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A New Method for solving Extreme Point Linear Fractional Programming Problems

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Abstract

This paper deals with the problem of finding the optimal value of the ratio of two linear functions subject to some linear restrictions. It is assumed that the feasible region of the solution is a convex polyhedron and the optimal value exists on an extreme point of the convex polyhedron. The basic idea of the proposed method is the search of an optimal solution by moving from one extreme point to the other of the feasible region until the optimal solution is reached. All the extreme points are first obtained by plotting the graphs of the corresponding linear inequalities and then, only those are considered which are feasible. At each feasible extreme point thus obtained, the value of the objective function is calculated, and the one, at which the objective function has the optimal value, is the desired solution. Results obtained by the developed method are compared with the results obtained by other methods with the help of illustrative examples.

Keywords: Linear Fractional Programming, Extreme Point, Optimal Solution

Mathematics Subject Classification: 90C05, 90C32

1. INTRODUCTION

An extreme point linear fractional programming (EPLFP) problem is the problem of maximization or minimization of a fraction of two linear functions subject to some linear restrictions. Because of its wide range of applications in real life, LFP is of considerable research and interest. It is used to achieve the highest ratio of outcome to cost, the highest ratio of net profit to capital invested, i.e., the ratio representing the highest efficiency.

An EPLFP problem can be written mathematically as follows :

$$\left.\begin{aligned} &\max z = \frac{c^T x + \alpha}{d^T x + \beta} \\ &\text{subject to} \\ &\quad Ax \le b \\ &\text{and } x \text{ is an extreme point of} \\ &\quad Px \le q \\ &\quad x \ge 0 \end{aligned}\right\} \qquad (1)$$

where x is an $(n \times 1)$ vector of decision variables; c, d are $(n \times 1)$ vectors of constants, A is an $(m \times n)$ matrix, b is an $(m \times 1)$ vector of constants, P is an $(p \times n)$ matrix, q is an $(p \times 1)$ vector of constants and α, β are real numbers and the denominator is supposed to be positive for all feasible solutions.

2. LITERATURE REVIEW

A number of methods have been developed for finding solution of linear fractional programming problems. Charnes and Cooper [1] transformed the linear fractional programming (LFP) problem into a linear programming (LP) problem and they derived the solution of the original LFP problem by solving the resulting LP problem. Martos [2] used the fundamental idea of simplex method, in which the optimal solution is found by moving from one extreme point to the other, till the criterion of optimality is satisfied. Kirby, Love and Swarup [3] solved such problems by an enumeration method. They also proposed a method [4] for solving such problems by introducing cuts and generating alternate solutions. Puri and Swarup developed an enumerative method [5].They also proposed a strong cut enumerative procedure [6] and a strong cut cutting place procedure [7]. Garg and Swarup [8] proposed a method for EPLFP problem using an additional complementarity condition. Bansal and Bakshi [9] established relation between primal and dual problems and they used it to propose a method for solving extreme point mathematical programming problem. Hossain, Arefin and Islam [10] used simplex method for solving extreme point linear programming and linear fractional programming problems. Murty[11] and Puri and Swarup[12]solved the Fixed Charge Problem using an extreme point method. Bitran and Novae [13] converted the LFP problem into a sequence of LP problems and solved these LP problems until two of them give identical solutions. Tantawy[14], [15] solved LFP problem using a feasible direction method and a duality method. Tantawy and Sallam [16] solved integer LFP problems using conjugate gradient projection method. Jagannathan [17] derived the condition for a LFP problem to be unbounded and asymptotic by studying few properties of programming problems in parametric form.

All the methods mentioned above involve a large number of computational steps and are time consuming. In the present paper, a very easy method to solve EPLFP problem is developed, which is similar to the graphical method of solving LP problem.

3. THEORETICAL DEVELOPMENT

The graphical method has been used to solve LP problem since long. But it has not been used to solve ELP problem so far. The following theorem proves that an EPLFP problem attains its maximum (or minimum) value at the extreme points only, if the feasible region is a convex polyhedron. This leads to the conclusion that the graphical method can be used to find the optimal solution of an EPLFP problem.

