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Perfect if and only if Triangular

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Abstract

A number n is perfect when $\sigma(n) = \sum_{\substack{0 \le d \le n \\ d|n}} d = 2n$. It was Euclid who proved that if $(2^k - 1)$ is a prime number, Mersenne prime, then $N = 2^{k-1}(2^k - 1)$ is an even perfect number. Moreover, if N is an even perfect number then $N = T_m$ for some $m \in \mathbb{N}$ and $m \ge 3$ is a triangular number where $T_m = \sum_{i=1}^{m} i$.
In this paper we proved the necessary and sufficient condition for an even triangular number T_m to be a perfect number N= $2^{k-1}(2^k - 1)$besides $T_m \not\equiv 4 \, mod \, 10$ and $T_m \not\equiv 2 \, mod \, 10$.

Keywords: Perfect Numbers, Triangular Numbers and Mresenne Primes.

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INTRODUCTION

A Perfect Number is a positive integer with the property that it coincides with the sum of all its positive divisors other than the number itself [1]. Thus, an integer $n \ge 1$ is a perfet number if

$$\sum_{\substack{0 \le d < n \\ d|n}} d = n$$

The nth triangular number is the number of dots composing a triangle with n dots on a side, and is equal to the sum of the n natural numbers from 1 to n. [2]

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Example 1:

Triangular Numbers: 1, 3, 6,10,15,21,28,36,45. . .

Perfect Numbers: 6, 28,496, 8126, 33550336, 8589869056,137438691328, 2305843008139952128, 2658455991569831744654692615953842176,...

The number 6 is unique in that 6 = 1 + 2+ 3 where 1, 2 and 3 are all of the proper divisors of 6. The number 28 also shares this property, for 28 = 1+ 2 +4 +7 + 14.

These perfect numbers have been a great deal of mathematical study- indeed, many of the basic theorems of numbers theory stem from the investigation of the Greeks into the problem of perfect and Pythagorean numbers. The Pythagoreans introduced the name perfect and there are speculations that there could be religious or astrological origins because the earth was created in 6 and the moon needs 28 days to circle the earth, mystical associations are natural. The early Hebrews also studied perfect numbers [3].

Definition 1: The sum of divisors is the function $\sigma(n) = \sum_{d|n} d$,where d runs over the positive divisors of n including 1 and n itself.

Definition 2: The number n is called *perfect* if $\sigma(n) = 2n$, when $\sigma(n) < 2n$ we say n is *deficient*, $\sigma(n) > 2n$ we say n is *abundant.*

Example 2:

6 and 28 are perfect as $\sigma(6) = 1 + 2 + 3 + 6 = 12 = 2(6)$ and

$$\sigma(28) = 1 + 2 + 4 + 7 + 14 + 28 = 56 = 2(28).$$

Euclid was the first mathematician who categorized even perfect numbers. He noticed that

$$6 = 2^1.3^1 = 2^1(2^2 - 1)$$
$$28 = 2^2.7 = 2^2(2^3 - 1)$$
$$496 = 16.31 = 2^4(2^5 - 1)$$
$$8126 = 64.127 = 2^6(2^7 - 1)$$

***Theorem 3* (Euclid)**[4,9]: *If* $(2^n - 1)$ *is prime then* $N = 2^{n-1}(2^n - 1)$ *is perfect.*

Proof: The only prime divisors of N are $(2^n - 1)$ and 2. Since $(2^n - 1)$ occurs as a single prime, we have that $\sigma(2^n - 1) = (1 + (2^n - 1)) = 2^n$, and thus

$$\sigma(N) = \sigma(2^{n-1})\sigma(2^n - 1) = \left(\frac{2^n - 1}{2 - 1}\right)2^n = 2^n(2^n - 1) = 2.2^{n-1}(2^n - 1) = 2N$$

So N is perfect.

