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Finite Propagation Speed and Finite Time Blowup of the Euler Equations for Generalized Chaplygin Gas

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Abstract

The blowup phenomenon for the N-dimensional isentropic compressible Euler equations for generalized chaplygin gas (GCG), which arises in a cosmology model related to dark matter and dark energy, is investigated. First, we establish the finite propagation speed property for the system. This allows one to apply the integration method to study the blowup problem. More precisely, by deriving a differential inequality, we show the any C^1 solution in a designed non-empty space blows up on finite time provided that the initial functional is sufficiently large.

Keywords: Blowup, Euler equations, Chaplygin, Finite Propagation Speed

MSC: 35Q31, 35B44, 76N15

1 INTRODUCTION AND MAIN RESULTS

The N-dimensional compressible isentropic Euler equations for the generalized Chaplygin gas (GCG) can be expressed as

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ \rho[u_t + (u \cdot \nabla)u] + \nabla p = 0, \end{cases} \tag{1}$$

where $\rho = \rho(t,x): [0,\infty) \times \mathbb{R}^N \to [0,\infty)$, $u = u(t,x): [0,\infty) \times \mathbb{R}^N \to \mathbb{R}^N$ and p are the density, the velocity, and the pressure function respectively. The pressure function is given by

$$p = -K\rho^{-\gamma} < 0, \tag{2}$$

for which the constants $K > 0$ and $0 < \gamma \leq 1$.

The first equation in (1) is derived from the mass conservation law while the second equation in (1) is a result of the momentum conservation law. Equation (2) is known as the state equation.

When the state equation is $p = K\rho^{\gamma} > 0, \gamma \geq 1$, system (1) constitutes the classical Euler equations for compressible fluids [5, 6]. The blowup phenomena and global existence results of the classical Euler equations have been examined extensively [1, 3, 7, 8, 12, 13, 14, 16, 17, 18].

In this article, we consider the GCG model with negative pressure (2), which was introduced in [15] in 2012 as a unification of dark matter and dark energy. To be specific, type Ia supernova (SNIa) observations [10] have shown that our universe has entered into a phase of accelerating expansion since 1998. During these years from that time, many additional observational results, including current Cosmic Microwave Background (CMB) anisotropy measurement [11], and the data of the Large Scale Structure (LSS) [9], also strongly support this suggestion. And these cosmic observations indicate that baryon matter component is about 4% for total energy density, and about 96% energy density in the universe is invisible. Considering the four-dimensional standard cosmology, this accelerated expansion for universe predicts that dark energy (DE) as an exotic component with negative pressure is filled in the universe. And it is shown that DE takes up about two-thirds of the total energy density from cosmic observations. The remaining one-third is dark matter (DM). In theory, DE models have already been constructed. But there exists another possibility: the invisible energy component is a unified dark fluid. i.e. a mixture of dark matter and dark energy. The GCG model was introduced as a unification of dark matter and dark energy.

Recently, the authors in [4] considered the gas expansion problem for the GCG Euler equations (1) with $N = 2$. More precisely, the authors proved the global existence of solution to the expansion problem of a wedge of gas into vacuum with the half angel $\theta \in (0, \pi/2)$ for the generalized Chaplygin gas after obtaining some priori estimates. Subsequently, the author in [2] investigated the solutions of system (1) by starting with the following velocity form.

$$u(t, x) = c(t)|x|^{\alpha-1}x.$$

It was shown that there are only trivial solutions if $\alpha \neq 1$ and the solutions will be expanding and blowing up when the initial data $c(0)$ and $\dot{c}(0)$ satisfy some simple inequalities in the case of $\alpha = 1$.

In this paper, we consider system (1) with the following initial data.

$$\begin{cases} \left(\rho(0,x), u(0,x)\right) = \left(\overline{\rho} + \rho_0(x), u_0(x)\right), \\ \quad \operatorname{supp}(\rho_0, u_0) \subseteq \{x : |x| \le R\}, \end{cases} \tag{3}$$

for some positive constants $\overline{\rho}$ and R. We first establish in section 2 the finite propagation speed property, which allows one to apply integration method to study the corresponding blowup problem. In section 3, we derive a differential inequality which is used to be analyzed and leads to a finite time blowup result in section 4.

