Coefficient Inequality For Some Subclass Of Meromorphic Functions

Karthiyayini. O and Sivasankari. V

Department of Science and Humanities
PESIT-Bangalore South Campus
1km before Electronic City, Hosur Road, Bangalore-560100, INDIA
E-mail: karthiyayini. roy@pes. edu, sivasankariv@pes. edu

Abstract

In this paper we define two new subclass \( S_{2,\alpha}(A, B, k, \lambda) \) and \( C_{2,\alpha}(A, B, k, \lambda) \) of meromorphic functions associated with a linear operator. We give Sufficient conditions for a function \( g(z) \) to be in these subclasses. Connections to earlier known results are also indicated.

Keywords and Phrases: Analytic functions, Meromorphic functions, Meromorphic starlike functions, Meromorphic convex functions.

2000 Mathematics Subject Classification: Primary 30C45

1. Introduction

Let \( \Sigma_\alpha \) denote the class of functions of the form,

\[
g(z) = \frac{1}{z^{1+\alpha}} + \sum_{n=1}^{\infty} b_n z^{n+\alpha}, (0 \leq \alpha < 1),
\]

which are analytic and meromorphic univalent in the punctured unit disc

\( U^* = \{z: z \in \mathbb{C}: 0 < |z| < 1\} \)

Motivated by the linear operator defined by Fateh S. Aziz et al. in [3] we define for \( g(z) \) in the form (1.1), the following

\[
F^{0}_{\alpha,\lambda}g(z) = g(z) \left( 0 \leq \lambda < \frac{1}{n+1} \right)
\]

\[
F^{k}_{0,\alpha}g(z) = g(z) \quad (k = 0,1,2,...)
\]

\[
F^{1}_{\lambda,\alpha}g(z) = F_{\lambda,\alpha}g(z) = (1-\lambda)g(z) - \lambda zg'(z)
\]
\[
\left(1 + \lambda \alpha\right) z^{\alpha+1} + \sum_{n=1}^{\infty} \left[1 - (n + \alpha + 1) \lambda \right] b_n z^{n+\alpha} \left(0 \leq \lambda < \frac{1}{n+1}\right)
\]
and
\[
F^2_{\lambda, \alpha} g(z) = F_{\lambda, \alpha} \left[ F_{\lambda, \alpha} g(z) \right] = \left(1 + \lambda \alpha\right)^2 z^{\alpha+1} + \sum_{n=1}^{\infty} \left[1 - (n + \alpha + 1) \lambda \right] b_n z^{n+\alpha} \left(0 \leq \lambda < \frac{1}{n+1}\right)
\]
hence, it can be easily seen that
\[
F^k_{\lambda, \alpha} g(z) = \left(1 + \lambda \alpha\right)^k z^{\alpha+1} + \sum_{n=1}^{\infty} \left[1 - (n + \alpha + 1) \lambda \right]^k b_n z^{n+\alpha} \left(0 \leq \lambda < \frac{1}{n+1}, k \in N = 0, 1, 2, \ldots\right) \quad (1.2)
\]
In this paper, we introduce the subclasses of meromorphic function \(S_{\Sigma^{a}}(A, B, k, \lambda)\) and \(C_{\Sigma^{a}}(A, B, k, \lambda)\) associated with the linear operator \(F^k_{\lambda, \alpha} g(z)\) and coefficient inequalities for functions in these subclasses are obtained.

**Definition 1.1:** For the real constants \(A\) and \(B\) \((0 \leq B \leq 1; -B \leq A < B)\), \(F^k_{\lambda, \alpha} g(z) \neq 0\), let \(S_{\Sigma^{a}}(A, B, k, \lambda)\) consists of functions \(g \in \Sigma^{a}\) which satisfy the following condition:
\[
- \frac{z \left( F^k_{\lambda, \alpha} g(z) \right)^{'} }{ F^k_{\lambda, \alpha} g(z) } < \frac{1 + A z}{1 + B z} \left(0 \leq \lambda < \frac{1}{n+1}, k = 0, 1, 2, \ldots \right. \text{ and } \left. z \in U^{*}\right) \quad (1.3)
\]

