On the absolute convergence Haar series

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Abstract

We prove that for every $\epsilon > 0$, there exists a measurable set $E \subset [0, 1]$ with $|E| > 1 - \epsilon$, such that for every function $f(x) \in C[0, 1]$ one can find a function $g(x) \in C[0, 1]$, $g(x) = f(x)$, $x \in E$, such that the expansion of $g(x)$ in the Haar system is unconditionally, absolutely converge in $C[0, 1]$.

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1. Introduction

We will consider the unconditionally convergence of Fourier-Haar series and the behavior of Fourier-Haar coefficients after modification of functions. Note that the well-known theorems of N.N. Luzin [1], D.E. Menshov [2] and M. G. Grigoryan [3] about “correction of functions.” (see also [4]–[10])

We remind that the Haar system is a basis in space $L^p[0, 1]$, $p \geq 1$ (see. [11]). i.e. there is unique series $\sum_{n=1}^{\infty} c_n(f) h_n(x)$, $c_n(f) = \int_0^1 f(x) h_n(x) dx$, $n \geq 1$ for each function $f(x) \in L^p[0, 1]$, which converges to $f(x)$ in the $L^p[0, 1]$– norm. A.M. Olevscii [12] have constructed function $f(x) \in L^\infty[0, 1]$, whose Fourier-Haar series $\sum_{k=1}^{\infty} c_k(f) h_k(x)$ can be so rearranged as to become divergent a.e. Note that P.L. Ul’yanov in [13] constructed a function $f_0(x) \in L^1[0, 1]$, whose Fourier-Haar coefficients is not bounded. i.e. $\limsup_{k \to \infty} |c_k(f_0)| = +\infty$.

F.G. Arutunyan in [4] proved follows: Let $f(x)$ be an a.e. finite measurable function on $[0, 1]$. Then for each $\epsilon > 0$ one can find a function $g \in L^2[0, 1]$, $\text{mes}\{x \in [0, 1]; g \neq f\} < \epsilon$. 

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$f \} < \epsilon$, such that the Fourier-Haar series of $g(x)$ is absolutely convergent uniformly on $[0, 1]$. In this paper we prove

**Theorem 1.1.** Let $f(x)$ be an a.e. finite measurable function on $[0, 1]$. Then for each $\epsilon > 0$ one can find a function $g \in L^\infty[0, 1]$, $\text{mes}\{x \in [0, 1]; g \neq f\} < \epsilon$, such that the sequence \{c_k(g), k \in \text{spectrum}(g)\}, is monotonically decreasing and the Fourier-Haar series of $g(x)$ is unconditionally convergent uniformly on $[0, 1]$.

**2. Proof of the theorem**

The following lemma is the basic tool in the proof of Theorem 1.1.

**Lemma 2.1.** Let numbers $k_0 \geq 1, \epsilon \in (0, 1)$ and a Haar polynomial $f(x)$ with
\[
\int_0^1 |f(x)| \, dx < 1
\]
be given, where $H_S = \{h_k(x)\}_{k=1}^\infty$ is a Haar system. Then one can find a measurable sets $E \subset \Delta$ and a polynomial $P(x)$ in the Haar system of the form

\[
Q(x) = \sum_{k=k_0+1}^k a_k h_{n_k}, \quad n_k \not\rightarrow, \text{ which satisfies to following conditions:}
\]

1) $|E| > 1 - \epsilon$;
2) $Q(x) = f(x) E$;
3) $\epsilon > a_k \geq a_{k+1} > 0, k \in [k_0; \bar{k})$;
4) $\sum_{k=k_0+1}^k a_k |h_{n_k}(x)| < \frac{4 |f(x)|}{\epsilon}$ if $x \in [0, 1]$.

**Proof.** The proof of Lemma can be done analogously as lemma 2.1 of paper [14] (see pp. 132–136).

Let $f_0(x)$ be an a.e. finite measurable function on $[0, 1]$, and let $\epsilon \in (0, 1)$. From N.N. Luzin’s Theorem exists a function $f \in C[0, 1]$ such that $\text{mes}\{x \in [0, 1]; f \neq f_0\} < \epsilon$.

It is easy to see that one can choose a sequence \{f_n(x)\}_{n=1}^\infty of polynomials in the Walsh systems such that

\[
\lim_{N \to \infty} \left\| \sum_{n=1}^N f_n(x) - f(x) \right\|_\infty = 0, \quad ||f_n(x)||_\infty \leq \epsilon \cdot 2^{-2(n+1)}, \quad n \geq 2.
\]

(1)

Applying repeatedly Lemma 2.1, we obtain sequences of sets \{E_n\}_{n=1}^\infty and polynomials in the Walsh systems \{\varphi_n(x)\}

\[
Q_n(x) = \sum_{k=m_n-1}^{m_n-1} a_k h_{s_k}(x), \quad n \geq 1, m_n \not\rightarrow.
\]

(2)
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which for all \( n \geq 1 \) satisfy the following conditions:

\[
Q_n(x) = f_n(x), \quad \text{for } x \in E_n, \quad (3)
\]

\[
|E_n| > 1 - \varepsilon 2^{-n}, \quad (4)
\]

\[
\max_{m \in [m_{n-1}; m_n)} \left| \sum_{k=m_{n-1}}^{m} a_k |h_{sk}(x)| \right| \leq 3\varepsilon^{-1}2^n \cdot ||f_n||_{\infty}, \quad (5)
\]

\[
|a_k| < |a_{k-1}| < \min\{|a_{m_{n-1}-1}|, 2^{-n}\}, \quad \text{for all } k \in [m_{n-1}; m_n). \quad (6)
\]

We put

\[
\sum_{k=1}^{\infty} a_k h_{sk}(x) = \sum_{n=1}^{\infty} \sum_{k=m_{n-1}}^{m_n-1} a_k h_{sk}(x), \quad (7)
\]

\[
g(x) = \sum_{n=1}^{\infty} Q_n(x). \quad (8)
\]

From (1)–(4), (8) we get \( g(x) \in L^\infty[0, 1) \) and

\[ g(x) = f(x), \quad \text{for } x \in \bigcap_{n=1}^{\infty} E_n, \quad |\bigcap_{n=1}^{\infty} E_n| > 1 - \varepsilon. \]

It is easy to see that the series (7) converges to the function \( g(x) \) uniformly on \([0, 1]\)
(see (1),(2),(5)) and therefore

\[
a_k = \int_0^1 g(x)h_{sk}(x)dx = c_{sk}(g), \quad k = 1, 2, \ldots \quad (9)
\]

Obviously \( \{c_{k}(g), \ k \in \text{spec}(g)\} \) is monotonically decreasing (see (6)).

\[
\sum_{k=1}^{\infty} a_k |h_{sk}(x)| = \sum_{n=1}^{\infty} \sum_{k=m_{n-1}}^{m_n-1} a_k |h_{sk}(x)| < \infty, \ x \in [0, 1].
\]

Theorem is proved.

References