2 THE FINITE PROPAGATION SPEED PROPERTY

In [13], Sideris et al proved the finite propagation speed property for system (1) with a damping term and positive pressure $p = K\rho^{\gamma}$, $\gamma > 1$ in the case of $N = 3$. In what follows, we amend the proof to establish the finite propagation speed property for system (1) with negative pressure (2) and general dimension N.

Theorem 1. Let (ρ, u) be a C^1 solution of the N-dimensional Euler equations (1) for the GCG with life span $T > 0$ and initial data (3). Then,

$$(\rho, u) = (\overline{\rho}, 0)$$

for all $t \in [0, T)$ and $|x| \ge R + \sigma t$, where $\sigma := \sqrt{K\gamma\overline{\rho}^{-\gamma-1}} > 0$.

Proof. Define

$$v := \frac{2}{-\gamma - 1}\left(\sqrt{p'(\rho)} - \sigma\right),$$

where p is regarded as a function of ρ. Then, $(1)_1$ and $(1)_2$ are transformed into

$$v_t + \sigma\nabla \cdot u = -u \cdot \nabla v - \frac{-\gamma - 1}{2} v\nabla \cdot u \tag{4}$$

and

$$u_t + \sigma\nabla v = -(u \cdot \nabla)u - \frac{-\gamma - 1}{2} v\nabla v \tag{5}$$

respectively. Multiply (4) and (5) by v and u respectively and add the results together. After rearranging terms, one obtains

$$\left(\frac{v^2+|u|^2}{2}\right)_t + \nabla\cdot(\sigma v u)$$
$$= -vu\cdot\nabla v - u\cdot(u\cdot\nabla u) - \frac{-\gamma-1}{2}v^2\nabla\cdot u - \frac{-\gamma-1}{2}vu\cdot\nabla v, \quad (6)$$

where $u\cdot\nabla u := \sum_{i=1}^{N} u_i\nabla u_i$ and $u := (u_1,\ldots,u_N)$.

The result is then followed by a similar approach of lemma 3.2 in [13]. For the sake of completeness, we provide the details as follows.

Fix $(x,t)\in\mathbb{R}^N\times(0,T]$ and $\mu\in[0,t)$. Define the truncated cone

$$C_\mu := \{(y,s): |y-x|\le\sigma(t-s), 0\le s\le\mu\}.$$

Note that the cross sections of C_μ are

$$U(s) := \{y: |y-x|\le\sigma(t-s)\} \text{ for } s\in[0,\mu].$$

Take integration on both sides of (6) over C_μ to get

$$\int_0^\mu\int_{U(s)}\left[\left(\frac{v^2+|u|^2}{2}\right)_t + \nabla\cdot(\sigma v u)\right]dyds \quad (7)$$

$$= \int_0^\mu\int_{U(s)}\left[-vu\cdot\nabla v - u\cdot(u\cdot\nabla u) - \frac{-\gamma-1}{2}v^2\nabla\cdot u - \frac{-\gamma-1}{2}vu\cdot\nabla v\right]dyds. \quad (8)$$

Lastly, we define

$$e(s) := \int_{U(s)}\frac{v^2+|u|^2}{2}(s,y)dy \quad (9)$$

and divide the proof into steps.

Step 1. By the Differentiation Formula for Moving Regions, the Fundamental Theorem of Calculus and the Divergence Theorem, (7) becomes

$$\int_{U(\mu)}\left(\frac{v^2+|u|^2}{2}\right)(\mu,y)dy - \int_{U(0)}\left(\frac{v^2+|u|^2}{2}\right)(0,y)dy +$$

$$\int_0^\mu \int_{\partial U(s)} \left[\sigma\left(\frac{v^2+|u|^2}{2}\right) + \frac{y-x}{|y-x|}\cdot \sigma v u\right] dSds$$

$$= e(\mu) - e(0) + \ \sigma \int_0^\mu \int_{\partial U(s)} \left(\frac{v^2+|u|^2}{2} + \frac{y-x}{|y-x|}\cdot v u\right) dSds$$

$$\geq e(\mu) - e(0),$$

where dS is the surface element with respect to the variable y and $\partial U(s)$ is the boundary of $U(s)$.