Theorem. If the feasible region of an EPLFP problem is a convex polyhedron, then there exists an optimal solution to the EPLFP problem and the objective function attains the optimal value at one of the vertices of the convex polyhedron.

Proof. Let the EPLFP problem be to determine x so as to maximize

$$z = \frac{c^T x + \alpha}{d^T x + \beta} \quad \text{subject to} \quad Ax \le b \ \text{ and } \ x \ge 0,$$

where $c, d, x \in R^n$, $b \in R^m$, A is an $(m \times n)$ matrix and α, β are scalars and the denominator is supposed to be positive for all feasible solutions.

The feasible region of the EPLFP problem is given by

$$S = \{x \in R^n : Ax \le b \ \text{ and } x \ge 0\}$$

Since S is a convex polyhedron, it is non-empty, closed and bounded. Thus the objective function z is continuous on a non-empty, closed and bounded set S, therefore z attains its maximum at S, i.e., the optimal solution exists.

Now, since S is a convex polyhedron, it has finite number of extreme points. Let these be $x^{(1)} = \left(x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}\right)$, $x^{(2)} = \left(x_1^{(2)}, x_2^{(2)}, \ldots, x_n^{(2)}\right), \ldots, x^{(m)} = \left(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}\right) \in R^n$.

Therefore, any $x = (x_1, x_2, \ldots, x_n) \in S$ can be written in the form of a convex combination of the extreme points, say,

$$x = \sum_{j=1}^{m} \lambda_j x^{(j)}, \quad \text{where } \lambda_1, \lambda_2, \ldots, \lambda_m \ge 0 \ \text{ and } \ \sum_{j=1}^{m} \lambda_j = 1.$$

Let $$z^{(O)} = \max\left\{\frac{c^T x^{(1)} + \alpha}{d^T x^{(1)} + \beta}, \frac{c^T x^{(2)} + \alpha}{d^T x^{(2)} + \beta}, \ldots, \frac{c^T x^{(m)} + \alpha}{d^T x^{(m)} + \beta}\right\} = \frac{c^T x^{(O)} + \alpha}{d^T x^{(O)} + \beta}$$

We shall prove that the value of z at any point x of S cannot exceed $z^{(O)}$.

Since $$\max\left\{\frac{c^T x^{(1)} + \alpha}{d^T x^{(1)} + \beta}, \frac{c^T x^{(2)} + \alpha}{d^T x^{(2)} + \beta}, \ldots, \frac{c^T x^{(m)} + \alpha}{d^T x^{(m)} + \beta}\right\} = \frac{c^T x^{(O)} + \alpha}{d^T x^{(O)} + \beta},$$ therefore

$$\frac{c^T x^{(j)}+\alpha}{d^T x^{(j)}+\beta} \le \frac{c^T x^{(O)}+\alpha}{d^T x^{(O)}+\beta} \quad \forall\ j=1,2,\ldots,m.$$

i.e.,

$$\frac{c_1x_1^{(j)}+c_2x_2^{(j)}+\ldots+c_nx_n^{(j)}+\alpha}{d_1x_1^{(j)}+d_2x_2^{(j)}+\ldots+d_nx_n^{(j)}+\beta} \le \frac{c_1x_1^{(O)}+c_2x_2^{(O)}+\ldots+c_nx_n^{(O)}+\alpha}{d_1x_1^{(O)}+d_2x_2^{(O)}+\ldots+d_nx_n^{(O)}+\beta} \quad \forall\ j=1,2,\ldots,m.$$

This implies that $\forall\ j=1,2,\ldots,m$ and $\forall\ i=1,2,\ldots,n$, we have

$$\frac{\left(\sum_{i=1}^{n} c_i x_i^{(j)}\right)+\alpha}{\left(\sum_{i=1}^{n} d_i x_i^{(j)}\right)+\beta} \le \frac{\left(\sum_{i=1}^{n} c_i x_i^{(O)}\right)+\alpha}{\left(\sum_{i=1}^{n} d_i x_i^{(O)}\right)+\beta}$$

i.e., we have

$$\left(\sum_{i=1}^{n} c_i x_i^{(j)}\right)+\alpha \le \frac{\left(\sum_{i=1}^{n} c_i x_i^{(O)}\right)+\alpha}{\left(\sum_{i=1}^{n} d_i x_i^{(O)}\right)+\beta}\left[\left(\sum_{i=1}^{n} d_i x_i^{(j)}\right)+\beta\right] \qquad (2)$$

