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Sinc collocation solution for integral algebraic equations of index-1

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Abstract

In this paper, we consider a collocation procedure based on sinc method to solve Integral Algebraic Equations (IAEs) of index-1. The global convergence analysis is given to guarantee the efficiency of the method. Numerical examples are presented to illustrate the validity and applicability of the method.

AMS subject classification:
Keywords: Volterra integral equations, integral algebriac equations, sinc collocation method, index of IAEs.

1. Introduction

Mixed systems of the first and second kind Volterra Integral Equations (VIEs) have been discussed by many authors because of their pivotal effect in various mathematical modeling processes, e.g. identification problem for time-dependent memory kernels in linear heat conduction and viscoelasticity (see [1] and [2]) and evolution of a chemical reaction within a small cell [3].

Many techniques have been adopted to solve IAEs numerically. Some of these techniques are based on polynomials approximation, e.g. the polynomial spline collocation method has been used by Kauthen [4] to solve (IAEs) of index-1. Shiri [5] implemented the direct and indirect Galerkin method while Katatbeh [6] used Discretized collocation method to solve the same system. Moreover, the Jacobi spectral method has been used in [7], [8] and [9] to solve IAEs of index-1, index-2 and of singular kernel, respectively.

The sinc collocation scheme has been investigated by several authors, e.g. Malknejed [10] applied the scheme for Fredholm integral equation of the first kind and investigated the applicability of that scheme; on the other hand, Rashidinia [11] used the scheme to present a numerical solution of a Volterra integral equation. Furthermore, Shamloo [12] gave a numerical solution based on sinc function for a Fredholm-Volterra equation.

In fact, Sinc methods can be applied for problems with singularities, and problems over infinite or semi-infinite ranges. For more information about the differences between sinc and polynomial approximations, refer to [13].

The main purpose of the present paper is to develop the sinc method to solve the (IAEs) of index-1. More precisely, we consider

$$y(x) = f(x) + \int_0^x k_{11}(x,t)y(t)dt + \int_0^x k_{12}(x,t)z(t)dt \tag{1}$$

$$0 = g(x) + \int_0^x k_{21}(x,t)y(t)dt + \int_0^x k_{22}(x,t)z(t)dt \tag{2}$$

Here, we assume that the functions f, g and $k_{i,j}(i, j = 1, 2)$ are sufficiently smooth for $x \in \Gamma = [a, b]$. In addition, if $g(0) = 0$ and $|k_{2,2}(t, t)| > 0$, then the Volterra integral equation of the first kind in (1) can be converted to be of the second kind.

The system of equations (1) is a special case of the integral algebraic equations

$$a(y) = a(y_0) + \int_0^t k(t, s, y)ds \tag{3}$$

which has a unique solution if $\|a_y^{-1}\|_2$ and k are smooth functions. For more details, refer to [14].

The properties of (IAEs) are found to be very similar to those of Differential Algebraic Equations (DAEs), e.g. the solution of (IAEs) is related to its index where the index m is defined to be the minimum number of derivatives of the system required to solve for the unknown in the form a standard VIE with a second kind for each unknown function. Provided that $\left\| \frac{\partial a_i^{-1}}{\partial y} \right\|$ is singular for $i = 1, 2, \ldots, m-1$, and $\left\| \frac{\partial a_m^{-1}}{\partial y} \right\| < \infty$.

The outline of this paper follows. In Section 2 we apply Sinc collocation method to the (IAEs) of index-1. An error analysis is presented in Section 3. We conclude with a numerical illustration in Section 4.

2. The Sinc collocation scheme

This section is devoted to applying the sinc collocation method to numerically solve the IAEs system of index-1. The sinc function is defined by Borel [15] as

$$sinc(x) = \frac{sin(\pi x)}{\pi x}$$

For computational purposes, a more convenient Sinc function is used instead [16]

$$S(k, h) \circ (x) = Sinc(k, h)(x) = \frac{sin(\pi(x - kh)/h)}{\pi(x - kh)/h}$$

If $h > 0$ and k is an integer, then $Sinc(k, h)(x)$ is called the kth Sinc function with step size h evaluated at x.
In order to get further explanation about our procedure, we present the following definitions and theorems.