**Definition 1.2:** For the real constants \(A\) and \(B\) \((0 \leq B \leq 1; -B \leq A < B)\), \(F^k_{\lambda, \alpha} g(z) \neq 0\), let \(C_{\Sigma^{a}}(A, B, k, \lambda)\) consists of functions \(g \in \Sigma^{a}\) which satisfy the following condition:
\[
- \left(1 + \frac{z \left( F^k_{\lambda, \alpha} g(z) \right)^{'} }{ F^k_{\lambda, \alpha} g(z) } \right) < \frac{1 + A z}{1 + B z} \left(0 \leq \lambda < \frac{1}{n+1}, k = 0, 1, 2, \ldots \right. \text{ and } \left. z \in U^{*}\right) \quad (1.4)
\]

**Remark 1.3.** By giving special values for \(\alpha, A, B, k, \lambda\), we obtain the following subclasses of meromorphic functions that were studied earlier in the literature.
1. \(S_{\Sigma^{a}}(A, B, 0, 0) = S_{\alpha}(A, B)\) and \(C_{\Sigma^{a}}(A, B, 0, 0) = C_{\alpha}(A, B)\)
2. \(S_{\Sigma^{x}}(A, B, 0, 0) = \Sigma^{x}(A, B)\) and \(C_{\Sigma^{x}}(A, B, 0, 0) = \Sigma^{x}(A, B)\)
3. \(S_{\Sigma^{y}}(2\alpha - 1, 1, 0, 0) = T^{*}_{\alpha}(A, B)\) and \(C_{\Sigma^{y}}(2\alpha - 1, 1, 0, 0) = C_{\alpha}(A, B)\)

The class \(S_{\alpha}(A, B)\) and \(C_{\alpha}(A, B)\) were studied by Rabha W. Ibrahim and Masilna Darus [6].
The class \(\Sigma^{x}(A, B)\) and \(\Sigma^{x}(A, B)\) were studied by S. Latha and L. Shivarudrappa [8].
The classes \(T^{*}_{\alpha}(A, B)\) and \(C_{\alpha}(A, B)\) were studied by Shigeyoshi Owa and Nicolae N. Pascu [7].
2. Main Results
We prove Sufficient conditions for \( g(z) \in \Sigma_{\alpha} \) to be in the classes \( S_{\alpha}(A,B,k,\lambda) \) and \( C_{\alpha}(A,B,k,\lambda) \).

**Theorem 2.1:** Let \( g(z) \in \Sigma_{\alpha} \). If
\[
\sum_{n=1}^{\infty} [(n+\alpha)(1+B) + (1+A)](1-(n+\alpha+1)\lambda) b_n |z| \leq [(B-A) - \alpha(1-B)](1+\lambda \alpha) z, (z \in U^*) \quad (2.1)
\]
holds and \( B(1+\alpha) > A+\alpha \), then \( g(z) \in S_{\alpha}(A,B,k,\lambda) \).

**Proof:** Assume that \( g(z) \in \Sigma_{\alpha} \) and satisfies 2.1. It is sufficient to show that
\[
\left| 1 + \frac{z(F_{\lambda,\alpha}^k g(z))'}{F_{\lambda,\alpha} g(z)} - B \frac{z(F_{\lambda,\alpha}^k g(z))'}{F_{\lambda,\alpha} g(z)} + A \right| < 1
\]
that is
\[
\left| \frac{F_{\lambda,\alpha}^k g(z) + z(F_{\lambda,\alpha}^k g(z))'}{AF_{\lambda,\alpha}^k g(z) + Bz(F_{\lambda,\alpha}^k g(z))'} \right| < 1
\]
Consider
\[
\left| \frac{F_{\lambda,\alpha}^k g(z) + z(F_{\lambda,\alpha}^k g(z))'}{AF_{\lambda,\alpha}^k g(z) + Bz(F_{\lambda,\alpha}^k g(z))'} \right| \leq \frac{\alpha(1+\lambda \alpha)^k + \sum_{n=1}^{\infty} (1+n+\alpha)(1-(n+\alpha+1)\lambda)^k b_n z^{n+2+\alpha+1}}{[B(1+\alpha) - A](1+\lambda \alpha)^k - \sum_{n=1}^{\infty} (B(n+\alpha) + A)(1-(n+\alpha+1)\lambda)^k b_n |z|}
\]
Hence (2.2) is bounded by 1, if
\[
\alpha(1+\lambda \alpha)^k + \sum_{n=1}^{\infty} (1+n+\alpha)(1-(n+\alpha+1)\lambda)^k b_n |z|
\]
\[
\leq [B(1+\alpha) - A](1+\lambda \alpha)^k - \sum_{n=1}^{\infty} (B(n+\alpha) + A)(1-(n+\alpha+1)\lambda)^k b_n |z|
\]
Equivalently,
\[
\sum_{n=1}^{\infty} (1+n+\alpha)(1-(n+\alpha+1)\lambda)^k b_n |z| + \sum_{n=1}^{\infty} (B(n+\alpha) + A)(1-(n+\alpha+1)\lambda)^k b_n |z|
\]
\[ \sum_{n=1}^{\infty} [(n + \alpha)(1 + B) + (1 + A)](1 - (n + \alpha + 1)\lambda) |b_n| \leq [(B - A) - \alpha(1 - B)](1 + \lambda \alpha)^k \]

