Approximation of Singular Integrals by a Mixed Quadrature of Anti-Gauss and Steffensen’s Quadrature Rules in the Adaptive Environment

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Abstract

An anti-Gauss 3-point rule is derived from the Gauss-Legendre 2-point rule. Then a new mixed quadrature rule is constructed blending anti-Gauss 3-point rule with the Steffensen’s 4-point rule. An algorithm using the mixed quadrature rule is designed for adaptive integration scheme and a number of singular as well as non-singular real definite integrals are evaluated. These results are compared with those obtained by using the constituent rules and another mixed quadrature rule developed earlier. The relative effectiveness of the scheme using the new mixed quadrature rule is discussed.

\textbf{Index terms:} Gauss-Legendre quadrature, anti-Gauss quadrature, Steffensen’s quadrature, mixed quadrature and adaptive integration scheme

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I. INTRODUCTION

Numerical integration is concerned with developing algorithms to approximate the integral of a function $f(x)$. The most commonly used algorithms are Newton-Cotes formulas, Romberg’s method, Gaussian quadrature, and to lesser extents Hermite’s formulas and certain adaptive techniques. Adaptive algorithms are now used widely for the numerical evaluation of integrals. An adaptive integration scheme based on

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Simpson’s \( \frac{1}{3} \) rule is discussed in detail in the book by S. Conte and C. de Boor[1]. W. Gander and W. Gautschi[12] developed a new adaptive integration scheme for approximation of real definite integrals. They took three rules such as: Lobatto quadrature rule and two successive Kronrod extensions of Lobatto quadrature rule for framing the termination criterion of adaptive quadrature. An alternative termination criterion is also given in the book of B. Bradie[3]. Recently R. B. Dash and D. Das [14, 15, 16] modified the termination criterion in which they used different types of mixed quadrature rules in place of Kronrod extension of a quadrature rule.

A new method of enhancing precision of quadrature rules is the method of mixed quadrature which was first coined by R. N. Das and G. Pradhan[10]. In this method a quadrature rule of higher precision is formed by taking linear/convex combination of two or more quadrature rules of equal lower precision. In literature, precision of quadrature rules are enhanced through either Richardson extrapolation or Kronrod extension. Richardson extrapolation gives us a family of formulae of higher precision taking into account trapezoidal formula as the base formula {refer to Bauer et al. [4], Ralston, pp. 131–133[2]; Lyness and Puri[5], K. E. Atkinson[7]}. On the other hand, Patternson[9] and Kronrod [8] presented quadrature rules of higher precision by taking Gaussian quadrature as base rule. These methods of precision enhancement, each having single base rule, are very much cumbersome. But the enhancement of precision by mixed quadrature approach with the aid of two rules is very simple and easy to handle. D. Das and R. B. Dash[13] are first to use mixed quadrature rule for approximation of real definite integrals in adaptive environment.

If the integrand \( f(x) \) has singularity at points \( a \) or \( b \) or \( \frac{a+b}{2} \) in the interval \( a, b \), then the quadrature rules of closed type are not applicable for that integrand. Thus open type quadrature rules are suitable for approximation of integrals in the interval \( a, b \) which such integrands. Keeping these difficulties in mind, D. Das and R. B. Dash[16] constructed an open type mixed quadrature rule of precision five by using two open type quadrature rules such as Steffensen’s 4-point rule and Gauss-Legendre 2-point rule having equal precision (i.e. three) to evaluate singular integrals.

The idea of anti-Gaussian quadrature is the maiden work of D. P. Laurie[11], in which he has given a theory of construction of anti-Gaussian \( n+1 \) point quadrature rule from the \( n \)-point Gaussian quadrature rule.

In this paper, we derive an anti-Gauss 3-point rule from the Gauss-Legendre 2-point rule using the idea of Laurie. This is an open type rule. An open type mixed quadrature rule of precision five is constructed blending the anti-Gauss 3-point rule with the Steffensen’s 4-point rule each having precision three. This mixed rule is used for fixing termination criterion in the adaptive integration scheme. Many singular as well as non-singular integrals are evaluated using this mixed rule in the adaptive and non-adaptive environments. The results are compared with those obtained by using the earlier established mixed quadrature rule[16] using the adaptive integration scheme. The benefit of mixing anti-Gauss 3-point rule instead of Gauss-Legendre 2-point rule with Steffensen’s 4-point rule is also discussed. The specialty of this paper is that, it is a maiden attempt by us to introduce effectively the anti-Gauss rule to
design a mixed quadrature rule which is subsequently used in the adaptive integration scheme.

II. BASIC QUADRATURE RULES

The general problem of numerical integration/quadrature rule is to find an approximate value of the integral

\[ I = \int_{a}^{b} w(x)f(x) \, dx, \quad (2.1) \]

where \( w(x) > 0 \) on \( a,b \) and \( w(x)f(x) \) is integrable in Riemann sense in \( a,b \).

The quadrature rule for (2.1) can be written in the from

\[ I \approx \sum_{i=0}^{n} \lambda_i f(x_i), \quad (2.2) \]

where \( x_i, i = 0(1)n \) are called the nodes distributed within the limits of integration.