Note that by Cauchy-Schwarz inequality,

$$-\frac{y-x}{|y-x|}\cdot vu \leq \left|\frac{y-x}{|y-x|}\cdot vu\right| \leq |vu| = |v||u| \leq \frac{v^2+|u|^2}{2}.$$

Step 2. By Cauchy-Schwarz inequality and the following two inequalities,

$$|u\cdot \nabla u| \leq |u|\sqrt{\sum_{i=1}^N |\nabla u_i|^2} \text{ and } |\nabla\cdot u| \leq \sqrt{\sum_{i=1}^N |\nabla u_i|^2},$$

the integrand of (8) can be estimated as follows.

$$-vu\cdot\nabla v - u\cdot(u\cdot\nabla u) - \frac{-\gamma-1}{2}v^2\nabla\cdot u - \frac{-\gamma-1}{2}vu\cdot\nabla v$$

$$\leq |v||u||\nabla v| + |u||(u\cdot\nabla u)| + \frac{\gamma+1}{2}v^2|\nabla\cdot u| + \frac{\gamma+1}{2}|v||u||\nabla v|$$

$$\leq 2[|v||u||\nabla v| + |u||(u\cdot\nabla u)| + v^2|\nabla\cdot u|]$$

$$\leq 2\left[\frac{v^2+|u|^2}{2}|\nabla v| + \frac{v^2+|u|^2}{2}\left(2\sqrt{\sum_{i=1}^N |\nabla u_i|^2}\right)\right]$$

$$= 2\left(\frac{v^2 + |u|^2}{2}\right)\left(|\nabla v| + 2\sqrt{\sum_{i=1}^{N} |\nabla u_i|^2}\right).$$

Thus, expression (8) is less than or equal to

$$C \int_0^{\mu} e(s)ds,$$

where

$$C = 2\max_{C_\mu}\left\{|\nabla v| + 2\sqrt{\sum_{i=1}^{N} |\nabla u_i|^2}\right\} < +\infty.$$

Step 3. Combining the results of Step 1 and Step 2, one has

$$e(\mu) - e(0) \leq C \int_0^{\mu} e(s)ds\,.$$

By Gronwall's Inequality and the definition (9) of $e(s)$, we see that

$$0 \leq e(\mu) \leq e(0)\exp(Ct).$$

If $|x| > R + \sigma t$, then $|y| > R$ for $y \in U(0)$.

$\Rightarrow e(0) = 0$ and $e(\mu) = 0$ for $|x| > R + \sigma t$.

$\Rightarrow v(\mu, x) = u(\mu, x) = 0$ for $|x| > R + \sigma t$.

$\Rightarrow (\rho, u)(\mu, x) = (\overline{\rho}, 0)$ for $|x| > R + \sigma t$.

As $\mu \in [0, t)$ is arbitrary, the result follows by continuity.

3. A DIFFERENTIAL INEQUALITY

In [12], Sideris considered the functional

$$F(t) = \int_{\mathbb{R}^3} \rho x \cdot u dx,$$

which is well-defined by the finite propagation speed property [12] for the three-dimensional classical Euler equations when $\gamma > 1$ and the initial data are non-vacuum. Sideris established a finite time blowup result by investigating $F'(t)$ and deriving a differential inequality of F of the form

$$F'(t) \geq CF^2(t)$$

for some positive constant C. Then, the blowup result followed provided the initial datum $F(0)$ is large enough.

In this section, we consider

$$F(t) := \int_{\mathbb{R}^N} f\rho x \cdot u dx, \tag{10}$$

for any given C^1 increasing function $f = f(r)$ on $[0, +\infty)$ such that $f(0) = 0$, where $r = |x|$.