Mersenne primes: Monk Martin Mersenne, a colleague of Descartes, Fermat and Pascal created with investigating these unique primes as early as 1644. He knew $(2^n - 1)$ is prime for n = 2, 3, 5, 7, 11, 13,17 and 19. [5, 6]

***Definition 4*:** A Mersenne prime is a prime number of the form $M_n = 2^{P_n} - 1$ where P_n is a prime number.

***Proposition 5*:**[5] *(Cateldi – Fermat) If* $(2^n - 1)$ *is prime, then* n *itself is prime.*

Proof: $x^n - 1 = (x - 1)(x^{n-1} + \ldots + x + 1)$. Suppose we can write $n = rs$ where $r, s > 1$. Then

$2^n - 1 = (2^r)^s - 1 = (2^r - 1)((2^r)^{s-1} + \ldots + 2^r + 1)$ so that $(2^r - 1)|(2^n - 1)$ which is prime, a contradiction. ■

***Theorem 6*:** *If* N *is an even perfect number, then* $N = 2^{n-1}(2^n - 1)$ *where* $(2^n - 1)$ *is prime*.

Proof: Since $2^n m = (2^n - 1)\sigma(m)$, every prime divisor of $(2^n - 1)$ must also divide m, for it is odd and cannot divide 2^n. So, suppose p^α divides $(2^n - 1)$ with p prime.

From the fact that if $a|b$, then $\frac{\sigma(a)}{a} \leq \frac{\sigma(b)}{b}$ where equality holds only if $a = b$ we have,

$$\frac{\sigma(m)}{m} \leq \frac{\sigma(p^\alpha)}{p^\alpha} = \frac{1+p+\ldots+p^\propto}{p^\propto} \geq \frac{p^{\propto-1}+p^\propto}{p^\propto} = \frac{1+p}{p}$$. Hence

$$1 = \frac{\sigma(N)}{2N} = \frac{\sigma(2^{n-1})\sigma(m)}{2^n m} \geq \frac{(2^n-1)(1+p)}{2^n p} = 1 + \frac{(2^n-1)-p}{2^n p}.$$

This is only satisfied when the fraction on the right is zero, so that $p = (2^n - 1), \propto = 1$ and $m = p$.

Hence $N = 2^{n-1}(2^n - 1)$. ■

Proposition 7: *Even perfect number ends in either 6 or 8.*

Proof: Every prime number $p \geq 2$ is of the form $p = 4m + 3$ or $p = 4m + 1$. In the former case,

$$N = 2^{n-1}(2^n - 1) = 2^{4m}(2^{4m+1} - 1) = (16)^m(2.(16)^m - 1)$$
$$\equiv 6^m(2(6)^m - 1) \equiv 6(mod 10)$$

Since by induction one can show that $6^m \equiv 6(mod 10)$ for all m .

Similarly in the latter case,

$$N = 2^{n-1}(2^n - 1) = 2^{4m+2}(2^{4m+3} - 1) = 4(16)^m(8.(16)^m - 1)$$
$$\equiv (4)(6)(8(6) - 1) \equiv 4(8 - 1) \equiv 8(mod 10).$$

Finally, if $n = 2$, $N = 2^{2-1}(2^2 - 1) = 6$ and so we have the result that even perfect number ends in either 6 or 8. ■

MAIN RESULTS

Let $A(x) = \sum_{i=0}^{m} a_i x^i$ and $B(x) = \sum_{i=0}^{n} b_i x^i$. Then $A(x)B(x) = C(x) = \sum_{k=0}^{n+m} c_k x^k$ where

$$C_k = \sum_{i=0}^{k} a_i b_{k-i} \text{ for } 0 \leq k \leq m + n .$$

The decimal expansion of a positive integer N, $0 \leq a_k < 10$ where a_0 is the unit digit of N is given by

$$N = A(10) = a_m 10^m + a_{m-1}10^{m-1} + \ldots + a_1 10^1 + a_0 10^0$$
$$= (a_m a_{m-1} a_{m-2} \ldots a_1 a_0)_{10}$$
$$= \sum_{i=0}^{m} a_i 10^i$$

***Theorem 8* [7]:** *A triangular number* $T_m = \sum_{i=1}^{m} i$ *is even if and only if* $m = (4k-1)$ *or* $m = 4k$ *for some* $\epsilon\, \mathbb{Z}^+$.

