Rotational and Self-similar Blowup Solutions for the Euler Equations for Generalized Chaplygin Gas

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Abstract

In this paper, we establish a family of rotational and self-similar solutions for the Euler equations for generalized Chaplygin gas which occurs in cosmology model as a unification of dark matter and dark energy. By analyzing a system of ordinary differential equations, we show that our family of solutions blows up on finite time in 4 cases out of 6 according to the classification of parameters.

Keywords: Rotational, Self-similar, Blowup, Euler equations, Generalized Chaplygin Gas
MSC: 35Q, 35C06, 76N10,

1. INTRODUCTION AND MAIN RESULTS

In this paper, we consider the 3-dimensional Euler equations for Generalized Chaplygin Gas (GCG):

\[
\begin{align*}
\rho_t + \nabla \cdot (\rho \mathbf{u}) &= 0, \\
\rho [\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u}] + \nabla p &= 0, \\
p &= -Ap^{-\gamma}, \quad A > 0, \quad 0 < \gamma < 1,
\end{align*}
\]

(1)

Where \( \rho(t, \mathbf{x}): (\mathbb{R}_{\geq 0}, \mathbb{R}^3) \to \mathbb{R}_{\geq 0}, \mathbf{u}(t, \mathbf{x}): (\mathbb{R}_{\geq 0}, \mathbb{R}^3) \to \mathbb{R}^3 \) represent the density and velocity of the substance considered respectively. \( p \) is the pressure function which is governed by the state equation (1)\(_3\). (1)\(_1\) is derived from the mass conservation law while (1)\(_2\) is a result of the momentum conservation law.

Since 1998, type Ia supernova (SNIa) observations [4] have shown that our universe

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has entered into a phase of accelerating expansion. During these years from that time, many additional observational results, including current Cosmic Microwave Background anisotropy measurement [5], and the data of the Large Scale Structure [3], also strongly support this suggestion. And these cosmic observations indicate that baryon matter component is about 4% for total energy density and about 96% energy density in universe is invisible. Considering the four-dimensional standard cosmology, this accelerated expansion for universe predicts that dark energy (DE) as an exotic component with negative pressure is filled in universe. And it is shown that DE takes up about two-third of the total energy density from cosmic observations. The remaining one third is dark matter (DM). In theory mounting DE models have already been constructed. But there exists another possibility: that the invisible energy component is a unified dark fluid. i.e. a mixture of dark matter and dark energy. The GCG model was introduced in [6] in 2012 as a unification of dark matter and dark energy.

For recent researches of system (1), readers may refer to [1] and [2]. On another aspect, it is known that in [7], the first equation of (1) enjoys a rotational and self-similar solutions of the following form.

\[
\begin{align*}
\rho &= \frac{f(s)}{a^2(t)b(t)}, \\
u_1 &= \frac{\dot{a}(t)}{a(t)}x - G(t)y, \\
u_2 &= G(t)x + \frac{\dot{a}(t)}{a(t)}y, \\
u_3 &= \frac{\dot{b}(t)}{b(t)}z,
\end{align*}
\]

with a self-similar variable \( S = \frac{x^2 + y^2}{a^2(t)} + \frac{z^2}{b^2(t)} \) and arbitrary \( C^1 \) functions \( f(s) \geq 0, G(t), a(t) > 0 \) and \( b(t) > 0 \).

Using (2), the author of [7] constructed a family of blowup solutions and solutions that are global in time for the compressible classical Euler equations. In this paper, we aim at constructing a family of rotational and self-similar solutions for the 3-dimensional Euler equations for GCG model. It is shown that our family of solutions includes both blowup type and global in time type.

For 3-dimensional case, we adopt the following notations.

\[
\mathbf{u} = (u_1, u_2, u_3),
\]

\[
\mathbf{x} = (x, y, z).
\]
**Theorem 1.** For system (1), we have the following family of rotational and self-similar exact solutions.

\[
\begin{align*}
\rho &= \frac{f(s)}{a^2(t)b(t)} s^2 = \frac{x^2 + y^2}{a^2(t)} + \frac{z^2}{b^2(t)}, \\
u_1 &= \frac{\dot{a}(t)}{a(t)} x - \frac{\xi}{a^2(t)} y, \\
u_2 &= \frac{\xi}{a^2(t)} x + \frac{\dot{a}(t)}{a(t)} y, \\
u_3 &= \frac{\dot{b}(t)}{b(t)} z,
\end{align*}
\]  

where \( f(s) \) is positive and is given by

\[
\frac{1}{f^{\gamma+1}(s)} = \alpha + \frac{\lambda(\gamma + 1)}{2K\gamma} s > 0
\]

for any positive \( \alpha \).

