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Some Trends In Line Graphs

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Abstract

In this paper, focus on some trends in line graphs and conclude that we are solving some graphs to satisfied for connected and maximal sub graphs, Further we present a general bounds relating to the line graphs.

Introduction

In 1736, Euler first introduced the concept of graph theory. In the history of mathematics, the solution given by Euler of the well known Konigsberg bridge problem is considered to be the first theorem of graph theory. This has now become a subject generally regarded as a branch of combinatorics. The theory of graph is an extremely useful tool for solving combinatorial problems in different areas such as geometry, algebra, number theory, topology, operations research, and optimization and computer science.

Graphs serve as mathematical models to analyze many concrete real-worldproblems successfully. Certain problems in physics, chemistry, communicationscience, computer technology, genetics, psychology, sociology, and linguistics canbe formulated as problems in graph theory. Also, many branches of mathematics, such as group theory, matrix theory, probability, and topology, have closeconnections with graph theory.Some puzzles and several problems of a practical nature have been instrumentalin the development of various topics in graph theory. The famous K¨onigsbergbridge problem has been the inspiration for the development of Eulerian graphtheory. The challenging Hamiltonian graph theory has been developed from the“Around the World” game of Sir William Hamilton. The theory of acyclic graphswas developed for solving problems of electrical networks, and the study of “trees”was developed for enumerating isomers of organic compounds. The well-knownfour-color problem formed the very basis for the development of planarity in graphtheory and combinatorial topology. Problems of linear programming and operationsresearch (such as maritime traffic problems) can be tackled by the theory of flows innetworks. Kirkman’s schoolgirl problem and scheduling problems are

examples ofproblems that can be solved by graph colorings. The study of simplicial complexescan be associated with the study of graph theory. Many more such problems can beadded to this list.
We define graph theory terminology and concepts that we will need in subsequent chapters. The reader who is familiar with graph theory will no doubt be acquainted with the terminology in the following Sections. However since graph theory terminology sometimes varies, we clarify the terminology that will be adopted in this paper.

Some Basic Definitions

Definition 2.1

A ***graph*** G=(V, E, φ) consists of a non empty set V={ V_1, V_2, …., } called the set of nodes (points, vertices)of the graph, E={ e_1, e_2, …., } is said to be the set of edges of the graph, and φ is a mapping from the set of edges E to set of ordered or unordered pairs of elements of V. The vertices are represented by the points and each edge is represented by a line diagrammatically.

Definition 2.2

If there is an edge from a vertex to itself then the edge is called ***self loop*** or simply ***loop***. A graph H is said to be a ***sub graph*** of G if all the vertices and all the edges of H are in G and if the adjacency is preserved in H exactly as in G.

Definition 2.3

If the vertex v_i is an end vertex of some edge e_k the e_k is said to be ***incident*** with v_i. Two edges are said to be adjacent if they are incident on a common vertex. Two vertices v_i and v_j are said to be ***adjacent*** if v_iv_j is an edge of the graph.

Definition 2.4

The number of edges incident at the vertex v_i is called the ***degree*** of the vertex with self loops counted twice and it is denoted by $d(v_i)$.

Definition 2.5

A ***walk*** in a graph G is an alternating sequence of vertices and edges beginning and ending with vertices in which all the vertices are distinct then the walk is called a ***path***. The length of a walk is the number of edges in it.

Definition 2.6

A ***walk*** is a trail if all the edges appearing in the walk are distinct. A ***cycle*** is a closed trail in which the vertices are all distinct.

Definition 2.7

Let G be a graph. Two vertices u and v of G are said to be connected if there exists u-v path in G. The subgraphs $G(V_1)$, $G(V_2)$, … are called components of G. If the

component is one then the graph is ***connected***graph. Otherwise the graph G is disconnected.

Definition 2.8
A simple graph G is said to be ***complete*** if every pair of distinct vertices are adjacent in G.

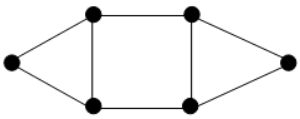
Definition 2.9
A graph G' is said to be ***spanning graph*** of G if the vertex set of G and G' are same.

Definition 2.10
A sub graph H of G is a ***spanning subgraph*** of G if V(H)=V(G).

Definition 2.11
Two graphs G_1 and G_2 are said to be isomorphic to each other, if there exists a one-to-one correspondence between the vertex sets which preserves adjacency of the vertices.

3. Line Graphs

Definition 3.1
Let G be a loopless graph. We construct a graphL(G) in the following way:
The vertex set of L(G) is in 1-1 correspondence with the edge set of G and two vertices of L(G) are joined by an edge if and only if the corresponding edges of G are adjacent in G. The graphL(G) (which is always a simple graph) is called the line graph or the edge graph of G.



