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A nonlocal diffusion equation revisited

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Abstract

We consider a class of elliptic problems where the diffusion depends on the range of nonlocal interactions. In a radial setting, we address the issue of existence of a continuum of solutions, giving a generalization of Chipot-Lovat theorem [7].

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1. Introduction

The aim of this paper is to study the elliptic equation following:

$$(P_r)\begin{cases} -div(a(L_r(u))\nabla u) = f \quad in \quad \Omega \\ u \in H_0^1(\Omega). \end{cases}$$

In the above problem Ω is a bounded open set of $\mathbb{R}^n$, a be a continuous function satisfying

$$\exists m, M \quad \text{such that} \quad 0 < m \leq a(\epsilon) \leq M \quad \forall \epsilon \in \mathbb{R} \tag{1}$$

and $f \in L^2(\Omega)$. The nonlocal functional L_r is defined such that

$$L_r(.)(x) : L^2(\Omega) \to \mathbb{R}, \quad u \to L_r(u)(x) = \int_{\Omega \cap B(x,r)} u(y)dy. \tag{2}$$

Here $B(x, r)$ is the closed ball of $\mathbb{R}^n$ with radius r.

This problem has been studied in various forms (see [6], [7], [8], [9], [10], [11]) and we refer the reader to [6] for more details of modelisation.

Our main focus will be on the various problems generated by this type of equations, including the bifurcation theory (see [1]). Indeed when $r = 0$, (P_0) has a unique solution.

However, when $r = d$, where d represents the diameter of Ω the number of solutions is determined by the following theorem due to Chipot and lovat (see [7]).

Let ϕ be the unique solution to the problem

$$\begin{cases} -\Delta\phi = f \quad in \quad \Omega \\ \phi \in H_0^1(\Omega), \end{cases} \tag{3}$$

We have

Theorem 1.1. [7] Assume (1) and $f \in L^2(\Omega)$. Then (P_d) has as many solutions as the equation in μ

$$a(\mu)\mu = L_d(\phi), \quad \mu \in \mathbb{R}.$$

As a consequence, $u = \dfrac{\phi}{a(\mu)}$ is a solution to (P_d) and $\mu = L_d(u)$.

The above theorem describes us the number of solutions of (P_d) taking into account the monotonicity of a. Assume for example that (P_d) admits several solutions. The question that arose is to describe the change in the structure of (P_r) when r varies from 0 to d. Several questions have been solved among them the issue when (P_d) admits a finite numbers of solutions (see [2], [5], [6]). We recall that the main difficulty in studying non-local problems lies in the lack of adequate theory of bifurcation.

Let a be a function with the following properties.

$$a(\mu) = \begin{cases} a_1 & if \quad \mu \leq \mu_1, \\ \dfrac{L_d(\phi)}{\mu} & if \quad \mu_1 \leq \mu \leq \mu_2, \\ a_2 & if \quad \mu_2 \leq \mu, \end{cases} \tag{4}$$

with a_1 and a_2 two positive constants. As a illustration of Theorem 1.1, we deduce that (P_d) admits a continuum of solutions (see [5], [6]).

The rest of this paper is to give the generalization of Theorem 1.1, in radial setting for all $r \in [0, d]$.

2. General case $r \in [0, d]$

The main result of this section is Theorem 2.4, which gives us the existence of a continuum of solutions to (P_r) with respect to the parameter r. In order to be in radial symmetric setting, we will assume that Ω is the open ball of $\mathbb{R}^n$ with radius $d/2$ and centered at zero. We denote by $L_r^2(\Omega)$ the subspace of $L^2(\Omega)$ consisting in radial solutions and following assumptions:

$$f \in L_r^2(\Omega), \quad f \geq 0 \quad \text{a.e. in} \quad \Omega \tag{5}$$

$$a \in C(\mathbb{R}, \mathbb{R}), \quad \inf_{\mathbb{R}} a > 0, \quad \sup_{\mathbb{R}} a < \infty. \tag{6}$$

We get an existence result

Lemma 2.1. [5] Assume (5) and (6) hold true. Then (P_r) admits at least a radial solution.