***Theorem 9*:** *Even triangular numbers* T_m *end not with 2 or 4. That is neither* $T_m \equiv 4 \; mod\; 10$ *nor* $T_m \equiv 2\; mod\; 10$.

Proof: Suppose T_m is an even triangular number. Then either $m = (4k-1)$or $= 4k$.

a) Suppose $m = 4k - 1$.

This implies $T_m = T_{4k-1} = \sum_{i=1}^{4k-1} i$

$$=\frac{(4k-1)(4k)}{2} = 2k(4k-1).$$

Let$A\,(10) = 2k = \sum_{i=0}^{m} a_i 10^i = (a_m a_{m-1} a_{m-2} \ldots a_1 a_0)_{10}$ be decimal expansion of the factor$(2k)$ of an even triangular number T_{4k-1} where the unit digit $a_0 \in \{0, 2, 4, 6, 8\}$.

Let b_0 be the unit digit of the factor $B(10) = (4k-1)$ of T_m where the decimal expansion is

$B(10) = 4k - 1 = \sum_{i=0}^{n} b_i 10^i = (b_n b_{n-1} \ldots b_1 b_0)_{10}$.

Consider $T_m = T_{4k-1} = (2k)(4k-1) = (2k)(2(2k)-1) = A(10)\,B(10)$

$$= (\sum_{i=0}^{m} a_i 10^i)(\sum_{i=0}^{n} b_i 10^i) = C(10)$$

$$= \sum_{i=0}^{n+m} c_i\, 10^i = (c_{m+n} c_{m+n-1} \ldots c_1 c_0)_{10}$$

The constant term c_0 of $C(10) = T_m = T_{4k-1}$ is $c_0 = a_0 b_0$.

We consider each unit digit $a_0 \in \{0, 2, 4, 6, 8\}$ of $2k = A(10)$ to determine unit digits b_0 of

$B(10) = (4k - 1)$ and c_0 of $C(10)$.

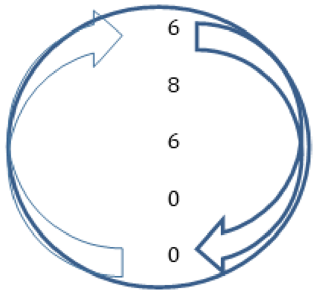
1) a_0= 0 ⇒ $c_0 = 0$

2) a_0 = 2, $b_0 =$ 2*2 – 1 = 3 ⇒ $c_0 = 6$

3) a_0 = 4, $b_0 =$ 2*4 – 1= 7 ⇒ $a_0 b_0 = c_0$ = 8 (Because (4)*(7) = 28 = $2 * 10^1 + 8 * 10^0$)

4) a_0= 6, $b_0 = 1$, because 2*6 – 1 = 11 = $1 * 10^1 + 1 * 10^0$ ⇒ $c_0 = 6$

5) a_0= 8, the unit digit $b_0 = 5$ because 2*8 – 1 = 1 5 = $1 * 10^1 + 5 * 10^0$ ⇒ $c_0 = 0$.



Cycle for unit digits in

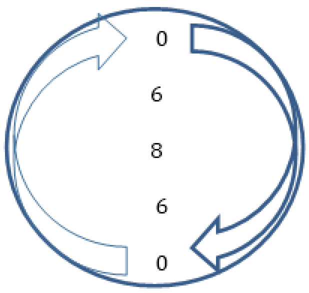
$T_m = T_{4k-1}$

T_3	T_7	T_{11}	T_{15}	T_{19}
6	28	66	120	190

T_{23}	T_{27}	T_{31}	T_{35}	T_{39}
276	378	496	630	780

Table I: $T_m = T_{4k-1}$

b) For $m = 4k$, in similar approach one can show that an even triangular number T_m has the following sequence of unit digits.