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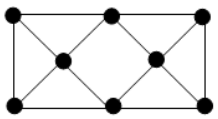


L(G)

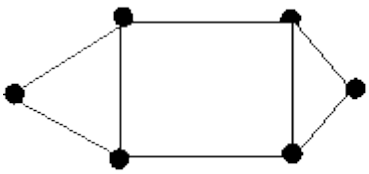
3.2 Properties of line graph

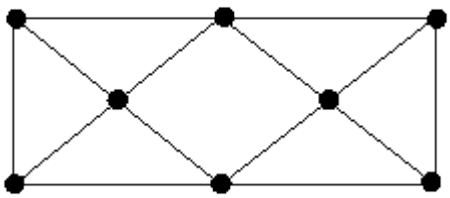
Property 3.2.1 If G is connected if any only of L(G) is connected.

Proof:
Necessary Part
Assume That G is connected. To Prove that L(G) is connected.
Since G is connected, there exists a path betweenevery pair of vertices.
Then by definition of line graphs, the adjacent edges of G are in adjacent vertices in L(G)
Then there is a path between every pair of vertices in L(G).
⇨ L(G) is connected.

Conversely
Assume that L(G) is connected. To prove that G is connected.
Suppose G is not connected, atleast any twoof its vertices are not connected by a path. Then atleast two components we had. If we consider one edge in 1st component then it doesn't have any adjacent edges in second component.
So, if we draw line graphs for both the two components then they do not have any path between them.ie) there are two components in L(G) also. That is, L(G) is disconnected.
Which is a contradiction. Therefore, G is connected.

Property 3.2.2 If H is subgraph of G, then L(G) is subgraph of L(G).
Consider the graph G and the corresponding L(G) given below





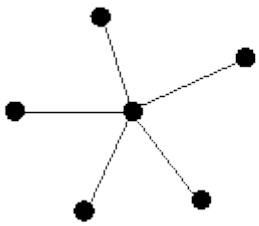
Property 3.2.3 The edges incident at a vertex of G give rise to a maximal complete sub graph of L(G).

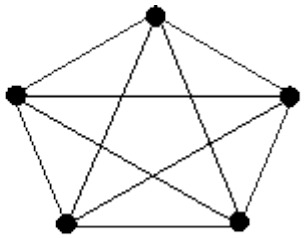
Property 3.2.4 If e=***uv*** is an edge of a simple graph G, the degree of e in L(G) is the same as the number of edges of G adjacent to e in G. The number is $d_G(u)+d_G(v)-2$. Hence, dl(g)=dg(u)+dg(v)-2

Property 3.2.5 If G is a simple graph,

Exercise 3.3. Show thatthe line graph of the star $K_{1,n}$ is the complete graph K_n.
Proof: In the star graph $K_{1,n}$ the vertex v is adjacent to all other n vertices u_1, u_2, …., u_n. That is v has an edge to every other n vertices. So in the corresponding line graph$L(K_{1,n})$, all the vertices are adjacent and connected to every vertices by an edge. This means that it is complete. Hence the result.

Example:





$K_{1,5}$complete Graph$L(K_{1,5})=K_5$

Exercise 3.4. Show that $L(C_n) \cong C_n$, $n \geq 3$.

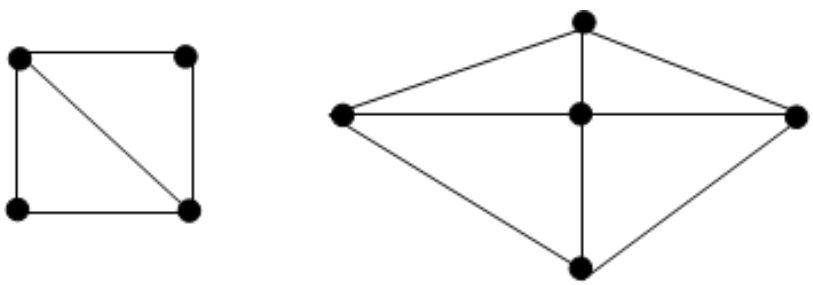
Proof: Consider a cycle with n vertices, it has n edges in its path. By these edges as vertices in $L(C_n)$, we form a cycle with n edges. This implies that, both C_n and $L(C_n)$ has same number of vertices and edges and also have the same degree sequence. So, there exists a 1-1 correspondence between $V(C_n)$ into $V(L(C_n))$ and $E(C_n)$ and $E(L(C_n))$ and preserves adjacency.

Example:

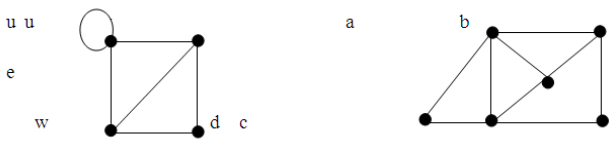
Theorem 3.5. The line graph of a simple graph G is a path if and only if G is a path.