$\forall\ j=1,2,\ldots,m$ and $\forall\ i=1,2,\ldots,n.$

Now, the objective function z at any point $x=(x_1,x_2,\ldots,x_n)$ of S is given by

$$z=\frac{c^T x+\alpha}{d^T x+\beta}=\frac{c_1(\sum_{j=1}^{m}\lambda_j x^{(j)})+c_2(\sum_{j=1}^{m}\lambda_j x^{(j)})+\ldots+c_n(\sum_{j=1}^{m}\lambda_j x^{(j)})+\alpha\sum_{j=1}^{m}\lambda_j}{d_1(\sum_{j=1}^{m}\lambda_j x^{(j)})+d_2(\sum_{j=1}^{m}\lambda_j x^{(j)})+\ldots+d_n(\sum_{j=1}^{m}\lambda_j x^{(j)})+\beta\sum_{j=1}^{m}\lambda_j}$$

$$=\frac{\lambda_1\left[\left(\sum_{i=1}^{n} c_i x_i^{(1)}\right)+\alpha\right]+\lambda_2\left[\left(\sum_{i=1}^{n} c_i x_i^{(2)}\right)+\alpha\right]+\ldots+\lambda_m\left[\left(\sum_{i=1}^{n} c_i x_i^{(m)}\right)+\alpha\right]}{\lambda_1\left[\left(\sum_{i=1}^{n} d_i x_i^{(1)}\right)+\beta\right]+\lambda_2\left[\left(\sum_{i=1}^{n} d_i x_i^{(2)}\right)+\beta\right]+\ldots+\lambda_m\left[\left(\sum_{i=1}^{n} d_i x_i^{(m)}\right)+\beta\right]}$$

$$\le\frac{\lambda_1\frac{\left(\sum_{i=1}^{n} c_i x_i^{(O)}\right)+\alpha}{\left(\sum_{i=1}^{n} d_i x_i^{(O)}\right)+\beta}\left[\left(\sum_{i=1}^{n} d_i x_i^{(1)}\right)+\beta\right]+\lambda_2\frac{\left(\sum_{i=1}^{n} c_i x_i^{(O)}\right)+\alpha}{\left(\sum_{i=1}^{n} d_i x_i^{(O)}\right)+\beta}\left[\left(\sum_{i=1}^{n} d_i x_i^{(2)}\right)+\beta\right]+\ldots+\lambda_m\frac{\left(\sum_{i=1}^{n} c_i x_i^{(O)}\right)+\alpha}{\left(\sum_{i=1}^{n} d_i x_i^{(O)}\right)+\beta}\left[\left(\sum_{i=1}^{n} d_i x_i^{(m)}\right)+\beta\right]}{\lambda_1\left[\left(\sum_{i=1}^{n} d_i x_i^{(1)}\right)+\beta\right]+\lambda_2\left[\left(\sum_{i=1}^{n} d_i x_i^{(2)}\right)+\beta\right]+\ldots+\lambda_m\left[\left(\sum_{i=1}^{n} d_i x_i^{(m)}\right)+\beta\right]}$$

[using equation (2)]

$$= \frac{\left(\sum_{i=1}^{n} c_i x_i^{(O)}\right) + \alpha}{\left(\sum_{i=1}^{n} d_i x_i^{(O)}\right) + \beta} \cdot \frac{\lambda_1\left[\left(\sum_{i=1}^{n} d_i x_i^{(1)}\right) + \beta\right] + \lambda_2\left[\left(\sum_{i=1}^{n} d_i x_i^{(2)}\right) + \beta\right] + \ldots + \lambda_m\left[\left(\sum_{i=1}^{n} d_i x_i^{(m)}\right) + \beta\right]}{\lambda_1\left[\left(\sum_{i=1}^{n} d_i x_i^{(1)}\right) + \beta\right] + \lambda_2\left[\left(\sum_{i=1}^{n} d_i x_i^{(2)}\right) + \beta\right] + \ldots + \lambda_m\left[\left(\sum_{i=1}^{n} d_i x_i^{(m)}\right) + \beta\right]}$$

$$= \frac{\left(\sum_{i=1}^{n} c_i x_i^{(O)}\right) + \alpha}{\left(\sum_{i=1}^{n} d_i x_i^{(O)}\right) + \beta}.$$

$= z^{(O)}$

i.e., $z \le z^{(O)}$

Thus the objective function z at any point x of S cannot have its value greater than $z^{(O)}$. Therefore the maximum (or minimum) value of the objective function cannot be achieved at the interior points, i.e., the objective function attains its maximum (or minimum) value at one of the extreme points.

