Some Trends In Line Graphs

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Abstract

In this paper, focus on some trends in line graphs and conclude that we are solving some graphs to satisfied for connected and maximal sub graphs. Further we present a general bounds relating to the line graphs.

Introduction

In 1736, Euler first introduced the concept of graph theory. In the history of mathematics, the solution given by Euler of the well known Konigsberg bridge problem is considered to be the first theorem of graph theory. This has now become a subject generally regarded as a branch of combinatorics. The theory of graph is an extremely useful tool for solving combinatorial problems in different areas such as geometry, algebra, number theory, topology, operations research, and optimization and computer science. Graphs serve as mathematical models to analyze many concrete real-world problems successfully. Certain problems in physics, chemistry, communications science, computer technology, genetics, psychology, sociology, and linguistics can be formulated as problems in graph theory. Also, many branches of mathematics, such as group theory, matrix theory, probability, and topology, have close connections with graph theory. Some puzzles and several problems of a practical nature have been instrumental in the development of various topics in graph theory. The famous K"onigsberg bridge problem has been the inspiration for the development of Eulerian graph theory. The challenging Hamiltonian graph theory has been developed from the “Around the World” game of Sir William Hamilton. The theory of acyclic graph was developed for solving problems of electrical networks, and the study of “trees” was developed for enumerating isomers of organic compounds. The well-known four-color problem formed the very basis for the development of planarity in graph theory and combinatorial topology. Problems of linear programming and operations research (such as maritime traffic problems) can be tackled by the theory of flows in networks. Kirkman’s schoolgirl problem and scheduling problems are
examples of problems that can be solved by graph colorings. The study of simplicial complexes can be associated with the study of graph theory. Many more such problems can be added to this list.

We define graph theory terminology and concepts that we will need in subsequent chapters. The reader who is familiar with graph theory will no doubt be acquainted with the terminology in the following Sections. However, since graph theory terminology sometimes varies, we clarify the terminology that will be adopted in this paper.

Some Basic Definitions

Definition 2.1
A graph \( G=(V, E, \varphi) \) consists of a non-empty set \( V=\{ V_1, V_2, \ldots \} \) called the set of nodes (points, vertices) of the graph, \( E=\{ e_1, e_2, \ldots \} \) is said to be the set of edges of the graph, and \( \varphi \) is a mapping from the set of edges \( E \) to the set of ordered or unordered pairs of elements of \( V \). The vertices are represented by the points and each edge is represented by a line diagrammatically.

Definition 2.2
If there is an edge from a vertex to itself then the edge is called self loop or simply loop. A graph \( H \) is said to be a subgraph of \( G \) if all the vertices and all the edges of \( H \) are in \( G \) and if the adjacency is preserved in \( H \) exactly as in \( G \).

Definition 2.3
If the vertex \( v_i \) is an end vertex of some edge \( e_k \) the \( e_k \) is said to be incident with \( v_i \). Two edges are said to be adjacent if they are incident on a common vertex. Two vertices \( v_i \) and \( v_j \) are said to be adjacent if \( v_i, v_j \) is an edge of the graph.

Definition 2.4
The number of edges incident at the vertex \( v_i \) is called the degree of the vertex with self loops counted twice and it is denoted by \( d(v_i) \).

Definition 2.5
A walk in a graph \( G \) is an alternating sequence of vertices and edges beginning and ending with vertices in which all the vertices are distinct then the walk is called a path. The length of a walk is the number of edges in it.

Definition 2.6
A walk is a trail if all the edges appearing in the walk are distinct. A cycle is a closed trail in which the vertices are all distinct.

Definition 2.7
Let \( G \) be a graph. Two vertices \( u \) and \( v \) of \( G \) are said to be connected if there exists a \( u-v \) path in \( G \). The subgraphs \( G(V_1), G(V_2), \ldots \) are called components of \( G \). If the
component is one then the graph is connected graph. Otherwise the graph G is disconnected.

**Definition 2.8**  
A simple graph G is said to be complete if every pair of distinct vertices are adjacent in G.

**Definition 2.9**  
A graph G’ is said to be spanning graph of G if the vertex set of G and G’ are same.

**Definition 2.10**  
A sub graph H of G is a spanning subgraph of G if V(H)=V(G).

**Definition 2.11**  
Two graphs G₁ and G₂ are said to be isomorphic to each other, if there exists a one-to-one correspondence between the vertex sets which preserves adjacency of the vertices.