Note that $F(t)$ is well-defined by the finite propagation speed property, i.e. Theorem 1.

We are going to investigate $F'(t)$.

By Stokes' Theorem,

$$F'(t) = \int_{\mathbb{R}^N} (p - \overline{p}) \nabla \cdot (fx) dx + \int_{\mathbb{R}^N} \sum_{i=1}^{N} (\rho u_i u) \cdot \nabla (f x_i) dx,$$

where

$$u := (u_1, u_2, u_3, \ldots, u_N) \text{ and } x := (x_1, x_2, x_3, \ldots, x_N).$$

As

$$\sum_{i=1}^{N} (\rho u_i u) \cdot \nabla (f x_i) = f\rho |u|^2 + \frac{f'\rho (x \cdot u)^2}{r},$$

one has the following relation.

$$F'(t) = \int_{B(t)} (p - \overline{p}) \nabla \cdot (fx) dx + \int_{B(t)} \left[f\rho |u|^2 + \frac{f'\rho (x \cdot u)^2}{r} \right] dx, \tag{11}$$

where

$$B(t) := \{x \in \mathbb{R}^N | 0 < |x| < R + \sigma t\}.$$

On the other hand, by Cauchy-Schwarz inequality,

$$F^2(t) = \left(\int_{B(t)} f\rho x \cdot u dx\right)^2$$
$$\leq \left(\int_{B(t)} f\rho |u|^2 dx\right)\left(\int_{B(t)} f\rho |x|^2 dx\right)$$
$$\leq \left(\int_{B(t)} f\rho |u|^2 dx\right)\left[(R+\sigma t)^2 f(R+\sigma t)\int_{B(t)} \rho dx\right]$$
$$= \left(\int_{B(t)} f\rho |u|^2 dx\right)\left[(R+\sigma t)^2 f(R+\sigma t)\left(m(t) + \int_{B(t)} \overline{\rho} dx\right)\right]$$
$$= \left(\int_{B(t)} f\rho |u|^2 dx\right)[(R+\sigma t)^2 f(R+\sigma t)(m(0) + \overline{\rho}|B(t)|)], \tag{12}$$

where $|B(t)|$ is the volume of B(t) and $m(t)$, which is equal to $m(0)$ as in the 3-dimensional case in [12], is defined by

$$m(t) := \int_{\mathbb{R}^N} (\rho - \overline{\rho}) dx.$$

It follows from (12) that

$$\int_{B(t)} f\rho |u|^2 dx \geq \frac{F^2(t)}{(R+\sigma t)^2 f(R+\sigma t)(m(0) + \overline{\rho}|B(t)|)}. \tag{13}$$

By (11) and (13), one obtains the following differential inequality.

$$F'(t) \geq \int_{B(t)} (p - \overline{p}) \nabla \cdot (fx) dx + \frac{F^2(t)}{(R+\sigma t)^2 f(R+\sigma t)(m(0) + \overline{\rho}|B(t)|)}. \tag{14}$$

4. FINITE TIME BLOWUP

In this section, we shall prove a finite time blowup result for any solution of system (1) in Ω, which is defined by

$$\Omega \coloneqq \{(\rho, u)\ C^1\ solutions\ of\ system\ (1)\ with\ life\ span\ T > 0 \mid \rho \geq \rho(0, x)\}. \quad (15)$$

One needs to prove that Ω is non-empty before proceeding.

In [2], the author constructed an exact solution of system (1) as follows.

$$\begin{cases} u = c(t)x, \\ \dfrac{1}{\rho^{\gamma+1}} = g(t) + h(t)r^2, \end{cases}$$

where

$$g(t) := \frac{1}{\rho^{\gamma+1}(0,0)} e^{(\gamma+1)N \int_0^t c(s)ds},$$

$$h(t) := \frac{\gamma+1}{2K\gamma}(\dot{c} + c^2)$$

and $c(t)$ satisfies the following ordinary differential equation.