Definition 2.1. Given a function f defined and bounded for all x on $(-\infty, \infty)$, the cardinal function of the function f is defined by

$$C(f, h)(x) = \sum_{k \in Z} f(kh) S(k, h) \circ (x)$$

Definition 2.2. Let $h > 0$, and let $W(\pi/h)$ denote the family of all functions f that are analytic in $\mathbb{C}$, such that

$$\int_{\Re} |f(t)|^2 dt < \infty$$

and such that for all z in $\mathbb{C}$

$$|f(z)| \leq c e^{\pi |z|/h}$$

with c a positive constant.

Definition 2.3. Let $h > 0$, every f in $W(\pi/h)$ has the cardinal representation

$$f(x) \approx \sum_{k=-\infty}^{\infty} f(kh) S(k, h) \circ (x)$$

Definition 2.4. Let $d > 0$, and let D_d denote the domain

$$D_d = \{w \in \mathbb{C} : |Im(w)| < d\}$$

Definition 2.5. Let D be a simply-connected domain having boundary ∂D. Let a and b denote two distinct points of ∂D, and let ϕ denote a conformal map of D onto D_d, such that $\phi(a) = -\infty$ and $\phi(b) = \infty$.

Now, let $\psi = \phi^{-1}$ denote the inverse map, and let Γ be defined by

$$\Gamma = \{z \in \mathbb{C} : z = \psi(u), u \in \mathbb{R}\}$$

Also, if $h > 0$, then $z_k = z_k(h) = \psi(kh), k = 0, 1, 2, \ldots$

Definition 2.6. Let $L_\alpha(D_d)$ denote the family of all analytic functions F in D_d for which there exists a constant c such that

$$|F(z)| \leq c \frac{|e^{\phi(z)}|^\alpha}{(1 + |e^{\phi(z)}|)^{2\alpha}}$$

for all $z \in D_d$ and where $0 < \alpha < 1$.

Let us define $\delta_{kj}^{(-1)}$ to be

$$\delta_{kj}^{(-1)} = \frac{1}{2} + \int_0^{k-j} sinc(x)dx$$

Consider the conformal mapping

$$\phi(x) = ln\frac{x-a}{b-x}$$

which is a one to one function that maps the eye-shaped region $\left\{z \in \mathbb{Z} : \left|arg\left(\frac{z-a}{b-z}\right)\right| < d < \frac{\pi}{2}\right\}$ onto D_d. Therefore, the Sinc grid points x_k are defined by $\frac{a+be^{kh}}{1+e^{kh}}$, $k = 0, \mp 1, \mp 2, \ldots$.

The following are some useful theorems, the proofs of Theorems 1 and 2 can be found in [17] and [18] while the proofs of Theorems 3 and 4 can be found in [19].

Theorem 2.7. Let $y(x)$, $z(x) \in L_\alpha(D_d)$, then there exist positive constants c_0 and r_0 independent of N, such that

$$\begin{aligned} \sup_{x\in\Gamma}\left|y(x) - \sum_{j=-N}^{N} y(x_j)S(j,h)\circ\phi(x)\right| &< c_0 exp(-(\pi d\alpha N)^{1/2}) \\ \sup_{x\in\Gamma}\left|z(x) - \sum_{j=-N}^{N} z(x_j)S(j,h)\circ\phi(x)\right| &< c_0 exp(-(\pi d\alpha N)^{1/2}). \end{aligned} \tag{4}$$

where $h = \sqrt{\frac{\pi d}{\alpha N}}$.

Theorem 2.8. Let y/ϕ' and $z/\phi' \in L_\alpha(D_d)$, then there exist positive constants c_1 and r_1 independent of N, such that

$$\begin{aligned} \left|\int_a^{x_j} y(t)dt - \sum_{j=-N}^{N} h\delta_{kj}^{-1}\frac{y(x_j)}{\phi'(x_j)}\right| &< c_1 exp(-(\pi d\alpha N)^{1/2}) \\ \left|\int_a^{x_j} z(t)dt - \sum_{j=-N}^{N} h\delta_{kj}^{-1}\frac{z(x_j)}{\phi'(x_j)}\right| &< r_1 exp(-(\pi d\alpha N)^{1/2}) \end{aligned} \tag{5}$$