### 3. Line Graphs

**Definition 3.1**  
Let G be a loopless graph. We construct a graph L(G) in the following way:  
The vertex set of L(G) is in 1-1 correspondence with the edge set of G and two vertices of L(G) are joined by an edge if and only if the corresponding edges of G are adjacent in G. The graph L(G) (which is always a simple graph) is called the line graph or the edge graph of G.

![Graphs G and L(G)](image)

**3.2 Properties of line graph**  
**Property 3.2.1** If G is connected if any only of L(G) is connected.

**Proof:**  
**Necessary Part**  
Assume that G is connected. To prove that L(G) is connected.  
Since G is connected, there exists a path between every pair of vertices.  
Then by definition of line graphs, the adjacent edges of G are in adjacent vertices in L(G).  
Then there is a path between every pair of vertices in L(G).

\[ \Rightarrow \quad \text{L(G) is connected.} \]
Conversely
Assume that L(G) is connected. To prove that G is connected.
Suppose G is not connected, at least any two of its vertices are not connected by a path.
Then at least two components we had. If we consider one edge in 1st component then it
doesn’t have any adjacent edges in second component.
So, if we draw line graphs for both the two components then they do not have any
path between them. i.e., there are two components in L(G) also. That is, L(G) is
disconnected.
Which is a contradiction. Therefore, G is connected.

Property 3.2.2 If H is subgraph of G, then L(G) is subgraph of L(G).
Consider the graph G and the corresponding L(G) given below

Property 3.2.3 The edges incident at a vertex of G give rise to a maximal complete
subgraph of L(G).

Property 3.2.4 If e=uv is an edge of a simple graph G, the degree of e in L(G) is the
same as the number of edges of G adjacent to e in G. The number is \(d_G(u) + d_G(v) - 2\).
Hence, \(d_L(g) = d_G(u) + d_G(v) - 2\)

Property 3.2.5 If G is a simple graph,

Exercise 3.3. Show that the line graph of the star \(K_{1,n}\) is the complete graph \(K_n\).
Proof: In the star graph \(K_{1,n}\), the vertex \(v\) is adjacent to all other \(n\) vertices \(u_1, u_2, \ldots, u_n\).
That is \(v\) has an edge to every other \(n\) vertices. So in the corresponding line
graph \(L(K_{1,n})\), all the vertices are adjacent and connected to every vertices by an edge.
This means that it is complete. Hence the result.

Example:
Some Trends In Line Graphs

**K**₁,₅complete Graph L(K₁,₅) = K₅

**Exercise 3.4.** Show that L(Cₙ) ≅ Cₙ, n ≥ 3.

**Proof:** Consider a cycle with n vertices, it has n edges in its path. By these edges as vertices in L(Cₙ), we form a cycle with n edges. This implies that, both Cₙ and L(Cₙ) have same number of vertices and edges and also have the same degree sequence. So, there exists a 1-1 correspondence between V(Cₙ) into V(L(Cₙ)) and E(Cₙ) and E( L(Cₙ)) and preserves adjacency.

**Example:**

![Image](image1.png)

**Theorem 3.5.** The line graph of a simple graph G is a path if and only if G is a path.

**Proof.** Let G be the path Pₙ on n vertices. Then clearly, L(G) is the path Pₙ₋₁ on n-1 vertices. Conversely, let L(G) be path. Then no vertex of G can have degree greater than 2 because if G has a degree greater than 2, the edges incident to v would form a complete subgraph of L(G) with a least three vertices. Hence, G must be either a cycle or a path. But G cannot be cycle or a path. But G cannot be a cycle, because the line graph of a cycle is again a cycle.

**Exercise 3.6.** Let H = L²(G) be defined as L(L(G)). Find m(H) if G is the graph given below.

![Image](image2.png)

**Exercise 3.7.** Give an example of a graph G to show that the relation d_{L(G)}(uv) = d_G(u) + d_G(v) - 2 may not be valid if G has a loop.
Consider the edge set of \( G \) by \( a=(u, v), b=(v, x), c=(x, w), d=(w, u), e=(w, v) \) and \( f=(u, u) \).

Now \( d_{L(G)}(e) = 3 \)

ie) \( d_{L(G)}(w, v) = d_G(w) + d_G(v) - 2 \)

\[ = 3 + 3 - 2 \]

\[ = 4 \]

Hence, \( d_{L(G)}(uv) \neq d_G(u) + d_G(v) - 2 \) when the graph has a loop.

**Exercise 3.8** Prove that a simple connected graph \( G \) is isomorphic to its line graph if and only if it is a cycle.