\( a(t) \) and \( b(t) \) satisfy the following ordinary differential equations (ODEs).

\[
\begin{align*}
\dot{a} - \frac{\xi^2}{a^3} &= \lambda a^{2\gamma+1}b^{\gamma+1}, \\
\dot{b} &= \lambda b^{\gamma} a^{2\gamma+2}, \\
a(0) &=: a_0 > 0, \dot{a}(0) =: a_1, \\
b(0) &=: b_0 > 0, \dot{b}(0) =: b_1,
\end{align*}
\]

where \( \lambda \) and \( \xi \) are arbitrary constants.

**Remark 2.** It can be easily checked that the \( f \) given by (6) satisfies the following ODE.

\[
\lambda + \frac{2K\gamma}{f^{\gamma+2}} = 0.
\]

More importantly, the blowup and global existence results of the family of solutions in theorem are included in the following theorem.

**Theorem 3.** For the family of solutions (5)-(7), we have the following 6 cases.

**Case 1:** \( \lambda = 0 \) and \( \xi = 0 \).

If \( a_1 \) and \( b_1 \) are both nonnegative, then the solutions (5) exist globally. Otherwise, the solutions (5) blow up on finite time.

**Case 2:** \( \lambda = 0 \) and \( \xi \neq 0 \).

The system (7) can be solved by
\[
\begin{align*}
\left\{ \begin{array}{l}
a = \frac{\sqrt{\xi^2 + (\xi|\xi| - ht)^2}}{\sqrt{h}}, \\
b = b_0 + b_1 t
\end{array} \right. \\
\text{or}
\left\{ \begin{array}{l}
a \equiv \frac{|\xi|}{\sqrt{h}}, \\
b = b_0 + b_1 t,
\end{array} \right.
\end{align*}
\]

where
\[
h := a_1^2 + \left( \frac{\xi}{a_0} \right)^2 > 0.
\]

Moreover, if \( b_1 \geq 0 \), the solutions (5) exist globally. Otherwise, the solutions (5) blow up on finite time.

Case 3: \( \lambda < 0 \) and \( \xi = 0 \).
the solutions (5) blow up on finite time.

Case 4: \( \lambda < 0 \) and \( \xi \neq 0 \).
the solutions (5) blow up on finite time.

Case 5: \( \lambda > 0 \) and \( \xi = 0 \).
the solutions (5) exist globally.

Case 6: \( \lambda > 0 \) and \( \xi \neq 0 \).
then the solutions (5) exist globally for the following two cases.

Case i: \( b_1 \geq 0 \).

Case ii: \( b_1 < 0 \) and
\[
b_1^2 < \frac{2\lambda \xi^{2\gamma + 2} b_0^{\gamma + 1}}{h^{\gamma + 1} (\gamma + 1)},
\]

where
\[
h := a_1^2 + \left( \frac{\xi}{a_0} \right)^2 > 0.
\]

2. PROOF OF THEOREM 1

To start the proof of Theorem 1, we need the following lemma which is given by Lemma 4 in [7].

Lemma 4. For the mass equation \((1)_1\), we have the following solutions in a general form.
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\[
\rho = \frac{f(s)}{a^2(t)b(t)}, \quad s = \frac{x^2 + y^2}{a^2(t)} + \frac{z^2}{b^2(t)},
\]

\[
\begin{align*}
  u_1 &= \frac{\dot{a}(t)}{a(t)} x - G(t)y \\
  u_2 &= G(t)x + \frac{\dot{a}(t)}{a(t)} y \\
  u_3 &= \frac{\dot{b}(t)}{b(t)} z,
\end{align*}
\]

(14)

Now, we present the proof of Theorem 1.

