On Computing the Inverse of Vandermonde Matrix

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Abstract

We present a novel simple approach for computing the inverse of Vandermonde matrix via synthetic divisions. This approach does not require matrix multiplication, computing determinant or solving a system of linear equations for determining the entries of the inverse of the given Vandermonde matrix. Some numerical examples are provided.

Keywords: Vandermonde matrix, matrix inverse, synthetic divisions, polynomial interpolation.

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1. INTRODUCTION

Vandermonde matrix (VDM) has important applications in areas like polynomial interpolation, curve fitting, signal processing, statistics, coding theory and control theory [1, 3, 4, 5, 7, 9]. The study of efficient approach for computing the inverse of VDM is still an important and interesting topic in various mathematics and engineering disciplines. In this paper, we present a simple and efficient approach for computing the inverse of VDM via synthetic divisions. This approach is suitable for either hand or machine computation.

2. MAIN RESULTS

For a set of distinct numbers $\lambda_i$, a Vandermonde matrix $V$ is defined as

$$V = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\lambda_1 & \lambda_2 & \cdots & \lambda_n \\
\lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1}
\end{pmatrix}.$$
Since $\lambda_i$ are distinct, so the determinant of $V$ is non-zero and hence $V^{-1}$ exists. According to [10], there is a simple useful formula for computation of $V^{-1}$.

**Theorem 2.1** The inverse of $V$ can be computed by applying the formula $V^{-1} = W \times A$, where $W$ and $A$ are square matrices defined by

$$W = \begin{pmatrix} \prod_{j=1}^{n} (\lambda_i - \lambda_j) & \prod_{j=1}^{n} (\lambda_i - \lambda_j) & \cdots & \prod_{j=1}^{n} (\lambda_i - \lambda_j) \\ \prod_{j=2}^{n} (\lambda_i - \lambda_j) & \prod_{j=2}^{n} (\lambda_i - \lambda_j) & \cdots & \prod_{j=2}^{n} (\lambda_i - \lambda_j) \\ \vdots & \vdots & \ddots & \vdots \\ \prod_{j=n}^{n} (\lambda_i - \lambda_j) & \prod_{j=n}^{n} (\lambda_i - \lambda_j) & \cdots & \prod_{j=n}^{n} (\lambda_i - \lambda_j) \end{pmatrix}; \quad A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \lambda_1 & 1 & 0 & \cdots & 0 \\ a_2 & a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & 1 \end{pmatrix}.$$ 

and $a_1 = -\sum_{j=1}^{n} \lambda_j$, $a_2 = \sum_{j=2}^{n} \lambda_j \lambda_1$, $a_3 = -\sum_{j=3}^{n} \lambda_j \lambda_1 \lambda_2$, ..., $a_n = (-1)^n \prod_{j=1}^{n} \lambda_j$.

By Theorem 2.1, we can easily derive the closed form of $V^{-1}$ below.

**Corollary 2.2** The closed form of $V^{-1}$ is given by

$$V^{-1} = \begin{pmatrix} \lambda_1^{n-1} + a_1 \lambda_2^{n-2} + \cdots + a_{n-1} \lambda_2 + a_{n-1} & \lambda_1^{2} + a_1 \lambda_1 + a_2 & \lambda_1 + a_1 & 1 \\ \lambda_1^{n-1} + a_1 \lambda_2^{n-2} + \cdots + a_{n-1} \lambda_2 + a_{n-1} & \lambda_1^{2} + a_1 \lambda_1 + a_2 & \lambda_1 + a_1 & 1 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} + a_1 \lambda_2^{n-2} + \cdots + a_{n-1} \lambda_2 + a_{n-1} & \lambda_1^{2} + a_1 \lambda_1 + a_2 & \lambda_1 + a_1 & 1 \end{pmatrix}.$$ 

Define $f(x) = \prod_{i=1}^{n} (x - \lambda_i) = x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$ and $f_i(x) = \prod_{j=1}^{i} (x - \lambda_j)$, where $1 \leq i \leq n$. 
Theorem 2.3

\[ f_i(x) = x^{n-1} + (\lambda_i + a_1)x^{n-2} + (\lambda_i^2 + a_1\lambda_i + a_2)x^{n-3} + \cdots + (\lambda_i^{n-1} + a_1\lambda_i^{n-2} + \cdots + a_{n-2}\lambda_i + a_{n-1}) \]

Proof. Consider

\[ (x - \lambda_i)[x^{n-1} + (\lambda_i + a_1)x^{n-2} + (\lambda_i^2 + a_1\lambda_i + a_2)x^{n-3} + \cdots + (\lambda_i^{n-1} + a_1\lambda_i^{n-2} + \cdots + a_{n-1})] \]

By expansion and then collecting the like powers of \( x \), we obtain \( x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \), which is equal to \( f(x) \). Since \( f_i(x) = f(x)/(x - \lambda_i) \), so the proof is completed.

According to Theorem 2.3, we can see that the numerators of the entries in each row of \( V^{-1} \) are exactly the same as the coefficients of \( f_i(x) \), except in the reverse order. Thus, we can apply synthetic divisions to \( f(x)/(x - \lambda_i) \) to compute the numerators of the entries of \( V^{-1} \) row by row. Also, since \( f_i(\lambda_i) = \prod_{j \neq i}(\lambda_i - \lambda_j) \), we can apply synthetic division again to \( f_i(x)/(x - \lambda_i) \) to compute the common denominator of the entries of \( V^{-1} \) in each row, namely \( \prod_{j \neq i}(\lambda_i - \lambda_j) \), where \( 1 \leq i \leq n \). In other words, each row of \( V^{-1} \) can be computed efficiently by two successive synthetic divisions. Some numerical examples are provided in the section below.