**Proof:** Suppose a simple graph is isomorphic to its line graph then by definition, there is a one-to-one and onto function from vertex set of \( G \) and \( L(G) \) with adjacency preserved. Then definitely both the graphs have same number of vertices and edges, this is possible only when the graph is cycle.

Conversely, if the graph is cycle of \( n \) vertices and \( n-1 \) edges (may be \( n \) edges) the line graph is also have the same number of vertices and edges as in \( G \), so, obviously there is an isomorphism between the graphs.

**Exercise 3.9.** If the graph \( H \) is a spanning sub graph of a graph \( G \), then \( L(H) \) is a spanning sub graph of \( L(G) \).

**3.10 Isomorphism On Line Graphs**

**Theorem:3.10.1**

If the simple graphs \( G_1 \) and \( G_2 \) are isomorphic, then \( L(G_1) \) and \( L(G_2) \) are isomorphic.

**Proof:** Let \( f(\varphi, \theta) \) be isomorphism between the simple graphs \( G_1 \) and \( G_2 \). Then \( \theta \) is a bijection of \( E(G_1) \) and \( E(G_2) \). We show that \( \theta \) is an isomorphism of \( L(G_1) \) and \( L(G_2) \) by \( \theta \) preserves adjacency and non-adjacency.

Then there exists a vertex \( v \) of \( c \) incident with both \( e_i \) and \( e_j \), and so \( \varphi \ (v) \) is a vertex incident with both \( \varphi \ (e_i) \) and \( \varphi \ (e_j) \). Hence \( \varphi \ (e_i) \) and \( \varphi \ (e_j) \) are adjacent vertices in \( L(G_2) \).

Now, let \( \varphi \ (e_i) \) and \( \varphi \ (e_j) \) be adjacent vertices in \( L(G_2) \). This means that they are adjacent edges in \( G_2 \) and hence there exists a vertex \( v' \) of \( G_2 \) incident to both \( \varphi \ (e_i) \) and \( \varphi \ (e_j) \) in \( G_2 \). Then \( \varphi^{-1}(v') \) is a vertex of \( G_1 \) incident to both \( e_i \) and \( e_j \), so that \( e_i \) and \( e_j \) are adjacent vertices of \( G_1 \).
Thus, $e_i$ and $e_j$ are adjacent vertices of $G_1$ if and only if $\varphi(e_i)$ and $\varphi(e_j)$ are adjacent vertices in $L(G_2)$. Hence, $\theta$ is an isomorphism of $L(G_1)$ and $L(G_2)$.

**Remark 3.10.2** The line graphs of simple graphs $G_1$ and $G_2$ are isomorphic, then it is not necessary that the two simple graphs $G_1$ and $G_2$ are isomorphic. Consider the graphs $K_{1,3}$ and $K_3$. Their line graphs are $K_3$. But $K_{1,3}$ is not isomorphic to $K_3$ since there is a vertex of degree 3 in $K_{1,3}$, whereas there is no such vertex in $K_3$.

**Theorem 3.10.3** Let $G$ and $G'$ be simple connected graphs with isomorphic line graphs. Then $G$ and $G'$ are isomorphic unless one of them is $K_{1,3}$ and the other is $K_3$.

**Definition 3.10.4** A graph $H$ is called a forbidden subgraph for a property $P$ of graphs if it satisfies the following condition: If a graph $G$ has property $P$, then $G$ cannot contain an induced subgraph isomorphic to $H$.

**Theorem 3.10.5** If $G$ is a line graph then $K_{1,3}$ is a forbidden subgraph of $G$.

Proof: Suppose that $G$ is the line graph of $H$ and that $G$ contains a $K_{1,3}$ as an induced subgraph. If $v$ is the vertex of degree 3 in $K_{1,3}$ and $v_1$, $v_2$, and $v_3$ are the neighbors of $v$ in this $K_{1,3}$, then the edge $e$ corresponding to $v$ in $H$ is adjacent to three edges $e_1$, $e_2$, and $e_3$ corresponding to the vertices $v_1$, $v_2$, and $v_3$. Hence, one of the end vertices of $e$ must be the end vertex of at least two of $e_1$, $e_2$, and $e_3$ in $H$, and hence $v$ together with two of $v_1$, $v_2$, and $v_3$ form a triangle in $G$. This means that the subgraph of $G$ considered above is not an induced subgraph of $G$, a contradiction.

**Conclusion**

In this paper, we are defining and verified for some line graphs. Now we are conclude that the above properties of some trends in line graphs are cycle or connected to the simple graphs and its satisfied for isomorphic.

**References**