Now, the exact solution of the system of the equations (1) can be approximated by

$$y(x) = \sum_{j=-N}^{N} y(x_j)\gamma_j, z(x) = \sum_{j=-N}^{N} z(x_j)\gamma_j$$

where

$$\gamma_j = \begin{cases} \omega_a(x) = \dfrac{1}{1+e^{\phi(x)}}, \; j=-N \\ S(j,h)\circ\phi(x), \; j=-N+1, -N+2, \cdots, N-1 \\ \omega_b(x) = \dfrac{e^{\phi(x)}}{1+e^{\phi(x)}}, \; j=N \end{cases} \tag{6}$$

Theorem 2.9. Let $y(x)$, $z(x) \in L_\alpha(D_d)$, then there exist positive constants c_2 and r_2 independent of N, such that

$$\begin{aligned} \sup_{x\in\Gamma} |y(x) - \sum_{j=-N}^{N} y(x_j)\gamma_j| &\le c_2 N^{1/2} exp(-(\pi d\alpha N)^{1/2}) \\ \sup_{x\in\Gamma} |z(x) - \sum_{j=-N}^{N} z(x_j)\gamma_j| &\le r_2 N^{1/2} exp(-(\pi d\alpha N)^{1/2}). \end{aligned} \tag{7}$$

Theorem 2.10. If $\dfrac{k_{ij}}{\phi'}\gamma_j \in L_\alpha(D_d)$ and h is defined as in Theorem 1, then

$$\begin{aligned} \int_0^{x_k} k_{11}(x_k,t)y(t)dt &= hy(t_{-N}) \sum_{j=-N}^{N} \delta_{kj}^{(-1)} \frac{k_{11}(x_k,t_j)}{\phi'(t_j)}\omega_a(t_j) \\ &+h \sum_{j=-N+1}^{N-1} \delta_{kj}^{(-1)} \frac{k_{11}(x_k,t_j)}{\phi'(t_j)} y(tj) \\ &+hy(t_N) \sum_{j=-N}^{N} \delta_{kj}^{(-1)} \frac{k_{11}(x_k,t_j)}{\phi'(t_j)}\omega_b(tj) \\ &+O(exp(-(\pi dN\alpha)^{\frac{1}{2}})) \end{aligned}$$

$$\begin{aligned} \int_0^{x_k} k_{21}(x_k,t)y(t)dt &= hy(t_{-N}) \sum_{j=-N}^{N} \delta_{kj}^{(-1)} \frac{k_{21}(x_k,t_j)}{\phi'(t_j)}\omega_a(t_j) \\ &+h \sum_{j=-N+1}^{N-1} \delta_{kj}^{(-1)} \frac{k_{21}(x_k,t_j)}{\phi'(t_j)} y(tj) \\ &+hy(t_N) \sum_{j=-N}^{N} \delta_{kj}^{(-1)} \frac{k_{21}(x_k,t_j)}{\phi'(t_j)}\omega_b(tj) \\ &+O(exp(-(\pi dN\alpha)^{\frac{1}{2}})) \end{aligned}$$

$$\int_0^{x_k} k_{12}(x_k,t)z(t)dt = hz(t_{-N})\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{k_{12}(x_k,t_j)}{\phi'(t_j)}\omega_a(t_j) + h\sum_{j=-N+1}^{N-1}\delta_{kj}^{(-1)}\frac{k_{12}(x_k,t_j)}{\phi'(t_j)}z(tj) + hz(t_N)\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{k_{12}(x_k,t_j)}{\phi'(t_j)}\omega_b(tj) + O(exp(-(\pi dN\alpha)^{\frac{1}{2}}))$$

and

$$\int_0^{x_k} k_{22}(x_k,t)z(t)dt = hz(t_{-N})\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{k_{22}(x_k,t_j)}{\phi'(t_j)}\omega_a(t_j) + h\sum_{j=-N+1}^{N-1}\delta_{kj}^{(-1)}\frac{k_{22}(x_k,t_j)}{\phi'(t_j)}z(tj) + hz(t_N)\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{k_{22}(x_k,t_j)}{\phi'(t_j)}\omega_b(tj) + O(exp(-(\pi dN\alpha)^{\frac{1}{2}}))$$