Proof. Let G be the path P_n on n vertices.Then clearly, L(G) is the path P_{n-1} on n-1 vertices.
Conversely, let L(G) be path. Then no vertex of G can have degree greater than 2 because if G has a degree greater than 2, the edges incident to v would form a complete subgraph of L(G) with a least three vertices. Hence, G must be either a cycle or a path. But G cannot be cycle or a path. But G cannot be a cycle, because the line graph of a cycle is again a cycle

Exercise 3.6. Let $H=L^2(G)$ be defined as L(L(G). Find m(H) if G is the graph given below.



Exercise 3.7. Give an example of a graph G to show that the relation $d_{L(G)}(uv) = d_G(u)+d_G(v)-2$ may not be valid if G has a loop.



(G)L(G)

Consider the edge set of G by a=(u, v), b=(v, x), c=(x, w), d=(w, u), e=(w, v) and f=(u, u).
Now $d_{L(G)}(e) = 3$
ie) $d_{L(G)}(w, v) = d_G(w)+d_G(v)-2$
$= 3+3-2$
$= 4$
Hence, $d_{L(G)}(uv) \neq d_G(u)+d_G(v) - 2$ when the graph has a loop.

Exercise 3.8 Prove that a simple connected graph G is isomorphic to its line graph if and only if it is a cycle.

Proof: Suppose a simple graph is isomorphic to its line graph then by definition, there is a one-to-one and onto function from vertex set of G and L(G) with adjacency preserved. Then definitely both the graphs have same number of vertices and edges, this is possible only when the graph is cycle.
Conversely, if the graph is cycle of n vertices and n-1edges(may be n edges) the line graph is also have the same number of vertices and edges as in G. so, obviously there is an isomorphism between the graphs.

Exercise 3.9. If the graph H is a spanning sub graph of a graph G, then L(H) is a spanning sub graph of L(G).

3.10 Isomorphism On Line Graphs

Theorem:3.10.1
If the simple graphs G_1 and G_2 are isomorphic, then $L(G_1)$ and $L(G_2)$ are isomorphic.

Proof: Let $f(\varphi, \theta)$ be isomorphism between the simple graphs G_1 and G_2. Then θ is a bijection of $E(G_1)$ and $E(G_2)$.We show that θ is an isomorphism of $L(G_1)$ and $L(G_2)$ by θ preserves adjacency and non-adjacency.
Then there exists a vertex v of c incident with both e_i and e_j, and so φ (v) is a vertex incident with both φ(ei) and φ (e_j). Hence φ (e_i) and φ (e_j) are adjacent vertices in $L(G_2)$.
Now, let φ (e_i) and φ (e_j) be adjacent vertices in $L(G_2)$.This means that they are adjacent edges in G_2 and hence there exists a vertex v' of G_2incident to both φ (e_i) and φ (e_j) in G_2. Then $\varphi^{-1}(v')$ is a vertex of G_1incident to both e_i and e_j, so thate_i and e_j are adjacent vertices of G_1.

Thus, e_i and e_j are adjacent vertices of G_1if and only if φ (e_i) and φ (e_j) are adjacent vertices in $L(G_2)$. Hence, θ is an isomorphism of $L(G_1)$ and $L(G_2)$.

Remark 3.10.2 The line graphs of simple graphs G_1 and G_2 are isomorphic, then it is not necessary that the two simple graphs G_1 and G_2 are isomorphic.
Consider the graphs $K_{1,3}$ and K_3.Their line graphs are K_3. But $K_{1,3}$ is not isomorphic to K_3 since there is a vertex of degree 3 in $K_{1,3}$, whereas there is no such vertex in K_3.
Theorem:3.10.3 Let G and G' be simple connected graphs with isomorphic line graphs. Then G and G' are isomorphic unless one of them is $K_{1,3}$ and the other is K_3.

Definition 3.10.4 Agraph H is called ***forbidden sub graph*** for a property P of graphs it it satisfies the following condition: If a graph G has property P, then G cannot contain an induced sub graph isomorphic to H.

Theorem 3.10.5 If G is a line graph then $K_{1,3}$ is a forbidden sub graph of G.
Proof: Suppose that G is the line graphof H and that G contains a $K_{1,3}$as an induced sub graph. If v is the vertex of degree 3 in $K_{1,3}$and v_1, v_2, and v_3 are the neighbors of v in this $K_{1,3}$then the edge e corresponding to v in H is adjacent to three edgese_1, e_2, and e_3 corresponding to the verticesv_1, v_2, and v_3. Hence, one of the end vertices of e must be the end vertex of at least two of e_1, e_2, and e_3 in H, and hence v together with two of v_1, v_2, and v_3 form a triangle in G. This means that the subgraph of G considered above is not an induced sub graph of G, a contradiction

Conclusion

In this paper, we are defining and verified for some lines graphs. now we are conclude that the above properties of some trends in line graphs are cycle or connected to the simple graphs and its satisfied for isomorphic.

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