**Proof of Theorem 1.**

**Step 1.** By Lemma 4 with \( G(t) = \frac{\xi}{a^2(t)} \), the mass equation is satisfied by solutions (5). Note that if we define \( G(t) = \frac{\xi}{a^2(t)} \), then the following equation is satisfied.

\[
\dot{G} + 2G \frac{\dot{a}}{a} = 0.
\]

(15)

We are ready to check that the momentum equations are satisfied by solutions (5). Note that (1) is equivalent to

\[
\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho} \nabla p = 0.
\]

(16)

In component form, we have

\[
\begin{align*}
  u_{1t} + u_1 u_{1x} + u_2 u_{1y} + u_3 u_{1z} + \frac{1}{\rho} p_x &= 0, \\
  u_{2t} + u_1 u_{2x} + u_2 u_{2y} + u_3 u_{2z} + \frac{1}{\rho} p_y &= 0, \\
  u_{3t} + u_1 u_{3x} + u_2 u_{3y} + u_3 u_{3z} + \frac{1}{\rho} p_z &= 0.
\end{align*}
\]

(17) (18) (19)

**Step 2.** For (17), we have

\[
\begin{align*}
  u_{1t} &= u_1 u_{1x} + u_2 u_{1y} + u_3 u_{1z} + \frac{1}{\rho} p_x \\
  &= \left[ \frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 \right] x - \dot{G} y + \left( \frac{\dot{a}}{a} x - G y \right) \frac{\dot{a}}{a} + \left( G x + \frac{\dot{a}}{a} y \right) (-G) \\
  &\quad + 0 + \frac{1}{\rho} p_x \\
  &= \frac{\ddot{a}}{a} x - \dot{G} y - 2G \frac{\dot{a}}{a} y - G^2 x + \frac{1}{\rho} p_x \\
  &= \left( \frac{\ddot{a}}{a} - G^2 \right) x - \left( \dot{G} + 2G \frac{\dot{a}}{a} - G^2 \right) y + \frac{1}{\rho} p_x \\
  &= \left( \frac{\ddot{a}}{a} - G^2 \right) x + \frac{1}{\rho} p_x \quad \text{by (15)}
\end{align*}
\]

(20) (21) (22) (23) (24)
\[
\begin{align*}
(\ddot{a} - \frac{\xi^2}{a^4})x + \frac{1}{\rho} p_x & \quad \text{by } G(t) = \frac{\xi}{a^2(t)} \\
= (\ddot{a} - \frac{\xi^2}{a^4})x + \frac{1}{\rho} p_x & \quad \text{(25)} \\
= (\lambda a^{2\gamma+1} b^{\gamma+1}) \frac{x}{a} + \frac{1}{\rho} p_x & \quad \text{by (7)_1} \\
= (\lambda a^{2\gamma} b^{\gamma+1})x + K\gamma \frac{a^{2\gamma} b^{\gamma+1}}{f^{\gamma+2}} x & \quad \text{by (5)_1} \\
= a^{2\gamma} b^{\gamma+1}x \left( \lambda + 2K\gamma \frac{\dot{f}}{f^{\gamma+2}} \right) & \quad \text{(30)} \\
= 0 \quad \text{by (8).} & \quad \text{(31)}
\end{align*}
\]

Thus, the first momentum equation is satisfied.

**Step 3.** For (18), we have
\[
\begin{align*}
\dot{u}_{2t} + u_1 u_{2x} + u_2 u_{2y} + u_3 u_{2z} + \frac{1}{\rho} p_y & \quad (32) \\
= \dot{\dot{x}} + \left[ \frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 \right] y + \left( \frac{\ddot{a}}{a} x - G\gamma \right) \dot{G} + \left( G\gamma + \frac{\dot{a}}{a} y \right) \ddot{a} + 0 & \quad (33) \\
& \quad + \frac{1}{\rho} p_y \\
= (\dot{\gamma} + 2G \frac{\dot{\dot{a}}}{a}) x + \left( \frac{\ddot{a}}{a} - G^2 \right) y + \frac{1}{\rho} p_y & \quad (34) \\
& \quad + \left( \frac{\ddot{a}}{a} - \frac{\xi^2}{a^4} \right) y + \frac{1}{\rho} p_y & \quad (35) \\
& \quad = (\ddot{\dot{a}} - \frac{\xi^2}{a^4}) y + \frac{1}{\rho} p_y & \quad (36) \\
& \quad = (\lambda a^{2\gamma} b^{\gamma+1}) y + \frac{1}{\rho} p_y & \quad (37) \\
& \quad = (\lambda a^{2\gamma} b^{\gamma+1}) y + 2K\gamma \frac{a^{2\gamma} b^{\gamma+1}}{f^{\gamma+2}} y & \quad (38) \\
& \quad = a^{2\gamma} b^{\gamma+1} y \left( \lambda + 2K\gamma \frac{\dot{f}}{f^{\gamma+2}} \right) = 0. & \quad (39)
\end{align*}
\]

Thus, the second momentum equation is satisfied.