4. PROPOSED METHOD

To find the optimal solution of the EPLFP problem (1), all the extreme points are first obtained by plotting the graphs of the corresponding linear inequalities $Px \le q$, $x \ge 0$. Then we consider only those extreme points which are feasible, i.e., which satisfy the given constraints $Ax \le b$ also. At each feasible extreme point thus obtained, the value of the objective function is calculated, and the one, at which the objective function has the optimal value, is the desired optimal solution of EPLFP problem (1).

5. ALGORITHM FOR THE PROPOSED METHOD

Step1. Find all the extreme points by plotting the graphs of the corresponding linear inequalities $Px \le q$, $x \ge 0$.

Step2. Check the feasibility of the extreme points obtained in Step 1.

Step3. Find the value of the objective function at each feasible extreme point obtained in Step 2.

The extreme point, at which the objective function has the optimal value, is the desired optimal solution of the given EPLFP problem (1).

6. NUMERICAL EXAMPLES

Example 1. Max $z = \frac{2x_1 + x_2}{4x_1 + x_2 + 1}$ subject to $-2x_1 + x_2 \le 1$, $2x_1 + 5x_2 \le 23$, $2x_1 + x_2 \le 15$

and (x_1, x_2) is an extreme point of

$-3x_1 + 2x_2 \le 4, \quad x_1 + 4x_2 \le 22, \quad 5x_1 + 4x_2 \le 46, \quad x_1 - 2x_2 \le 5, \quad x_1, x_2 \ge 0.$

Solution : If the linear inequalities representing the extreme points are plotted on the graph taking them as equations, then the area bounded by all these inequalities is as shown in Figure 1,by the shaded area OABCDEO.

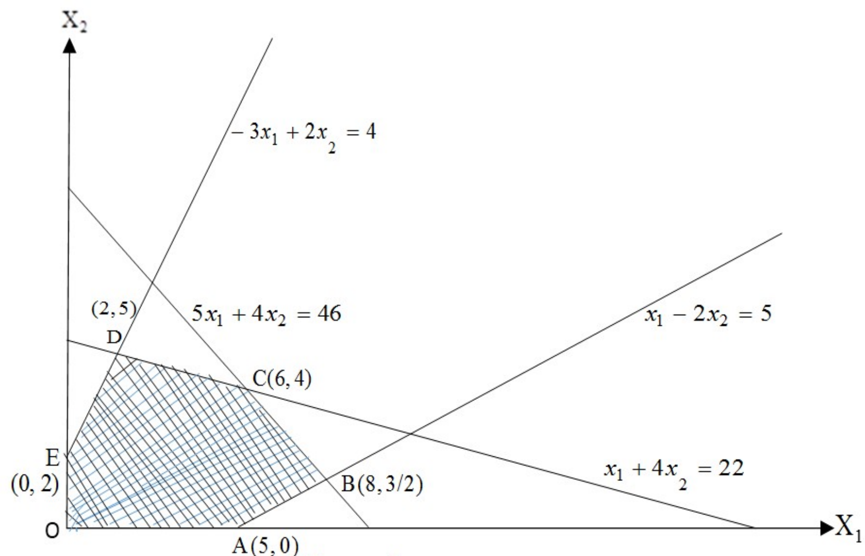

Figure 1

The coordinates of the extreme points are :

O = (0, 0), A = (5, 0), B = (8, 3/2), C = (6, 4), D = (2, 5), E = (0, 2).

Now we check whether the above extreme points are feasible or not.

