estimate

$$L_n((u-x)^m, x) = O\left(n^{-[(m+1)/2]}\right), n \to \infty,$$

implies that $C_{n,2,k}(x)$ is bounded for fixed x and k as $n \to \infty$. Hence, applying Theorem 3.6, we obtain the following result for G_α.
Theorem 3.7: Assuming the conditions of Theorem 3.6, we obtain the following result for the family G_α of linear positive operators:

$$\left|G_\alpha(f,x) - f(x) - \frac{x^2}{2\alpha}f''(x)\right|$$

$$\leq \frac{x^2}{2\alpha}\left[1 + \frac{\sqrt{2}}{\sqrt{x}} \cdot C_{\alpha,2,k}(x)\right]\omega_\varphi\left(f''; x\sqrt{\left(\frac{6}{\alpha^2} + \frac{3}{\alpha}\right)}\right),$$

where

$$C_{\alpha,2,k}(x) = 1 + \frac{1}{Q_{\alpha,3}^L(x)} \cdot \sum_{m=0}^{k} \binom{k}{m} x^{k-m} \frac{Q_{\alpha,m+3}^L(x)}{2^m}.$$

We consider

$$\frac{Q_{\alpha,m}^L}{Q_{\alpha,3}^L}, 4 \leq m \leq k,$$

is a bounded ratio for fixed x and k, when $\alpha \to \infty$ and

$$Q_{\alpha,r}^L = G_\alpha(|u-x|^r, x).$$

Corollary 3.8: If f and f'' satisfy the same assumptions as those stated in Theorem 3.7, then, for $x \in (0,\infty)$ we have:

$$\lim_{n\to\infty} \alpha[G_\alpha(f,x) - f(x)] = \frac{x^2}{2} f''(x).$$

For continuous functions on $[0,\infty)$ with exponential growth satisfying $\|f\|_A := \sup_{x\in[0,\infty)} |f(x)\mathrm{e}^{-Ax}| < \infty, A > 0$, the first-order modulus of continuity [18] is defined as follows

$$\omega_1(f,\delta,A) = \sup_{|g|\le\delta, 0\le x<\infty} |f(x) - f(x+g)|\mathrm{e}^{-Ax}.$$

A similar form of the second-order modulus of continuity was introduced by Ditzian in [4].

Remark 3.9: [18] For every positive number $g > 0$ and $m \in \mathbb{N}$, the following holds true

$$\omega_1(f,mg,A) \le m \cdot e^{A(m-1)h} \cdot \omega_1(f,g,A).$$

Our main result, expressed in terms of the exponential modulus of continuity, is stated in the following theorem.

Theorem 3.10: Let G_α, be a family of linear positive operators. If $f \in C^2[0,\infty) \cap E$ and $f'' \in \mathrm{Lip}(\beta,A)$, where $0 < \beta \le 1$, then, for $\alpha > 2Ax$ and $x \in [0,\infty)$,we have:

$$\left|G_\alpha(f,x) - f(x) - \frac{x^2}{2\alpha} f''(x)\right|$$

$$\le \left[e^{2Ax} + \frac{C(A,x)}{2} + \frac{\sqrt{C(2A,x)}}{2}\right] \cdot \mu_{\alpha,2}(x) \cdot \omega_1\left(f'', \sqrt{\frac{\mu_{\alpha,4}(x)}{\mu_{\alpha,2}(x)}}, A\right),$$

where $\mu_{\alpha,2}(x)$ and $\mu_{\alpha,4}(x)$ are given in Lemma 2.1. Here, $C(A,x) = 2^{2(Ax+1)}\left(1 + \frac{Ax}{2}\right)$ and the spaces $\mathrm{Lip}(\beta,A)$,$0 < \beta \le 1$, denote the class of all functions f satisfying $\omega_1(f,\delta,A) \le Q\delta^\beta$ for all $\delta < 1$.

Proof: Let $f \in C^2[0,\infty)$, then, the Taylor expansion of f about the point $x \in [0,\infty)$ is given by:

$$f(u) = f(x) + (u-x)f'(x) + \frac{(u-x)^2}{2!} f''(x) + \varepsilon_2(u,x),$$

(3.5)

where

$$\varepsilon_2(u,x) = \frac{f''(\xi) - f''(x)}{2}(u-x)^2.$$

Also ξ lies between x and u. Applying the operator G_α to both sides of (3.5) and using Lemma 2.1, we obtain:

$$\left|G_\alpha(f,x) - f(x) - \frac{x^2}{2\alpha} f''(x)\right| \le G_\alpha(|\varepsilon_2(u,x)|,x).$$

(3.6)