where \( B(1 + \alpha) > 1 + \alpha \). This completes the proof.

**Theorem 2.2:**
Let \( g(z) \in \Sigma_\alpha \). If
\[
\sum_{n=1}^{\infty} [n + \alpha](n + \alpha + 1)(B - A) \lambda |b_n| \leq (1 + \alpha)(B - A) - \alpha(1 - B) |l + \lambda \alpha|, (z \in U) \quad (2.3)
\]
holds and \( B(1 + \alpha) > A + \alpha \), then \( g(z) \in C_{\Sigma^*}(A, B, k, \lambda) \).

**Proof:**
Assume that \( g(z) \in \Sigma_\alpha \) and satisfies 2.3. It is sufficient to show that
\[
\left| \frac{1 + z(F_{k, \alpha}^k g(z))^\alpha}{F_{k, \alpha}^k g(z)} + 1 \right| < 1
\]
\[
B \left| \frac{1 + z(F_{k, \alpha}^k g(z))^\alpha}{F_{k, \alpha}^k g(z)} + A \right|
\]
that is
\[
\left| \frac{(zF_{k, \alpha}^k g(z))^\alpha + 2(F_{k, \alpha}^k g(z))^\alpha}{Bz(F_{k, \alpha}^k g(z))^\alpha + (A + B)(F_{k, \alpha}^k g(z))^\alpha} \right| < 1
\]
Consider,
\[
\left| \frac{(zF_{k, \alpha}^k g(z))^\alpha + 2(F_{k, \alpha}^k g(z))^\alpha}{Bz(F_{k, \alpha}^k g(z))^\alpha + (A + B)(F_{k, \alpha}^k g(z))^\alpha} \right| = \frac{\alpha(1 + \lambda \alpha)^k + \sum_{n=1}^{\infty} (n + \alpha)(1 + n + \alpha)(1 - (n + \alpha + 1)\lambda) b_n z^{-n+2\alpha+1}}{(\alpha + 1)(1 + \lambda \alpha)^k - \sum_{n=1}^{\infty} (n + \alpha)(1 + n + \alpha)(1 - (n + \alpha + 1)\lambda) b_n z^{-n+2\alpha+1}}
\]
\[
\leq \frac{\alpha(1 + \lambda \alpha)^k + \sum_{n=1}^{\infty} (n + \alpha)(1 + n + \alpha)(1 - (n + \alpha + 1)\lambda) b_n |l|}{(\alpha + 1)(1 + \lambda \alpha)^k - \sum_{n=1}^{\infty} (n + \alpha)(1 + n + \alpha)(1 - (n + \alpha + 1)\lambda) b_n |l|}
\]
Hence 2.4 is bounded by 1, if
\[
\sum_{n=1}^{\infty} [(n + \alpha)(1 + B) + (1 + A)](1 - (n + \alpha + 1)\lambda) |b_n| \leq (1 + \alpha)(B - A) - \alpha(1 - B) |l + \lambda \alpha|, (z \in U')
\]
where \( B(1 + \alpha) > 1 + \alpha \). This completes the proof.

**Remark 2.3:** By specializing on the parameters \( \alpha, A, B, k, \lambda \) as in Remark 1.1 in
Coefficient Inequality For Some Subclass Of Meromorphic Functions

Theorem 2.1 and Theorem 2.2, we obtain corresponding Theorem 2.1 and Theorem 2.2 in [6], [8], [7].

References