Extreme points (x_1, x_2)	Constraints $-2x_1 + x_2 \le 1,\ 2x_1 + 5x_2 \le 23,\ 2x_1 + x_2 \le 15$	Feasible/ Infeasible	Value of z
O (0, 0)	$-2(0)+0 \le 1,\ 2(0)+5(0) \le 23,\ 2(0)+0 \le 15$	Feasible	0
A(5, 0)	$-2(5)+0 \le 1,\ 2(5)+5(0) \le 23,\ 2(5)+0 \le 15$	Feasible	10/21
B(8,3/2)	$-2(8)+\frac{3}{2} \le 1,\ 2(8)+5\left(\frac{3}{2}\right) > 23,\ 2(8)+\frac{3}{2} > 15$	Infeasible	
C (6, 4)	$-2(6)+4 \le 1,\ 2(6)+5(4) > 23,\ 2(6)+4 > 15$	Infeasible	
D (2, 5)	$-2(2)+5 \le 1,\ 2(2)+5(5) > 23,\ 2(2)+5 \le 15$	Infeasible	
E (0, 2)	$-2(0)+2 > 1,\ 2(0)+5(2) \le 23,\ 2(0)+2 \le 15$	Infeasible	

Table 1

Clearly, the objective function z attains its optimal value at the point A (5, 0) and $\max z = 10/21 = 0.47619$.

Example 2. Max $z = \dfrac{5x_1 + 2x_2}{x_1 + 8x_2 + 1}$ subject to $x_1 + 2x_2 \le 5,\quad 2x_1 + x_2 \le 8$

and (x_1, x_2) is an extreme point of

$x_1 + x_2 \le 4, \qquad x_1 - x_2 \le 2, \qquad x_1, x_2 \ge 0.$

Solution : If the linear inequalities representing the extreme points are plotted on the graph taking them as equations, then the area bounded by all these inequalities is as shown in Figure 2, by the shaded area OABCO.

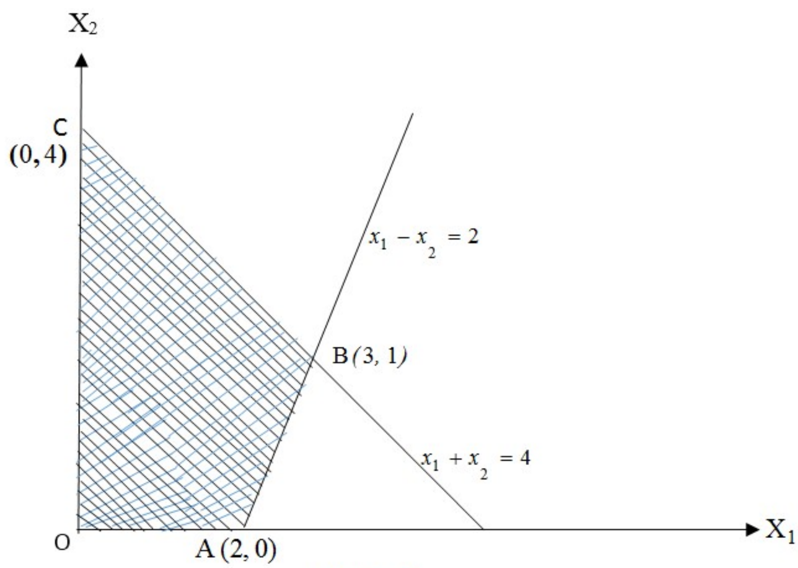

Figure 2

The coordinates of the extreme points are :

O = (0, 0), A = (2, 0), B = (3, 1), C = (0, 4).

Now we check whether the above extreme points are feasible or not.

Extreme points (x_1, x_2)	Constraint $x_1 + 2x_2 \le 5, 2x_1 + x_2 \le 8$	Feasible/ Infeasible	Value of z
O (0, 0)	$0 + 2(0) \le 5, 2(0) + 0 \le 8$	Feasible	0
A (2, 0)	$2 + 2(0) \le 5,\ 2(2) + 0 \le 8$	Feasible	10/3
B (3, 1)	$3 + 2(1) \le 5, 2(3) + 1 \le 8$	Feasible	17/12
C (0, 4)	$0 + 2(4) > 5,\ 2(0) + 4 \le 8$	Infeasible	

Table 2

Clearly, the objective function z attains its optimal value at the point A (2, 0) and max $z = 10/3 = 3.3333$.