To complete the proof of the theorem, we need to estimate $G_\alpha(|\varepsilon_2(u,x)|,x)$. By straightforward computations, we obtain:

$$|\varepsilon_2(u,x)| \leq \frac{1}{2}(\mathrm{e}^{2Ax}+\mathrm{e}^{Au})\left(1+\frac{|u-x|}{g}\right)\omega_1(f'',g,A)|u-x|^2.$$

Consequently,

$$G_\alpha(|\varepsilon_2(u,x)|,x) \leq \frac{1}{2}\left[G_\alpha\left((\mathrm{e}^{2Ax}+\mathrm{e}^{Au})\cdot\left(|u-x|^2+\frac{|u-x|^3}{g}\right);x\right)\right]\omega_1(f'',g,A).$$

(3.7)

Along with operators (1.1), we have:

$$G_\alpha(u^k\mathrm{e}^{Au},x) = \left(\frac{\alpha}{x}\right)^\alpha \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)}\left(\frac{\alpha}{x}-A\right)^{-k-\alpha}, Ax < \alpha.$$

(3.8)

By using (3.8), we get:

$$G_\alpha((u-x)^2\mathrm{e}^{Au},x) = \left(\frac{\alpha}{x}\right)^\alpha\left(\frac{\alpha}{x}-A\right)^{-\alpha-2}(A^2x^2+\alpha)$$

$$= \left(\frac{\alpha}{x}\right)^{\alpha+2}\left(\frac{\alpha}{x}-A\right)^{-\alpha-2}\left(1+\frac{A^2x^2}{\alpha}\right)\mu_{\alpha,2}(x)$$

$$= \left(1-\frac{Ax}{\alpha}\right)^{-\alpha-2}\left(1+\frac{A^2x^2}{\alpha}\right)\mu_{\alpha,2}(x) \quad .$$

(3.9)

For $\alpha > 2Ax$, (3.9) becomes:

$$G_\alpha((u-x)^2\mathrm{e}^{Au},x) \leq 2^{2(Ax+1)}\left(1+\frac{Ax}{2}\right)\mu_{\alpha,2}(x) := C(A,x)\mu_{\alpha,2}(x).$$

(3.10)

Applying, Cauchy-Schwarz inequality we get:

$$G_\alpha(|u-x|^3\mathrm{e}^{Au},x) \leq \sqrt{G_\alpha((u-x)^2\mathrm{e}^{2Au},x)}\cdot\sqrt{\mu_{\alpha,4}(x)}$$

$$\leq \sqrt{C(2A,x)\mu_{\alpha,2}(x)}\cdot\sqrt{\mu_{\alpha,4}(x)}.$$

(3.11)

Substituting $g = \sqrt{\frac{\mu_{\alpha,4}(x)}{\mu_{\alpha,2}(x)}}$ in (3.7) and combining (3.6), (3.10) and (3.11), we obtain the desired result.

In the following convergence estimate, let $C^*[0,\infty)$, denotes the linear space of all real-valued continuous functions on $[0,\infty)$ such that:

$\lim_{x\to\infty} f(x)$ exists and is finite, endowed with the uniform norm. Furthermore, for every $\delta \geq 0$, the modulus of continuity of f (see [6]) is defined by

$$\omega^*(f,\delta)=\sup_{\substack{x,u\geq 0\\ |\mathrm{e}^{-x}-\mathrm{e}^{-u}|\leq\delta}}|f(x)-f(u)|.$$

Theorem 3.11: Let $f,f''\in C^*[0,\infty)$, then, for $x\in[0,\infty)$, the following inequality holds:

$$\left|\alpha[G_\alpha(f,x)-f(x)]-\frac{x^2}{2}f''(x)\right|$$

$$\leq\frac{\omega^*\left(f'',\alpha^{-1/2}\right)}{2}\left[x^2+x^2\sqrt{\frac{3(\alpha+2)}{\alpha}}[\alpha^2G_\alpha((e^{-x}-e^{-u})^4,x)]^{1/2}\right].$$

Proof: Applying Taylor's formula, we obtain:

$$f(u)=f(x)+(u-x)f'(x)+(u-x)^2\frac{f''(x)}{2}+g(\xi,x)(u-x)^2,$$

where ξ lies between x and u and

$$g(\xi,x):=\frac{f''(\xi)-f''(x)}{2}.$$

On applying the operator G_α to the above equality and invoking Lemma 2.1, we obtain:

$$\left|G_\alpha(f,x)-\mu_{\alpha,0}(x)f(x)-\mu_{\alpha,1}(x)f'(x)-\frac{1}{2}\mu_{\alpha,2}(x)f''(x)\right|$$

$$=|G_\alpha(g(\xi,x)(u-x)^2,x)|.$$

Hence, we get:

$$\left|\alpha[G_\alpha(f,x)-f(x)]-\frac{x^2}{2}f''(x)\right|=|\alpha G_\alpha(g(\xi,x)(u-x)^2,x)|.$$