Cycles for unit digits in

$T_m = T_{4k}$

T_4	T_8	T_{12}	T_{16}	T_{20}	T_{24}
10	36	78	136	210	300

T_{28}	T_{32}	T_{36}	T_{40}	T_{44}	T_{48}
406	528	666	820	990	1176

Table II: $T_m = T_{4k}$

Hence if a triangular number is even, then its unit digit is either 0, 6 or 8 but not 2 and 4. This implies

$T_m \not\equiv 4 \; mod \; 10$ and $T_m \not\equiv 2 \; mod \; 10$. ■

Proposition 10 [8]: *Every even perfect number ends in either 6 or 8.*

Theorem 11: *An even triangular number T_{4k} for each $k \geq 1$cannot be written in the form of*

$2^{n-1}(2^n - 1)$ *for any* $n \geq 2$.

Proof: Suppose $T_{4k} = \frac{4k(4k+1)}{2} = 2k(4k + 1) = 2^{n-1}(2^n - 1)$ for some $n \geq 2$ and $k \geq 1$. Then,

$2k(4k + 1) = 2^{n-1}(2^n - 1)$ iff $4k^2 + k = 2^{n-2}(2^n - 1)$

$\Leftrightarrow 4k^2 + k - 2^{n-2}(2^n - 1) = 0$

$\Leftrightarrow (4k - (2^n - 1))(k + 2^{n-2}) = 0$

$\Leftrightarrow 4k = 2^n - 1$ or $k + 2^{n-2} = 0$

$\Leftrightarrow k = \frac{2^n - 1}{2^2}$ (not an integer) or $k = -2^{n-2}$(not a positive integer)

$\Rightarrow k \in \emptyset \rightarrow \leftarrow$

Hence $T_{4k} \neq 2^{n-1}(2^n - 1)$ for any $k \geq 1$ and $n \geq 2$. ■

***Corollary 12**: If an even triangular number T_m is perfect, then $m = (4k - 1)$ for some $k \geq 1$.*

6	10	28	36	66	78	120	136	190	210	276	300	378	406
2*3	2*5	4*7	4*9	6*11	6*13	8*15	8*17	10*19	10*21	12*23	12*25	14*27	14*29
t_3	t_4	t_7	t_8	t_{11}	t_{12}	t_{15}	t_{16}	t_{19}	t_{20}	t_{23}	t_{24}	t_{27}	t_{28}
$2^{2-1} * (2^2 - 1)$		$2^{3-1} * (2^3 - 1)$				$2^{4-1} * (2^4 - 1)$							

496	528	630	666	780	820	946	990	1128	1176	1326	1378	1540	1540
16*31	16*33	18*35	18*37	20*39	20*41	22*43	22*45	24*47	24*49	26*51	26*53	28*55	28*57
t_{31}	t_{32}	t_{33}	t_{36}	t_{35}	t_{40}	t_{37}	t_{44}	t_{39}	t_{48}	t_{41}	t_{52}	t_{43}	t_{56}
$2^{5-1} * (2^5 - 1)$													

Table III: **Even Triangular Numbers with some in $2^{n-1}(2^n - 1)$ form.**

Theorem 13**:** *An even triangular number* T_m *is perfect if and only if* $m = (2^t - 1)$ *for some prime number* t *.*

Proof: A triangular number T_m is even if and only if either $m = (4k-1)$ or $m = 4k$ for some $k \geq 1$ and an every prime number $p > 2$ is of the form $p = (4l+3)$ or $p = 4l+1$. [7]

Suppose T_m is perfect.

This implies $T_m = 2^{n-1}(2^n - 1)$ and $(2^n - 1)$ is prime. But $(2^n - 1)$ prime only if n is prime number. By

(Theorem 11) and the later remark above the only choice for m is $m = (4k-1)$ = $4(k-1)+3$ = $4l+3$

but not $m = 4k$.

Hence, $T_m = \sum_{i=1}^{m} i = \sum_{i=1}^{4k-1} i = (2k)(4k-1) = 2^{n-1}(2^n - 1)$ and

$T_m = 2^{n-1}(2^n - 1) \Leftrightarrow (4k-1)(2k) = 2^{n-1}(2^n - 1)$ (1)

It is easy to show that gcd$(2k, 4k-1)$ = gcd $(2^{n-1}, 2^n - 1)$ = 1

= gcd$(2k, 2^n - 1)$ = gcd$(2^{n-1}, 4k-1)$. (2)

We use (2) to show that $(4k-1)$ = $(2^n - 1)$.