\( \lambda_i, i = 0(1)n \) are called the weights of the quadrature rule.

The error of approximation is given as

\[ E_n f = I f - \sum_{i=0}^{n} \lambda_i f(x_i). \quad (2.3) \]

(2.1) Steffensen’s quadrature rule

An integration formula

\[ \int_{a}^{b} f(x) \, dx \approx \sum_{i=1}^{n} w_i f(x_i), \quad x_i < x_2 < \cdots < x_n \quad (2.1.1) \]

is said to be of closed type if the function evaluation is needed at the end points of the interval \( a,b \). An integration formula is said to be of open type if both of the end points are omitted from the function evaluation. Steffensen’s quadrature rules [6] are open type Newton-Cotes quadrature rules.

Steffensen’s 4-point rule

\[ \int_{a}^{b} f(x) \, dx = \frac{5h}{24} [11f(a + h) + f(a + 2h) + f(a + 3h) + 11f(a + 4h)] + E_4 \]

\( (2.1.2) \)

where \( h = \frac{b - a}{5} \)

and \( E_4 = \frac{95}{144} h^5 f^{(n)}(\xi) \quad a < \xi < b \)

\[ (2.1.3) \]

The degree of precision of the rule (2.1.2) is 3.

(2.2) Anti-Gaussian quadrature rule

An anti-Gaussian \( n+1 \) point quadrature rule is a rule whose degree of precision is \( 2n-1 \). It integrates polynomials of degree up to \( 2n+1 \) with an error equal in magnitude but of opposite in sign to that of \( n \)-point Gaussian rule.

Using this idea of Laurie, we wish to construct an anti-Gauss 3-point rule of precision
3 which will integrate polynomials of degree up to five with an error equal in magnitude but of opposite in sign to that of Gauss-Legendre 2-point rule.

**Construction of anti-Gauss 3-point rule from the Gauss-Legendre 2-point rule**

Denoting the anti-Gauss 3-point rule by \( R_{eG_3} f \), we write

\[
I f = \int x dx R_{eG_1} f .
\]  

(2.2.1)

Furthermore, we can express \( R_{eG_3} f \) in the form

\[
R_{eG_3} f = \sum_{i=0}^{3} w_i f x_i = w_0 f x_0 + w_1 f x_1 + w_2 f x_2 ,
\]  

(2.2.2)

where, \( w_i 's \) are weights and \( x_i 's \) are the distinct points (nodes) in the interval \(-1,1\).

The error associated with the anti-Gauss 3-point rule is \( I f - R_{eG_3} f \). This error is equal to the negative of the error associated with Gauss-Legendre 2-point rule. That is,

\[
I f - R_{eG_3} f = - I f - R_{eG_2} f
\]  

(2.2.3)

or, \( R_{eG_3} f = 2I f - R_{eG_2} f \)

(2.2.4)

or, \( w_0 f x_0 + w_1 f x_1 + w_2 f x_2 = 2 \int x dx f \left( -\frac{1}{\sqrt{3}} \right) + f \left( \frac{1}{\sqrt{3}} \right) \).  

(2.2.5)

It integrates polynomials of degree up to 5 with an error equal in magnitude but of opposite in sign to that of 2-point Gauss-Legendre rule. That is, for \( f x = x^i, i = 0,1,2,3,4,5 \), we get the following system of equations.

\[
w_0 + w_1 + w_2 = 2
\]  

(a)

\[
w_0 x_0 + w_1 x_1 + w_2 x_2 = 0
\]  

(b)

\[
w_0 x_0^2 + w_1 x_1^2 + w_2 x_2^2 = \frac{2}{3}
\]  

(c)

\[
w_0 x_0^3 + w_1 x_1^3 + w_2 x_2^3 = 0
\]  

(d)

\[
w_0 x_0^4 + w_1 x_1^4 + w_2 x_2^4 = \frac{26}{45}
\]  

(e)

\[
w_0 x_0^5 + w_1 x_1^5 + w_2 x_2^5 = 0
\]  

(f)

Solving the above system of equations, we get

\[x_0 = \frac{\sqrt{13}}{13}, x_1 = 0, x_2 = -\frac{\sqrt{13}}{13} \text{ and } w_0 = \frac{5}{13}, w_1 = \frac{16}{13}, w_2 = \frac{5}{13}.\]

Substituting these into Eq.(2.2.2), we get

\[
\int x dx R_{eG_3} f = \int \frac{5}{13} f \left( \frac{\sqrt{13}}{13} \right) + 16 f \left( 0 \right) + 5 f \left( -\frac{\sqrt{13}}{13} \right) .
\]  

(2.2.6)

This is anti-Gauss 3-point rule corresponding to Gauss-Legendre 2-point rule. The error associated with the rule (2.2.6) is given by

\[E_{eG_3} f = -\frac{1}{135} f^{"} \xi, \quad -1<\xi<1\]  

(2.2.7)

Hence the degree of precision of the anti-Gauss 3-point rule is 3.
III. CONSTRUCTION OF THE MIXED QUADRATURE RULE OF PRECISION FIVE

A mixed quadrature rule of precision five is constructed by using the following two quadrature rules.