By using the properties of $\omega^*(.\,,\delta)$ and the Cauchy–Schwarz inequality, we get:

$$\alpha G_\alpha(|g(\xi,x)|(u-x)^2,x)$$

$$\leq\frac{1}{2}\alpha\omega^*(f'',\delta)\mu_{\alpha,2}(x)+\frac{\alpha}{2\delta^2}\omega^*(f'',\delta)$$

$$\left[G_\alpha((\mathrm{e}^{-x}-\mathrm{e}^{-u})^4,x)\cdot\mu_{\alpha,4}(x)\right]^{1/2}$$

$$\leq \frac{\omega^*(f'',\delta)}{2}\left[x^2+\frac{\alpha}{\delta^2}\left[G_\alpha((\mathrm{e}^{-x}-\mathrm{e}^{-u})^4,x)\cdot\left(\frac{3(\alpha+2)x^4}{\alpha^3}\right)\right]^{1/2}\right].$$

Choosing $\delta=\alpha^{-1/2}$, we finally get the desired result.

4. Graphical and Numerical presentation: In this section, we provide a numerical section that directly verifies the theoretical results, especially the convergence as $\alpha\to\infty$, given in section 3.

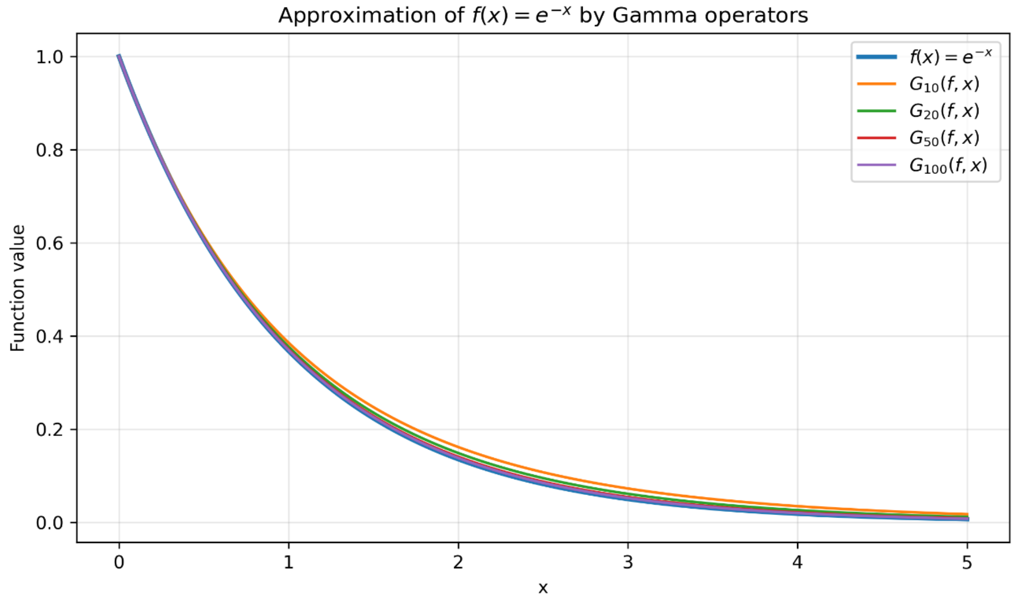

Figure 1: Approximation for $f(x)=e^{-x}$

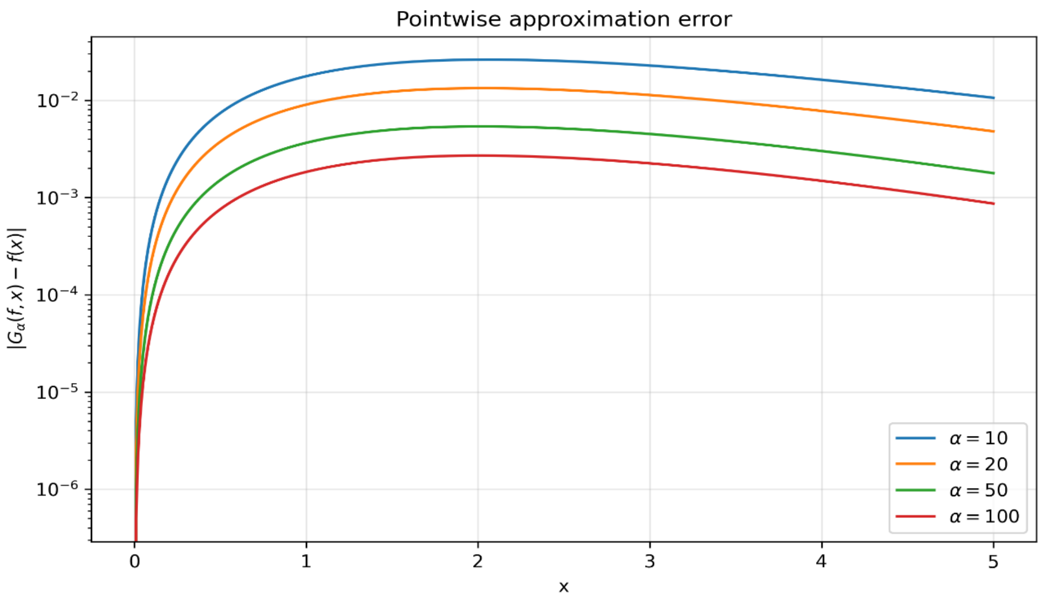

Figure 2: Pointwise approximation error

Following table gives $E_\alpha(x) = |G_\alpha(f,x) - f(x)|$

x	α	$G_\alpha(f,x)$	$\lvert G_\alpha(f,x) - f(x)\rvert$
0.5	10	0.613913	0.007383
0.5	20	0.610271	0.003740
0.5	50	0.608039	0.001508
0.5	100	0.607287	0.000756
1	10	0.385543	0.017664
1	20	0.376889	0.009010
1	50	0.371528	0.003648
1	100	0.369711	0.001832
2	10	0.161506	0.026170
2	20	0.148644	0.013308
2	50	0.140713	0.005377
2	100	0.138033	0.002698
3	10	0.072538	0.022751
3	20	0.061100	0.011313
3	50	0.054288	0.004501
3	100	0.052033	0.002246

Table: 1

Since $(x) = e^{-x}$, the table clearly shows that, $G_\alpha(f,x) \to f(x)$ as $\alpha \to \infty$.

References

[1] Aral, A., Gonska, H., Heilmann, M., and Tachev, G., 2016, "Quantitative Voronovskaya-type results for polynomially bounded functions," Results Math., 70(3–4), pp. 313–324.

[2] Deveci, S.N., Acar, T., and Alagoz, O., 2020, "Approximation by Gamma type operators," Math Meth Appl Sci., 43, pp. 2772–2782.

[3] DeVore, R.A., and Lorentz, G.G., 1993, "Constructive Approximation," Springer, Berlin.

[4] Ditzian, Z., 1979, "On global inverse theorem of Szasz and Baskakov operators," Can. J. Math. 31(2), pp. 255–263.