Because $(2^n - 1)$is prime, either gcd $(4k-1, 2^n - 1) = 1$ or $(2^n - 1) | (4k-1)$.

If the former is true, incorporating with what we have in (2),

gcd$(2k, 4k-1)$ = gcd $(2^{n-1}, 2^n - 1)$ = 1

= gcd$(2k, 2^n - 1)$ = 1 = gcd$(2^{n-1}, 4k-1)$

= gcd $(4k-1, 2^n - 1) = 1$ and is clear to see that $(4k-1) = (2^n - 1)$ and then $2k = 2^{n-1}$.

If the later holds, then there exists $d \in \mathbb{Z}^+$ such that $(4k-1) = d(2^n - 1)$. (3)

As $(4k-1)$ and $(2^n - 1)$ are both odd, this implies d is an odd integer too.

Substituting (3) into (1) we have,

$(4k-1)(2k) =$ (2k) $(2^n - 1)\, d = 2^{n-1}(2^n - 1)$. This implies (2k) d $= 2^{n-1}$ and either

$d = \frac{2^{n-1}}{2k} \in \mathbb{Q} \wedge d = \frac{2^{n-1}}{2k} \notin \mathbb{Z}^+$or $d = \frac{2^{n-1}}{2k} \in \mathbb{Z}^+$ and is an even integer for the only factors of

2^{n-1} are multiples of 2, which is a contradiction to d is an odd integer.

Hence$(2^n-1) \mid (4k-1)$ only when $d=1$ and hence $(4k-1)=(2^n-1)$ and $2k=2^{n-1}$.

Consequently,

$(2k)(4k-1) = 2^{n-1}(2^n-1) \Leftrightarrow (4k-1)=(2^n-1)$ and $2k = 2^{n-1}$

$\Leftrightarrow 4k=2^n$ and $k=2^{n-2} \Leftrightarrow k=2^{n-2}$ for some prime n .

Thus $m=4k-1=2^2 2^{n-2}-1= (2^n-1)$, and if an even Triangular number T_m is perfect, $m= (2^t-1)$ for some prime number t. ■

Alternative proof:

$(2k)(4k-1) = 2^{n-1}(2^n-1) \Leftrightarrow 8k^2-2k = 2^{n-1}(2^n-1)$

$\Leftrightarrow 4k^2-k = 2^{n-2}(2^n-1)$

$\Leftrightarrow 4k^2-k - 2^{n-2}(2^n-1) = 0 \Leftrightarrow 4k^2-k - 2^{n-2}(2^n-1) = 0$

$\Leftrightarrow 4k^2-2^n k + (2^n-1)k - 2^{n-2}(2^n-1) = 0$

$\Leftrightarrow 4k(k-2^{n-2}) + (2^n-1)(k - 2^{n-2}) = 0$

$\Leftrightarrow (k-2^{n-2})(4k+2^n-1) = 0 \Leftrightarrow k = 2^{n-2}$ or $4k = 1-2^n$

$\Leftrightarrow k = 2^{n-2}$ or $k = 2^2-2^{n-2} = 2^{-2}(1-2^n)$

$\Leftrightarrow k = 2^{n-2}$ or $k \epsilon \emptyset$ (because $(1-2^n)<0, \forall\, n \geq 3$ and $k=\frac{1-2^n}{2^2} \notin \mathbb{Z}^+$).

$\Rightarrow k = 2^{n-2}$.

Consequently, $m = 4k-1 = 4(2^{n-2})-1 = (2^n-1)$, where n is prime. ■

Conversely, suppose T_m is an even triangular number where $m = (2^t-1)$ for some prime number t .