**Remark 5.** It is not obvious that the steps in Step 3 have a similar structure to the steps in Step 2. Thus, we include the details above.
Step 4. For (19), we have

\[ u_{3t} + u_1u_{3x} + u_2u_{3y} + u_3u_{3z} + \frac{1}{\rho} p_z \]

(40)

\[ = \left[ \frac{\dot{b}}{b} - \left( \frac{\dot{b}}{b} \right)^2 \right] z + 0 + 0 + \frac{\dot{b}}{b} z \left( \frac{\dot{b}}{b} \right) + \frac{1}{\rho} p_z \]

(41)

\[ = \frac{\dot{b}}{b} z + \frac{1}{\rho} p_z \]

(42)

\[ = \lambda b^{\gamma-1} a^{2\gamma+2} z + \frac{1}{\rho} p_z \]

by (7)_2 (43)

\[ = \lambda b^{\gamma-1} a^{2\gamma+2} z + 2k\gamma \frac{a^{2\gamma+2} b^{\gamma-1} \dot{f}}{f^{\gamma+2}} \]

(44)

\[ = a^{2\gamma+2} b^{\gamma-1} \left( \lambda + 2k\gamma \frac{\dot{f}}{f^{\gamma+2}} \right) \]

(45)

\[ = 0. \]

(46)

Thus, the third momentum equation is satisfied.
The proof is hence complete.

3. BLOWUP OR GLOBAL EXISTENCE

For the proof of Theorem 3, we consider the system of ODEs (7). We have the following six cases to analyze.

Case 1: \( \lambda = 0 \) and \( \xi = 0 \).
Case 2: \( \lambda = 0 \) and \( \xi \neq 0 \).
Case 3: \( \lambda < 0 \) and \( \xi = 0 \).
Case 4: \( \lambda < 0 \) and \( \xi \neq 0 \).
Case 5: \( \lambda > 0 \) and \( \xi = 0 \).
Case 6: \( \lambda > 0 \) and \( \xi \neq 0 \).

Case 1 is trivial as both \( a \) and \( b \) are linear. More precisely, we have the following proposition.

**Proposition 6.** For Case 1, if \( a_1 \) and \( b_1 \) are both nonnegative, then the solutions (5) globally exist. Otherwise, the solutions (5) blow up on finite time.

**Proof.** If both \( a_1 \) and \( b_1 \) are nonnegative, then \( a(t) = a_0 + a_1 t \) and \( b(t) = b_0 + b_1 t \) are always positive. Hence, the solutions are global in time. If \( a_1 < 0 \), then \( u_1 = \frac{a_1}{a_0 + a_1 t} x \) will blow up at time \(-a_0/a_1 > 0\). Similarly, if \( b_1 < 0 \), then \( u_3 = \frac{b_1}{b_0 + b_1 t} z \) will blow up at time \(-b_0/b_1 > 0\).

For Case 2, we have the following proposition.
**Proposition 7.** For Case 2, the system (7) can be solved by

\[
\begin{aligned}
a &= \frac{\sqrt{\xi^2 + (C|\xi| - ht)^2}}{\sqrt{h}}, \\
b &= b_0 + b_1 t
\end{aligned}
\]  

(47)

or

\[
\begin{aligned}
a &= \frac{|\xi|}{\sqrt{h}}, \\
b &= b_0 + b_1 t,
\end{aligned}
\]  

(48)

where

\[h := a_1^2 + \left(\frac{\xi}{a_0}\right)^2 > 0.
\]

(49)

Moreover, if \(b_1 \geq 0\), the solutions (5) exist globally. Otherwise, the solutions (5) blow up on finite time.

**Proof.** In this case, the system (7) becomes

\[
\begin{aligned}
\dot{a} - \frac{\xi^2}{a^3} &= 0, \\
\dot{b} &= 0.
\end{aligned}
\]

(50)

From \((50)_1\), we have

\[
(\dot{a})^2 + \left(\frac{\xi}{a}\right)^2 = h,
\]

(51)

where

\[h := a_1^2 + \left(\frac{\xi}{a_0}\right)^2 > 0.
\]

(52)

Thus, we have

\[
\begin{aligned}
\dot{a}(t) &= \sqrt{h} \cos \theta(t), \\
\dot{\frac{|\xi|}{a(t)}} &= \sqrt{h} \sin \theta(t),
\end{aligned}
\]

(53)

for some \(\theta(t)\). From \((53)_2\), one has

\[
a = \frac{|\xi|}{\sqrt{h} \sin \theta}
\]

(54)

\[
\dot{a} = \frac{|\xi| - \cos \theta}{\sqrt{h} \sin^2 \theta} \theta_t.
\]

(55)
Combining (55) with (53), one has
\[
|\xi| - \cos \theta \frac{\theta_t}{\sqrt{h} \sin^2 \theta} = \sqrt{h} \cos \theta 
\]
(56)
\[
\cos \theta = 0 \text{ or } \frac{d\theta}{dt} = \frac{h}{|\xi|} \sin^2 \theta
\]
(57)
\[
\theta = \frac{\pi}{2} \text{ or } \theta = \cot^{-1} \left( \frac{C|\xi| - ht}{|\xi|} \right),
\]
(58)
for some arbitrary constant \( C \).