3. EXAMPLES

Example 3.1 Find the inverse of the following Vandermonde matrix.

\[ V = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix} \]

Solution. Define \( f(x) = (x + 1)(x - 2)(x - 3) = x^3 - 4x^2 + x + 6 \). By using synthetic divisions, we obtain

\[
\begin{array}{c|ccc}
-1 & 1 & -4 & 1 \\
 & & -1 & 5 \\
1 & -5 & 6 \\
 & & -1 & 6 \\
1 & -6 & 12 \\
\end{array}
\]

So, the first row of \( V^{-1} \) is equal to \((1/2, -5/12, 1/12)\).
Similarly,

\[
\begin{array}{ccc}
2 & 1 & -4 & 1 \\
 & 2 & -4 \\
1 & -2 & -3 \\
 & 2 & 0 \\
1 & 0 & -3
\end{array}
\]

Thus, the second row of \( V^{-1} \) is equal to (1 2/3 -1/3).

Also,

\[
\begin{array}{ccc}
3 & 1 & -4 & 1 \\
 & 3 & -3 \\
1 & -1 & -2 \\
 & 3 & 6 \\
1 & 2 & 4
\end{array}
\]

So, the third row of \( V^{-1} \) is equal to (-1/2 -1/4 1/4).

Hence,

\[
V^{-1} = \begin{pmatrix}
1/2 & -5/12 & 1/12 \\
1 & 2/3 & -1/3 \\
-1/2 & -1/4 & 1/4
\end{pmatrix}.
\]

**Example 3.2** Find a cubic interpolation polynomial through the points (0, 3), (1, 2), (2, 7) and (4, 59).

Solution. Let the interpolation polynomial be \( f(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3 \), where \( a_i \) are unknown coefficients to be determined. Since \( f(0) = 3, f(1) = 2, f(2) = 7, f(4) = 59 \), we can use a matrix equation to represent the data as follows:

\[
\begin{pmatrix}
3 \\
2 \\
7 \\
59
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
1 & 2 & 4 & 8 \\
1 & 4 & 16 & 64
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4
\end{pmatrix}.
\]
Now, let us consider the following Vandermonde matrix

\[
V = \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 2 & 4 \\
0 & 1 & 4 & 16 \\
0 & 1 & 8 & 64
\end{pmatrix}.
\]

Define \( h(x) = x(x - 1)(x - 2)(x - 4) = x^4 - 7x^3 + 14x^2 - 8x \).

By using synthetic divisions, we obtain

\[
\begin{array}{cccc}
0 & 1 & -7 & 14 & -8 \\
& 0 & 0 & 0 & 0 \\
1 & -7 & 14 & -8 & 0 \\
& 0 & 0 & 0 & 0 \\
1 & -7 & 14 & -8 & 0
\end{array}
\]

So, the first row of \( V^{-1} \) is equal to \( (1 \ -7/4 \ 7/8 \ -1/8) \).

Next,

\[
\begin{array}{cccc}
1 & 1 & -7 & 14 & -8 \\
& 1 & -6 & 8 & 0 \\
1 & -6 & 8 & 0 & 0 \\
& 1 & -5 & 3 & 0 \\
1 & -5 & 3 & 0 & 0
\end{array}
\]

So, the second row of \( V^{-1} \) is equal to \( (0 \ 8/3 \ -2 \ 1/3) \).

Next,

\[
\begin{array}{cccc}
2 & 1 & -7 & 14 & -8 \\
& 2 & -10 & 8 & 0 \\
1 & -5 & 4 & 0 & 0 \\
& 2 & -6 & -4 & 0 \\
1 & -3 & -2 & -4 & 0
\end{array}
\]

Thus, the third row of \( V^{-1} \) is equal to \( (0 \ -1 \ 5/4 \ -1/4) \).
Also,

\[
\begin{array}{c|cccc}
4 & 1 & -7 & 14 & -8 \\
   & 4 & -12 & 8  & \\
\hline
   & 1 & -3  & 2  & 0  \\
   & 4 & 4   & 24 & \\
\hline
   & 1 & 1   & 6  & 24 \\
\end{array}
\]

So, the last row of \( V^{-1} \) is equal to (0 1/12 -1/8 1/24).

Hence,

\[
V^{-1} = \begin{pmatrix}
1 & -7/4 & 7/8 & -1/8 \\
0 & 8/3 & -2 & 1/3 \\
0 & -1 & 5/4 & -1/4 \\
0 & 1/12 & -1/8 & 1/24 \\
\end{pmatrix}
\]

Therefore,

\[
\begin{pmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4 \\
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
-7/4 & 8/3 & -1 & 1/12 \\
7/8 & -2 & 5/4 & -1/8 \\
-1/8 & 1/3 & -1/4 & 1/24 \\
\end{pmatrix} \begin{pmatrix}
3 \\
2 \\
7 \\
59 \\
\end{pmatrix} = \begin{pmatrix}
3 \\
-2 \\
0 \\
1 \\
\end{pmatrix}
\]

Hence, the required cubic interpolation polynomial is \( f(x) = 3 - 2x + x^3 \).

4. CONCLUDING REMARKS

We have presented a simple and efficient method for computing the inverse of VDM via synthetic divisions, based on the recent related works of the author [6, 8, 10]. This method does not require matrix multiplication, computation of determinant or solving a system of linear equations. Since the complexity of synthetic divisions is the same as applying the Horner’s rule [2] for handling nested multiplications of polynomials, so the computational cost is comparatively less than that by applying direct matrix multiplication to W×A for computing the inverse of VDM. Thus, this method is suitable for either hand or machine calculation.

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REFERENCES