[5] Gupta, V., and Maheshwari, P., (2019), "Approximation with certain Post-Widder operators," Publications De L'institut Mathematique, 105(119), pp. 131-136.

[6] Holhos, A., 2010, "The rate of approximation of functions in an infinite interval by positive linear operators," Stud. Univ. Babes-Bolyai Math., 2, pp. 133–142.

[7] Karsli, H., 2011, "On convergence of general Gamma type operators," Anal. Theory Appl., 27(3), pp. 288-300.

[8] Maheshwari, P., 2006, "An inverse result in simultaneous approximation of Modified Beta operators," Georgian Math. J. 13(2), pp. 297-306.

[9] Mihesan, V., 2008, "Gamma approximating operators," Creat. Math. Inf. 17, pp. 466–472.

[10] Paltanea, R., 2011, "Estimates of approximation in terms of a weighted modulus of continuity," Bull. Transilv. Univ. Brasov, 4(53), pp. 67–74.

[11] Prakash, O., Sharma, D., and Maheshwari P.,2012, "On the Micchelli combinations of Modified Beta operators," Proc. Jangjeon Math. Soc., 15(4), pp. 437-446.

[12] Rathore, R.K.S., 1973, "Linear combinations of linear positive operators and generating relations on special functions," Ph. D. Thesis, Delhi.

[13] Sharma, P., and Sharma, D., 2023, "Some Statistical Approximation based on Post-Widder operators," J. of Advances in Mathematics, 22, pp.23-29

[14] Sharma, P. M., and Abid, M., 2020, "Approximation by (p,q) Szász-Beta–Stancu operators," Arab. J. Math., 9, pp. 191–200.

[15] Sharma, P. M., 2018, "Approximation properties of certain q-genuine Szasz operators," Complex Analysis and Operator theory, 12(7), pp. 27-36.

[16] Sharma, P., 2023, "Some Aspects of Approximation Theory Based on Various Positive Linear Operators," Eliva Press: Chişinău, Moldova.

[17] Maheshwari, P. M., and Sharma, R., 2016, "Some approximation properties of q-MBS operators," Advances in Theor. And Appl. Math., 11(1), pp. 69-77.

[18] Tachev, G., Gupta, V., and Aral A., 2020, "Voronovskaja's theorem for functions with exponential growth," Georgian Math. J., 27(3), pp. 459-468.

[19] Tachev, G., and Gupta, V., 2016, "General form of Voronovskaja's theorem in terms of weighted modulus of continuity," Results Math. 69(3–4), pp. 419–430.

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