Thus, system (53) can be solved by
\[
a \equiv \frac{|\xi|}{\sqrt{h}}
\]
(59)
or
\[
a = \frac{\sqrt{\xi^2 + (C|\xi| - ht)^2}}{\sqrt{h}}.
\]
(60)
Thus, \( a \) is always positive. As \( b \) is linear, the result follows.

For Case 3, we have the following proposition.

**Proposition 8.** For Case 3, the solutions (5) blow up on finite time.

**Proof.** Note the \( a \) and \( b \) are strictly concave as \( \lambda < 0 \). Suppose the solutions
\[
\begin{cases}
\dot{a} = \lambda a^{\gamma+1} b^{\gamma+1}, \\
\dot{b} = \lambda b^{\gamma} a^{2\gamma+2}, \\
a > 0, \\
b > 0
\end{cases}
\]
(61)
are global. If \( \dot{a} \geq 0 \) and \( \dot{b} \geq 0 \) for all \( t \). Then, \( a \) and \( b \) are increasing. However, as \( \lambda < 0 \),
\[
\dot{a} = a_1 + \lambda \int_0^t a^{\gamma+1}(s)b^{\gamma+1}(s)ds \to -\infty
\]
(62)
as \( t \to +\infty \). This is a contradiction. Thus, either \( \dot{a}(t_1) < 0 \) or \( \dot{b}(t_2) < 0 \) for some \( t_1 \) and \( t_2 \). It follows that \( \dot{a}(t) < 0 \) for all \( t \geq t_1 \) and \( \dot{b}(t) < 0 \) for all \( t \geq t_2 \). This means that either \( a \) or \( b \) will meet the \( t \)-axis eventually. Thus, the solutions are not global.

Next, we consider Case 4.
Proposition 9. For Case 4, the solutions (5) blow up on finite time.

Proof. In this case, we have
\[
\begin{cases}
\ddot{a} - \frac{\xi^2}{a^3} < 0, \\
\dot{b} = \lambda b^\gamma a^{2\gamma + 2}, \\
a \geq 0, b \geq 0.
\end{cases}
\] (63)

From (63)_1, we have
\[
(\dot{a})^2 + \frac{\xi^2}{a^2} \leq h,
\] (64)

where
\[
h := a_1^2 + \left(\frac{\xi}{a_0}\right)^2 > 0.
\] (65)

Thus, \(\frac{\xi^2}{a^2} \leq h\). That is,
\[
a \geq \frac{|\xi|}{\sqrt{h}} > 0.
\] (66)

Suppose \(b\) is global. Then, the solutions (63) are global. However, as \(\lambda < 0\),
\[
\dot{b} = b_1 + \lambda \int_0^t b^\gamma a^{2\gamma + 2} ds \to -\infty
\] as \(t \to +\infty\). Thus, \(b\) will eventually meet the \(t - axis\). This is a contradiction. Thus, \(b\) is not global. Thus we are done in this case. ■

Next, we have the following proposition.

Proposition 10. For Case 5, the solutions (5) exist globally.

Proof. In this case, consider
\[
\begin{cases}
\ddot{a} = \lambda a^{2\gamma + 1} b^{\gamma + 1}, \\
\dot{b} = \lambda b^\gamma a^{2\gamma + 2}, \\
a \geq 0, b \geq 0.
\end{cases}
\] (68)

Note that \(a\) and \(b\) are strictly convex. It is clear that if both \(a_1\) and \(b_1\) are nonnegative, then \(a\) and \(b\) are always positive and the solutions (5) exist globally.

Suppose both \(a_1\) is negative and \(0 < t_1 < +\infty\) is the first zero of \(a\). Note that we have
\[
a \ddot{a} = b \ddot{b}
\] (69)
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For all $t \leq t_1$. As $a$ is strictly decreasing on or before $t_1$, we have

$$\ddot{b} = a\ddot{a} < a_0\ddot{a}$$

(70)

$$\ddot{a} > \frac{b}{a_0}$$

(71)

$$\ddot{a}(t_1) > 0,$$

(72)

which is a contradiction.