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{c} + c^2) + [2 - (\gamma+1)N](\dot{c} + c^2)c = 0. \quad (16)$$

By part two of Theorem 3.2 in [2], if $\dot{c}(0) + c^2(0) > 0$, $c(0) < 0$ and $\dot{c}(0) + \frac{2-(\gamma+1)N}{2} c^2(0) < 0$, then $\dot{c}(t) + c^2(t) > 0$, $c(t) < 0$ for all t. Thus, from $c(t) < 0$, one sees that g is decreasing. From $\dot{c}(t) + c^2(t) > 0$ and (16), one obtains that h is decreasing.

Thus, $1/\rho$ is decreasing in t and

$$\frac{1}{\rho} \leq \frac{1}{\rho(0, x)}.$$

Thus, we have shown that the set Ω is non-empty.

From (14) in section 3, we have

$$F'(t) \geq \int_{B(t)} (p - \overline{p}) \nabla \cdot (fx) dx + \frac{F^2(t)}{(R+\sigma t)^2 f(R+\sigma t)(m(0) + \overline{\rho}|B(t)|)}$$

or

$$F'(t) \geq A_1(t) + \frac{F^2(t)}{A_2(t)} \qquad (17)$$

with

$$A_1(t) := \int_{B(t)} (p - \overline{p}) \nabla \cdot (fx) dx \qquad (18)$$

and

$$A_2(t) := (R+\sigma t)^2 f(R+\sigma t)(m(0) + \overline{\rho}|B(t)|). \qquad (19)$$

As

$$p = -K \frac{1}{\rho^\gamma} \geq -K \frac{1}{\rho^\gamma(0,x)} = p(0,x),$$

We have that $A_1(t) \geq A_1(0)$. Thus, (17) becomes

$$F'(t) \geq A_1(0) + \frac{F^2(t)}{A_2(t)}.$$

Note that $A_1(0)$ is negative as $\rho(0,x) \geq \overline{\rho}$ by (3).

Now, fix $\tau > 0$, then for all $t \in [0,\tau]$, one has

$$F'(t) \geq \frac{F^2(t)}{A_2(t)} + A_1(0)$$

$$= \frac{F^2(t)}{2A_2(t)} + \left(\frac{F^2(t)}{2A_2(t)} + A_1(0)\right)$$

$$\geq \frac{F^2(t)}{2A_2(t)} + \left(\frac{F^2(t)}{2A_2(\tau)} + A_1(0)\right).$$

Define $I(t) := \frac{F^2(t)}{2A_2(\tau)} + A_1(0)$. It follows that if $I(0) > 0$, then $I(t) \geq 0$ on $[0, \tau]$. Thus,

$$F'(t) \geq \frac{F^2(t)}{2A_2(t)} \qquad (20)$$

on $[0, \tau]$.

If one further requires that

$$F(0) > \left[\int_0^\tau \frac{1}{2A_2(s)} ds\right]^{-1}, \qquad (21)$$

then the solutions blow up on finite time. More precisely, from (20),

$$\frac{1}{F(0)} - \frac{1}{F(t)} \geq \int_0^t \frac{1}{2A_2(s)} ds$$

$$\frac{1}{F(t)} \leq \frac{1}{F(0)} - \int_0^t \frac{1}{2A_2(s)} ds \qquad (22)$$

on $[0, \tau]$. Note that if (21) is satisfied, then the right-hand side of (22) will be negative when t approaches to τ. However, from $F(0) > 0$ and (20), one has that $1/F(t) > 0$. Thus, the life span T must be less that τ and hence is finite.

In conclusion, we have obtained the following result.

Theorem 2. For any C^1 solution (ρ, u) of system (1) with life span $T > 0$ belonging to Ω, if

$$F(0) > max\left\{\left[\int_0^\tau \frac{1}{2A_2(s)} ds\right]^{-1}, \sqrt{-2A_1(0)A_2(\tau)}\right\},$$

then T is finite, where τ is any positive fixed real number. Ω, $F(t)$, $A_1(t)$ and $A_2(t)$ are defined in (15), (10), (18) and (19) respectively.

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