Then $T_m = T_{2^t-1} = \sum_{i=1}^{2^t-1} i$

$= \frac{(2^t-1)(2^t-1+1)}{2} = (2^{t-1})(2^t-1) = \mathrm{N}$ and which is perfect. ■

Corollary 14: *An even triangular T_M is perfect if and only if M is a* ***Mersenne prime.***

Theorem 15: *An even triangular number T_{4k-1} is not perfect if $k \equiv -1\, mod\, 5$ or $k \equiv 0\, mod\, 5$.*

Proof: Consider an even triangular number T_{4k-1}. Suppose $k \equiv -1\, mod\, 5$ or

$k \equiv 0 \, mod \, 5$. Then $k \equiv -1 \, mod \, 5$ if and only if $k \equiv 4 \, mod \, 5$ if and only if $5 \mid (k-4)$

if and only if $k = 4 + 5t$ for some $t \in \mathbb{Z}^+$.

Hence $T_{4k-1} = T_{4(4+5t)-1} = T_{15+20t} = \frac{(15+20t)(20t+16)}{2}$

$= \frac{20(3+4t)(4+5t)}{2} = 10(3+4t)(4+5t)$ and $10 \mid T_{4k-1}$.

Consequently T_{4k-1} is not a perfect number. (Proposition 10).

Similarly if $k \equiv 0 \, mod \, 5$ one can show that $10 \mid T_{4k-1}$ which implies it is not perfect. ■

Theorem 16 : *If a triangular number T_m is perfect , then*

$$T_{(2^n-1)} = T^2_{\left(2^{\frac{n+1}{2}}\right)} - 2^3 T^2_{\left(2^{\frac{n-1}{2}}\right)} = \sum_{k=1}^{\left(2^{\frac{n-1}{2}}\right)} (2k-1)^3 .$$

Proof: If a triangular number T_m is perfect then $m = (2^n - 1)$ where n is a prime number (Theorem 13).

$$T_n^2 = \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

$$\Rightarrow T^2_{\left(2^{\frac{n+1}{2}}\right)} = \sum_{i=1}^{\left(2^{\frac{n+1}{2}}\right)} i^3 = \left(\frac{2^{\left(\frac{n+1}{2}\right)}\left(2^{\left(\frac{n+1}{2}\right)}+1\right)}{2}\right)^2 = \frac{2^{(n+1)}\left(2^{\left(\frac{n+1}{2}\right)}+1\right)^2}{4} =$$

$$2^{n-1}\left(2^{\frac{n+1}{2}}+1\right)^2 \text{ and} \quad (4)$$

$$T^2_{\left(2^{\frac{n-1}{2}}\right)} = \sum_{i=1}^{\left(2^{\frac{n-1}{2}}\right)} i^3 = \left(\frac{2^{\left(\frac{n-1}{2}\right)}\left(2^{\left(\frac{n-1}{2}\right)}+1\right)}{2}\right)^2 = \frac{2^{(n-1)}\left(2^{\left(\frac{n-1}{2}\right)}+1\right)^2}{4} =$$

$$2^{(n-3)}\left(2^{\left(\frac{n-1}{2}\right)}+1\right)^2$$

$$\Rightarrow \quad 2^3 T^2_{\left(2^{\frac{n-1}{2}}\right)} = 2^3 . 2^{(n-3)}\left(2^{\left(\frac{n-1}{2}\right)}+1\right)^2 = 2^n\left(2^{\left(\frac{n-1}{2}\right)}+1\right)^2 \quad (5)$$

Combining (4) and (5) we have,

$$T^2_{\left(2^{\frac{n+1}{2}}\right)} \ - \ T^2_{\left(2^{\frac{n-1}{2}}\right)} \ = \ 2^{(n-1)}\left(2^{\left(\frac{n+1}{2}\right)}+1\right)^2 \ - \ 2^n\left(2^{\left(\frac{n-1}{2}\right)}+1\right)^2$$

$$= \ (2^{n-1})\left(\left(2^{\left(\frac{n+1}{2}\right)}+1\right)^2 - \ 2\left(2^{\left(\frac{n-1}{2}\right)}+1\right)^2\right)$$

$$= \ (2^{n-1})\left((2^{n+1}+2.2^{\left(\frac{n+1}{2}\right)}+1) - 2^1(\ 2^{n-1}+2.2^{\left(\frac{n-1}{2}\right)}+1)\right)$$

$$= (2^{n-1})\ ((2^{n+1}+.2^{\left(\frac{n+3}{2}\right)}+1\ ) - \ 2^1(\ 2^{n-1}+2^{\left(\frac{n+1}{2}\right)}+1))$$