Example 3. Max $z = \frac{2x_1 + 3x_2}{x_1 + x_2 + 7}$ subject to $3x_1 + 2x_2 \le 6$

and (x_1, x_2) is an extreme point of

$3x_1 + 5x_2 \le 15,\quad 4x_1 + 3x_2 \le 12,\quad x_1, x_2 \ge 0.$

Solution : If the linear inequalities representing the extreme points are plotted on the graph taking them as equations, then the area bounded by all these inequalities is as shown in Figure 3,by the shaded area OABCO.

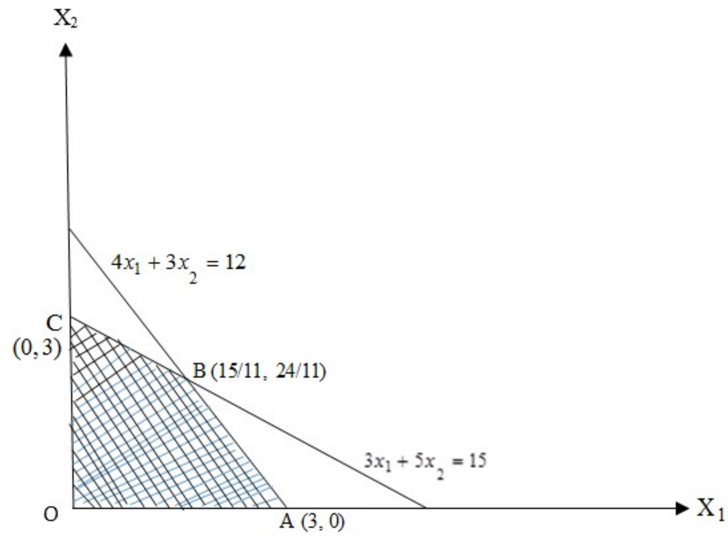

Figure 3

The coordinates of the extreme points are:

O = (0, 0), A = (3, 0), B = (15/11, 24/11), C = (0, 3).

Now we check whether the above extreme points are feasible or not.

Extreme points (x_1, x_2)	Constraint $3x_1 + 2x_2 \le 6$	Feasible/Infeasible	Value of z
O (0, 0)	$3(0) + 2(0) \le 6$	Feasible	0
A (3, 0)	$3(3) + 2(0) > 6$	Infeasible	
B (15/11, 24/11)	$3\left(\frac{15}{11}\right) + 2\left(\frac{24}{11}\right) > 6$	Infeasible	
C (0, 3)	$3(0) + 2(3) \le 6$	Feasible	9/10

Table 3

Clearly, the objective function z attains its optimal value at the point C (0,3) and max $z = 9/10 = 0.9$.

7. COMPARISON OF THE NUMERICAL RESULTS

The following table shows the comparison between the proposed method and other optimization methods:

Example	Reference	Optimal solution	Optimal value
Ex.1	Proposed	(5, 0)	0.47619
	Ref.[4]	(5, 0)	0.47619
	Ref.[7]	(5, 0)	0.47619
	Ref.[8]	(5, 0)	0.47619
Ex.2	Proposed	(2, 0)	3.3333
	Ref.[4]	(2, 0)	3.3333
	Ref.[5]	(2, 0)	3.3333
Ex.3	Proposed	(0, 3)	0.9
	Ref.[5]	(0, 3)	0.9

Table 4: Comparison of the Numerical Results

It is obvious that the results obtained by the proposed method are the same as those obtained by other methods for all the examples, which proves the validity of the proposed method. If we compare the computational steps of the proposed method with any existing method, it can be seen that there is a lot of difference between the two. In the proposed method, there is no clumsiness of computation like other methods. This helps to save our time.

8. CONCLUSION

The optimization technique proposed in this article finds the solution of linear fractional programming problem by using graphical method, which is very easy to understand and apply. The computational steps of the method are very easy in comparison to the other methods, which saves our time. Comparison of the proposed method with other methods for numerical examples shows that the proposed method gives identical results with those obtained by the other methods. This proves the validity of the method.

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