Now, to find an approximation for the exact solution of the system of equations (1), we consider the sinc function interpolation of the unknown functions $y(x)$ and $z(x)$ as follows

$$y_N(x) = \sum_{j=-N}^{N} y_j\gamma_j \tag{8}$$

and

$$z_N(x) = \sum_{j=-N}^{N} z_j\gamma_j \tag{9}$$

where the unknowns y_j and z_j need to be determined. Firstly, if we substitute the function approximation (8) and (9) into the system (1) and replace x by the Sinc grid points $x_k, k = -N, \ldots, N$ we get for the first equation of (1)

$$
\begin{aligned}
f(x_k) =& y_{-N}\left(\omega_a(x_k) - h\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{k_{11}(x_k,t_j)}{\phi'(t_j)}\omega_a(t_j)\right) \\
&+\sum_{j=-N+1}^{N-1}\left(S(j,h)\circ\phi(x_k) - h\delta_{kj}^{(-1)}\frac{k_{11}(x_k,t_j)}{\phi'(t_j)}\right)y_j \\
&+y_N\left(\omega_b(x_k) - h\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{k_{11}(x_k,t_j)}{\phi'(t_j)}\omega_b(t_j)\right) \\
&-z_{-N}h\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{k_{12}(x_k,t_j)}{\phi'(t_j)}\omega_a(t_j) \\
&-h\sum_{j=-N+1}^{N-1}\delta_{kj}^{(-1)}\frac{k_{12}(x_k,t_j)}{\phi'(t_j)}z_j - z_N h\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{k_{11}(x_k,t_j)}{\phi'(t_j)}\omega_b(t_j)
\end{aligned}
\tag{10}
$$

and for the second equation of (1), we have

$$
\begin{aligned}
g(x_k) =& -z_{-N}h\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{k_{21}(x_k,t_j)}{\phi'(t_j)}\omega_a(t_j) \\
&-h\sum_{j=-N+1}^{N-1}\left(\delta_{kj}^{(-1)}\frac{k_{21}(x_k,t_j)}{\phi'(t_j)}\right)y_j \\
&-z_N h\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{k_{21}(x_k,t_j)}{\phi'(t_j)}\omega_b(t_j) \\
&-z_{-N}h\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{k_{22}(x_k,t_j)}{\phi'(t_j)}\omega_a(t_j) \\
&-h\sum_{j=-N+1}^{N-1}\delta_{kj}^{(-1)}\frac{k_{22}(x_k,t_j)}{\phi'(t_j)}z_j \\
&-z_N h\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{k_{22}(x_k,t_j)}{\phi'(t_j)}\omega_b(t_j)
\end{aligned}
\tag{11}
$$

For simplicity let us introduce the following

$$\begin{aligned} A(j,k) &= y_{-N}\left(\omega_a(x_k) - h\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{k_{11}(x_k,t_j)}{\phi'(t_j)}\omega_a(t_j)\right) \\ &+ \sum_{j=-N+1}^{N-1}\left(S(j,h)\circ\phi(x_k) - h\delta_{kj}^{(-1)}\frac{k_{11}(x_k,t_j)}{\phi'(t_j)}\right)y_j \\ &+ y_N\left(\omega_b(x_k) - h\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{k_{11}(x_k,t_j)}{\phi'(t_j)}\omega_b(t_j)\right) \end{aligned}$$

$$\begin{aligned} B(j,k) = &-z_{-N}h\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{k_{12}(x_k,t_j)}{\phi'(t_j)}\omega_a(t_j) \\ &- h\sum_{j=-N+1}^{N-1}\delta_{kj}^{(-1)}\frac{k_{12}(x_k,t_j)}{\phi'(t_j)}z_j \\ &- z_N h\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{k_{12}(x_k,t_j)}{\phi'(t_j)}\omega_b(t_j), \end{aligned}$$

$$\begin{aligned} C(j,k) = &-y_{-N}h\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{k_{21}(x_k,t_j)}{\phi'(t_j)}\omega_a(t_j) \\ &- \sum_{j=-N+1}^{N-1}h\delta_{kj}^{(-1)}\frac{k_{21}(x_k,t_j)}{\phi'(t_j)}y_j \\ &- y_N h\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{k_{21}(x_k,t_j)}{\phi'(t_j)}\omega_b(t_j), \end{aligned}$$