Let ϕ be the solution to (3). We denote by I_r the interval

$$I_r := [\inf_{\Omega} L_r(\phi),\ \sup_{\Omega} L_r(\phi)] \quad \forall r \in [0, d].$$

Proposition 2.2. [5] Assume (5) and (6) hold true. Let $r \in [0, d]$ and suppose that the function a has the following properties. There exists $0 \leq m_1 \leq m_2$ such that

$$\begin{aligned} &a(m_1) = \max_{[m_1,m_2]} a \quad a(m_2) = \min_{[m_1,m_2]} a \\ &m_1\, a(m_1) \leq min\ I_r \quad max\ I_r \leq m_2\, a(m_2). \end{aligned} \tag{7}$$

Then (P_r) admits a radial solution u and

$$m_1 \leq L_r(u) \leq m_2 \quad a.e \quad in \quad \Omega.$$

Remark 2.3. The above Proposition proved in [5] gives us existence of a radial solution and an estimation of this radial solution. This reasoning is very crucial to prove the main result of this section.

We are now in position to give the main result.

Theorem 2.4. Assume (5) and (6) hold true. Let $r \in [0, d]$ and the function a satisfying the following properties.

There exists two positive constants m_0, m_1 $(m_0 < m_1)$ such that for all positive constants $\{m_n\}_{n=2,\cdots}$ with $m_n < m_{n+1}$ and $m_n, m_{n+1} \in]m_{n-2}, m_{n-1}[$, we have for all $i \in \mathbb{N}$,

$$\begin{aligned} &m_{i+3} \max_{[m_{i+3},m_{i+1}]} a \leq m_i \max_{[m_i,m_{i+2}]} a \leq min\ I_r \\ &max\ I_r \leq m_{i+1} \min_{[m_{i+3},m_{i+1}]} a \leq m_{i+2} \min_{[m_i,m_{i+2}]} a. \end{aligned} \tag{8}$$

Then (P_r) admits a continuum of solutions such that

$$\begin{aligned} &m_i \leq L_r(u_i) \leq m_{i+2} \quad \text{if } i \text{ is even,} \\ &m_{i+2} \leq L_r(u_i) \leq m_i \quad \text{if } i \text{ is odd.} \end{aligned} \tag{9}$$

Proof. From (8), we get

$$m_3 \max_{[m_3,m_1]} a \leq m_0 \max_{[m_0,m_2]} a \leq min\ I_r \tag{10}$$

and

$$min\ I_r \leq m_1 \min_{[m_3,m_1]} a \leq m_2 \min_{[m_0,m_2]} a. \tag{11}$$

We apply Proposition 2.2 to get existence of two solutions u_0 and u_1 such that

$$\begin{aligned} m_0 \le L_r(u_0) \le m_2, \\ m_3 \le L_r(u_1) \le m_1. \end{aligned} \tag{12}$$

Since $m_2 < m_3$ then u_0 and u_1 are distinct solutions. Again from (8), there exists two positive constants m_4 and m_5 with $m_4, m_5 \in]m_2, m_3[$ such that

$$m_5 \max_{[m_5,m_3]} a \le m_2 \max_{[m_2,m_4]} a \le min\, I_r \tag{13}$$

and

$$min\, I_r \le m_3 \min_{[m_5,m_3]} a \le m_4 \min_{[m_2,m_4]} a. \tag{14}$$

We again apply Proposition 2.2 to get existence of two distinct solutions u_2 and u_3 such that

$$\begin{aligned} m_2 \le L_r(u_2) \le m_4, \\ m_5 \le L_r(u_3) \le m_3. \end{aligned} \tag{15}$$

We can repeat this procedure to obtain a continuum of solutions $\{u_i\}_{i\in\mathbb{N}}$ satisfying (9). This completes the proof of theorem. ■

Remark 2.5. The previous theorem is very important especially when one wishes for the parabolic problems focus in asymptotic behaviour and the attractor theory (see [3], [4]).

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