$$= (2^{n-1})\left(2^{n+1}+2^{\left(\frac{n+3}{2}\right)}+1 - \ 2^n - 2^{\left(\frac{n+3}{2}\right)} - 2\right)$$

$$= \ (2^{n-1})(\ 2^{n+1} - 2^n - 1) = \ (2^{n-1})(\ 2.2^n - 2^n - 1)$$

$$= \ (2^{n-1})(\ 2^n \ - 1) \ = \frac{2^n(2^n-1)}{2} \ = \ T_{(2^n-1)} \qquad \blacksquare \qquad (6)$$

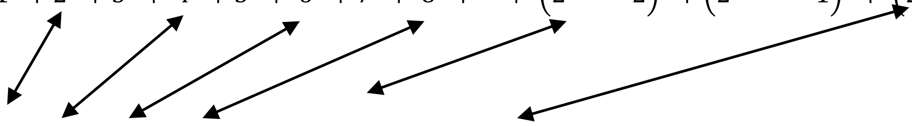
Next we show that, $T_{(2^n-1)} \ = \ \sum_{i=1}^{\left(2^{\frac{n-1}{2}}\right)}(2i-1)^3$.

$$T_{(2^n-1)} = \ T^2_{\left(2^{\frac{n+1}{2}}\right)} \ - \ 2^3\,T^2_{\left(2^{\frac{n-1}{2}}\right)} \ = \ \sum_{i=1}^{\left(2^{\frac{n+1}{2}}\right)}(i)^3 \ - \ 2^3\sum_{i=1}^{\left(2^{\frac{n-1}{2}}\right)}(i)^3$$

$$= \left(1^2+2^3+3^3+\ \ldots + \ \left(2^{\left(\frac{n+1}{2}\right)}\right)^3\right) - \ 2^3\left(1^2+2^3+3^3+\ \ldots + \ \left(2^{\left(\frac{n-1}{2}\right)}\right)^3\right)$$

$$= \left(1^2+2^3+3^3+\ \ldots + \ \left(2^{\left(\frac{n+1}{2}\right)}\right)^3\right) - \left(2^3+4^3+6^3+\ \ldots + \ \left(2^{\left(\frac{n+1}{2}\right)}\right)^3\right)$$

$$= \left(1^2+2^3+3^3+\ 4^3+5^3+\ 6^3+7^3+\ 8^3+\cdots+\left(2^{\frac{n+1}{2}}-2\right)^3+\left(2^{\left(\frac{n+1}{2}\right)}-1\right)^3+\left(2^{\frac{n+1}{2}}\right)^3\right)$$



$$- \left(2^3+\ 4^3+\ 6^3+\ 8^3\ldots+\left(2^{\frac{n+1}{2}}-2\right)^3+\ \left(2^{\frac{n+1}{2}}\right)^3\right)$$

$$=\left(1^2+3^3+5^3+\ \ 7^3+\cdots+\left(2^{\left(\frac{n+1}{2}\right)}-1\right)^3\right) = \left(1^2+3^3+5^3+\ \ 7^3+\cdots+\left(2.2^{\left(\frac{n-1}{2}\right)}-1\right)^3\right)$$

$= (1^2+3^3+5^3+\ \ 7^3+\cdots+(2k-1)^3)$ where $k = \ 2^{\left(\frac{n-1}{2}\right)}$

$$= \ \sum_{k=1}^{\left(2^{\frac{n-1}{2}}\right)}(2k-1)^3$$

This implies $T_{2^n-1} \ = \ \sum_{k=1}^{2^{\left(\frac{n-1}{2}\right)}}(2k-1)^3$. $\blacksquare$ (7)

From (6) and (7) it follows,

$$T_{(2^n-1)} = T^2_{\left(2^{\frac{n+1}{2}}\right)} - 2^3 T^2_{\left(2^{\frac{n-1}{2}}\right)} = \sum_{k=1}^{\left(2^{\frac{n-1}{2}}\right)} (2k-1)^3 \ . \qquad \blacksquare$$

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