$$\begin{aligned} D(j,k) = &-z_{-N}h\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{k_{22}(x_k,t_j)}{\phi'(t_j)}\omega_a(t_j) \\ &- \sum_{j=-N+1}^{N-1}h\delta_{kj}^{(-1)}\frac{k_{22}(x_k,t_j)}{\phi'(t_j)}z_j \\ &- z_N h\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{k_{22}(x_k,t_j)}{\phi'(t_j)}\omega_b(t_j), \end{aligned}$$

$$F = [f(x_{-N}), f(x_{-N+1}), f(x_{-N+2}), \dots, f(x_N)]^T$$

and

$$G = [g(x_{-N}), g(x_{-N+1}), g(x_{-N+2}), \dots, g(x_N)]^T$$

Having applied the previous equations, the systems (10) and (11) can be wrewriten in the matrix form as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} Y \\ Z \end{bmatrix} = \begin{bmatrix} F \\ G \end{bmatrix},$$

where $A = [A(i-N-1, j-N-1)]$, $B = [B(i-N-1, j-N-1)]$, $C = C(i-N-1, j-N-1)$ and $D = D(i-N-1, j-N-1)$, $i, j = 1, 2, \dots, 2N+1$ and where

$$Y = \begin{bmatrix} y_{-N} \\ y_{-N+1} \\ \cdot \\ \cdot \\ \cdot \\ y_N \end{bmatrix}$$

and

$$Z = \begin{bmatrix} z_{-N} \\ z_{-N+1} \\ \cdot \\ \cdot \\ \cdot \\ z_N \end{bmatrix}$$

The approximation of the functions Y and Z at the nodes x_j, $j = -N, \dots, N$ are denoted by Y_N and Z_N, respectively and they can be obtained by $Y_N = T_N Y$ and $Z_N = T_N Z$, where

$$T_N = \begin{bmatrix} \omega_a(x_{-N}) & 0 & \dots & 0 & \omega_b(x_{-N}) \\ \omega_a(x_{-N+1}) & 1 & \dots & 0 & \omega_b(x_{-N+1}) \\ \vdots & & \ddots & & \\ \omega_a(x_{N-1}) & 0 & \dots & 1 & \omega_b(x_{N-1}) \\ \omega_a(x_N) & 0 & \dots & 0 & \omega_b(x_N) \end{bmatrix}$$

3. Error Analysis

In this section, the convergence of the Sinc method for (IAE) of index-1 is discussed. The proofs for the following lemma and theorem are based on [19].

Lemma 3.1. Let $y(x)$ and $z(x)$ are the exact solutions of system (1) and $\frac{k_{ij}}{\phi}\gamma_j(x) \in L_\alpha(D_d)$, $i, j = 1, 2$ then for the step size $h = \sqrt{\frac{\pi d}{\alpha N}}$, there exists constants c_{ij}, $i, j =$

1, 2 independent of N such that

$$\|A\bar{Y} - F\|_2 \le c_{11} N^{1/2} exp(-(\pi d\alpha N)^{1/2})$$
$$\|B\bar{Z} - G\|_2 \le c_{12} N^{1/2} exp(-(\pi d\alpha N)^{1/2})$$
$$\|C\bar{Y} - F\|_2 \le c_{21} N^{1/2} exp(-(\pi d\alpha N)^{1/2})$$
$$\|D\bar{Z} - G\|_2 \le c_{22} N^{1/2} exp(-(\pi d\alpha N)^{1/2})$$

where $\bar{Y} = (y(x_{-N}), y(x_{-N+1}), \ldots, y(x_N))$ and $\bar{Z} = (z(x_{-N}), z(x_{-N+1}), \ldots, z(x_N))$.