Similarly, if $b_1$ is negative and $0 < t_2 < +\infty$ is the first zero of $b$. Then

As $b$ is strictly decreasing on or before $t_2$, we have

$$\ddot{a} = b\ddot{b} < b_0\ddot{b}$$

(73)

$$\ddot{b} > \frac{a}{b_0}\ddot{a}$$

(74)

$$\ddot{b}(t_1) > 0,$$

(75)

which is a contradiction. Thus, both $a$ and $b$ are always positive and hence the solutions (5) are global in time. ■

For the last case, we have the following proposition:

**Proposition 11.** For Case 6, the solutions (5) are global in time for the following two cases.

**Case i:** $b_1 \geq 0$.

**Case ii:** $b_1 < 0$ and

$$b_1^2 < \frac{2\lambda \xi^{2\gamma + 2} b_0^{\gamma + 1}}{h^{\gamma + 1}(\gamma + 1)},$$

(76)

where

$$h = a_1^2 + \frac{\xi^2}{a_0^2} > 0.$$

(77)
Proof. In this case, consider

\[
\begin{align*}
\ddot{a} - \frac{\xi^2}{a^3} &= \lambda a^{2\gamma+1}b^{\gamma+1}, \\
\dot{b} &= \lambda b^{\gamma} a^{2\gamma+2}, \\
a &> 0, b \geq 0.
\end{align*}
\] (78)

From (78)_1, one has

\[
\ddot{a} - \frac{\xi^2}{a^3} \geq 0
\] (79)

\[
\ddot{a}^2 + \frac{\xi^2}{a^2} \geq h.
\] (80)

Thus,

\[
\ddot{a} - \frac{\xi^2}{a^3} \geq 0
\] (81)

\[
a\ddot{a} \geq \frac{\xi^2}{a^2}
\] (82)

\[
a\ddot{a} \geq h - \dot{a}^2
\] (83)

\[
a\ddot{a} + \dot{a}^2 \geq h
\] (84)

\[
\frac{d}{dt}(a\ddot{a}) \geq h
\] (85)

\[
a\ddot{a} \geq a_0 a_1 + ht
\] (86)

\[
\frac{1}{2}a^2 \geq \frac{1}{2}ht^2 + a_0 a_1 t + \frac{1}{2}a_0^2
\] (87)

\[
a^2 \geq ht^2 + 2a_0 a_1 t + a_0^2.
\] (88)

As the global minimum of the right hand side of (88) is

\[
\frac{\xi^2}{h} > 0
\] (89)

one obtains

\[
a^2 \geq \frac{\xi^2}{h}
\] (90)
Thus, if $b_1 \geq 0$, then the solutions are global.

Suppose $b_1 < 0$ and $b_1^2 < \frac{2\lambda \xi^2 y + 2 b_0^y y^+1}{h y + 1 (y + 1)}$. If the solutions blow up and $0 < t_2 < +\infty$ is the first zero of $b$, then $\dot{b} < 0$ for all $t < t_2$ and $\dot{b}(t_2) \leq 0$. From $(78)_2$,

$$\dot{b} \geq \frac{\lambda \xi^2 y + 2}{h y + 1} b^y$$

(91)

$$\ddot{b} \dot{b} \leq \frac{\lambda \xi^2 y + 2}{h y + 1} b^y \dot{b}$$

(92)

$$\frac{1}{2} \ddot{b}^2 - \frac{1}{2} b_1^2 \leq \frac{\lambda \xi^2 y + 2}{h y + 1} \frac{1}{y + 1} \left( b^y y^+1 - b_0^y y^+1 \right)$$

(93)

$$\dot{b}^2 \leq \frac{2\lambda \xi^2 y + 2}{h y + 1 (y + 1)} b^y y^+1 + \left( b_1^2 \frac{2\lambda \xi^2 y + 2 b_0^y y^+1}{h y + 1 (y + 1)} \right)$$

(94)

$$\dot{b}^2 < \frac{2\lambda \xi^2 y + 2}{h y + 1 (y + 1)} b^y y^+1.$$  

(95)

From (95),

$$\dot{b}^2(t_2) < 0,$$

(96)

which is a contradiction. Thus, the solutions are global. ■

Combining the results from Propositions 6 to 11, we have thus established Theorem 3.

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REFERENCES