Proof. Let $|e_{1k}| = |e_1(x_k)| = |(AY - F)_k|$, $|e_{2k}| = |e_2(x_k)| = |(CY - F)_k|$, $|\epsilon_{1k}| = |\epsilon_1(x_k)| = |(BZ - G)_k|$ and $|\epsilon_{2k}| = |\epsilon_2(x_k)| = |(DZ - G)_k|$ then

$$\begin{aligned}|e_{1k}| \le & \left| y(x_k) - \sum_{j=-N}^{N} y(x_j)\gamma_j \right| + \int_0^{x_k} k_{11}(x_k, t) y(t) dt \\ & - hy(x_{-N}) \sum_{j=-N+1}^{N-1} \delta_{kj}^{(-1)} \frac{k_{11}(x_k, t_j)}{\phi'(t_j)} \omega_a(t_j) \\ & + hy(x_N) \sum_{j=-N+1}^{N-1} \delta_{kj}^{(-1)} \frac{k_{11}(x_k, t_j)}{\phi'(t_j)} \omega_b(t_j) \le c_{11} exp(-(\pi\alpha d N)^{1/2}),\end{aligned}$$

$$\begin{aligned}|e_{2k}| \le & \left| y(x_k) - \sum_{j=-N}^{N} y(x_j)\gamma_j \right| + \int_0^{x_k} k_{21}(x_k, t) y(t) dt \\ & - hy(x_{-N}) \sum_{j=-N+1}^{N-1} \delta_{kj}^{(-1)} \frac{k_{21}(x_k, t_j)}{\phi'(t_j)} \omega_a(t_j) \\ & + hy(x_N) \sum_{j=-N+1}^{N-1} \delta_{kj}^{(-1)} \frac{k_{21}(x_k, t_j)}{\phi'(t_j)} \omega_b(t_j) \le c_{21} exp(-(\pi\alpha d N)^{1/2}),\end{aligned}$$

$$\begin{aligned}|\epsilon_{1k}| \le & \left| z(x_k) - \sum_{j=-N}^{N} z(x_j)\gamma_j \right| + \int_0^{x_k} k_{12}(x_k, t) z(t) dt \\ & - hz(x_{-N}) \sum_{j=-N+1}^{N-1} \delta_{kj}^{(-1)} \frac{k_{12}(x_k, t_j)}{\phi'(t_j)} \omega_a(t_j) \\ & + hz(x_N) \sum_{j=-N+1}^{N-1} \delta_{kj}^{(-1)} \frac{k_{12}(x_k, t_j)}{\phi'(t_j)} \omega_b(t_j) \le c_{21} exp(-(\pi\alpha d N)^{1/2}),\end{aligned}$$

and

$$
\begin{aligned}
|\epsilon_{2k}| \le & \left| z(x_k) - \sum_{j=-N}^{N} z(x_j)\gamma_j \right| + \int_0^{x_k} k_{22}(x_k, t) z(t) dt \\
& - h z(x_{-N}) \sum_{j=-N+1}^{N-1} \delta_{kj}^{(-1)} \frac{k_{22}(x_k, t_j)}{\phi'(t_j)} \omega_a(t_j) \\
& + h z(x_N) \sum_{j=-N+1}^{N-1} \delta_{kj}^{(-1)} \frac{k_{22}(x_k, t_j)}{\phi'(t_j)} \omega_b(t_j) \le c_{22} exp(-(\pi \alpha d N)^{1/2}),
\end{aligned}
$$

Now, $\|A\bar{Y} - F\|_2 \le \left(\sum_{k=-N}^{N} |e_{1k}|^2 \right)^{1/2} \le c_{11} exp(-(\pi \alpha d N)^{1/2})$, and similarly for $\|C\bar{Y} - F\|_2$, $\|B\bar{Z} - G\|_2$ and $\|D\bar{Z} - G\|_2$. ■

Lemma 3.2. Let $y(x)$, $z(x)$ are in $L_\alpha(D_d)$ and $h = \sqrt{\frac{\pi d}{\alpha N}}$, then there exist constants c_3 and r_3 such that

$$
\sup_{x \in \Gamma} |y(x) - y_N(x)| \le c_3 \sqrt{\frac{N}{exp(\pi \alpha d N)}}
$$

and

$$
\sup_{x \in \Gamma} |z(x) - z_N(x)| \le r_3 \sqrt{\frac{N}{exp(\pi \alpha d N)}}
$$

Proof. Let $E_1 = \left| y(x) - \sum_{j=-N}^{N} y(x_j)\gamma_j \right|$ and $E_2 = \left| \sum_{j=-N}^{N} y(x_j)\gamma_j - \sum_{j=-N}^{N} y_j \gamma_j \right|$. By using Theorem 3, we obtain

$$
\sup_{x \in \Gamma} |E_1| \le c_2 N^{1/2} exp(-(\pi d \alpha N)^{1/2}) \tag{12}
$$

and

$$
|E_2| \le \left(\sum_{j=-N}^{N} |y(x_j) - y_j|^2 \right)^{1/2} \left(\sum_{j=-N}^{N} |\gamma_j|^2 \right)^{1/2} \le \eta \left(\sum_{j=-N}^{N} |y(x_j) - y_j|^2 \right)^{1/2} \tag{13}
$$

Now by using the Lemma, we have

$$\left(\sum_{j=-N}^{N} |y(x_j) - y_j|^2\right)^{1/2} = \|\bar{Y} - Y\|_2 = \|(A-C)^{-1}((A-C)\bar{Y} - (A-C)Y)\|_2$$
$$\le c_4 \|(A-C)^{-1}\|_2 N^{1/2} exp(-(\pi\alpha d N)^{1/2}) \tag{14}$$

where $c_4 = min\{c_3, c_{21}\}$. Therefore

$$\left|\sum_{j=-N}^{N} y(x_j) - y_j\right| \le \eta \|(A-C)^{-1}\|_2 c_4 exp(-(\pi\alpha d N)^{1/2}) \le c_5 exp(-(\pi\alpha d N)^{1/2}) \tag{15}$$

4. Numerical Examples

All the calculations were supported by the software Mathematica ®. In all examples, $d = \pi/2$ and $\alpha = 1$. The errors on the grids are defined as

$$|E_N| = \max_{-N \le j \le N} |y(x_j) - \bar{y}(x_j)|$$

and

$$|e_N| = \max_{-N \le j \le N} |z(x_j) - \bar{z}(x_j)|$$

Example 4.1. Consider equation with $k_{11}(x,t) = xt$, $k_{12}(x,t) = xt^2$, $k_{21}(x,t) = x - t$, $k_{22}(x,t) = xt + x + t + 1$ and $f(x) = (-1+x)x^2 + \frac{x^5}{2} - \frac{2x^6}{5}$, $g(x) = \frac{x^2}{2} + \frac{x^3}{2} - \frac{x^4}{6} - \frac{3x^5}{10}$ so that the exact solutions are $y(x) = x^2(x-1)$ and $z(x) = x(x-1)$.
We solved Example 1 for different values of N. The maximum of absolute errors on the Sinc grid are tabulated in Table 1. This table shows that as N increases the errors decreased rapidly.

Table 1: Errors on the Sinc grid for Example 1

N	1	2	4	6	10	20
$\lvert E_N \rvert$	0.000359135	0.0000431165	8.35822×10^{-6}	2.1748×10^{-6}	1.264708×10^{-7}	9.68021×10^{-10}
$\lvert e_N \rvert$	0.00481749	0.00320156	0.00104247	0.000984131	0.000131427	8.0828×10^{-6}

Example 4.2. Consider equation with $k_{11}(x,t) = sin(x-t)$, $k_{12}(x,t) = sin(x+t)$, $k_{21}(x,t) = cos(x-t)$, $k_{22}(x,t) = cos(x+t)$ and $f(x) = \frac{1}{30}(32cos(x) - 35cos(2x) + 3cos(6x)) + \frac{2}{3}(-1 + cos(x))sin(x)$, $g(x) = \frac{2}{3}(cos(x) - cos(2x)) + \frac{1}{30}(-32sin(x) +$

Table 2: Errors on the Sinc grid for Example 2

N	1	2	4	6	10	20
$\|E_N\|$	0.0912022	0.0127261	0.00747386	0.000985475	3.4551×10^{-6}	1.786×10^{-9}
$\|e_N\|$	0.10633	0.0984097	0.0231823	0.00703032	0.0018291	4.1123×10^{-6}

$25sin(2x) - 3sin(6x))$ so that the exact solutions are $y(x) = sin(2x)$ and $z(x) = cos(4x) - 1$.

The approximate solution is calculated for different values of N. The maximum of absolute errors on the Sinc grid are tabulated in Table 2. This table shows that as N increases the errors decreased rapidly.

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