(i) anti-Gauss 3-point rule \( R_{aG_3} f \)

(ii) Steffensen’s 4-point rule \( R_{St_4} f \)

The anti-Gauss 3-point rule \( R_{aG_3} f \) is
\[
I f = \int_a^b f(x) \, dx = \frac{1}{3} \left[ f(a) + \frac{2}{3} f \left( a + \frac{1}{3} \sqrt{15} \right) + 5 f \left( a + \frac{2}{5} \sqrt{15} \right) \right].
\] (3. 1)

The Steffensen’s 4-point rule \( R_{St_4} f \) is
\[
I f = \int_a^b f(x) \, dx = \frac{1}{12} \left[ 11 f \left( \frac{a + \sqrt{5}}{5} \right) + f \left( \frac{a}{5} \right) + f \left( \frac{a + \sqrt{5}}{5} \right) + 11 f \left( \frac{2}{5} \right) \right].
\] (3. 2)

Each of these rules (3. 1) and (3. 2) is of precision 3. Let \( E_{aG_3} f \) and \( E_{St_4} f \) denote the errors in approximating the integral \( I f \) by the rules (3. 1) and (3. 2) respectively.

Then,
\[
I f = R_{aG_3} f + E_{aG_3} f \quad (3. 3)
\]
\[
I f = R_{St_4} f + E_{St_4} f \quad (3. 4)
\]

Assuming \( f \) to be sufficiently differentiable in \(-1 \leq x \leq 1\), and using Maclaurin’s expansion of function \( f \) in \( x \), we can express the errors associated with the quadrature rules under reference as
\[
E_{aG_3} f = \frac{-1}{135} f^{(iv)}(0) - \frac{1016}{7! \times 675} f^{(iv)}(0) - \ldots
\] (3. 5)
\[
E_{St_4} f = \frac{38}{5625} f^{(iv)}(0) + \frac{13136}{7! \times 9375} f^{(iv)}(0) + \ldots
\] (3. 6)

Now multiplying the Eqs (3. 3) and (3. 4) by \( \frac{38}{125} \) and \( \frac{1}{3} \) respectively, then adding the results we obtain,
\[
I f = \frac{1}{239} 14R_{aG_3} f + 125R_{St_4} f + \frac{1}{139} 14E_{aG_3} f + 125E_{St_4} f
\]
or,
\[
I f = R_{aG_3} f + E_{aG_3} f + 125R_{St_4} f + 125E_{St_4} f \quad (3. 7)
\]
where
\[
R_{aG_3} f = \frac{1}{239} 14R_{aG_3} f + 125R_{St_4} f \quad (3. 8)
\]
and
\[
E_{aG_3} f = \frac{1}{239} 14E_{aG_3} f + 125E_{St_4} f \quad (3. 9)
\]

Eq. (3. 8) expresses the desired mixed quadrature rule for the approximate evaluation of \( I f \) and Eq (3. 9) expresses the error generated in this approximation.

Substituting Eqs(3. 5) and (3. 6) into Eq(3. 9), we obtain
\[
E_{aG_3} f = \frac{32}{7! \times 2151} f^{(iv)}(0) + \ldots
\] (3. 10)

As the first term of \( E_{aG_3} f \) contains 6th order derivative of the integrand, the degree
of precision of the mixed quadrature rule is 5. It is called a mixed type rule as it is constructed from two different types of rules of equal precision.

IV. ERROR ANALYSIS OF THE MIXED QUADRATURE RULE

An asymptotic error estimate and an error bound of the rule (3.8) are given in theorems 4.1 and 4.2 respectively.

Theorem 4.1
Let \( f \) be sufficiently differentiable function in the closed interval \([-1,1]\). Then the error \( E_{\omega_{ij}R_{ij}} f \) associated with the mixed quadrature rule \( R_{\omega_{ij}R_{ij}} f \) is given by
\[
\left| E_{\omega_{ij}R_{ij}} f \right| \leq \frac{32}{7 \times 2151} \left| f^{(n)} \right| 0
\]

Proof
The proof follows from the Eq (3.10)

Theorem 4.2
The bound for the truncation error
\[
E_{\omega_{ij}R_{ij}} f = I f - R_{\omega_{ij}R_{ij}} f
\]
is given by
\[
E_{\omega_{ij}R_{ij}} f \leq \frac{38M}{10755} \left| \eta_2 - \eta_1 \right|, \eta_1, \eta_2 \in -1,1
\]
where \( M = \max_{-1 < x < 1} \left| f^{(n)} \right| x \)

Proof
\[
E_{\omega_{ij}} f = \frac{-1}{135} f^{(n)} \eta_1, \eta_1 \in -1,1
\]
\[
E_{R_{ij}} f = \frac{38}{5625} f^{(n)} \eta_2, \eta_2 \in -1,1
\]
Hence
\[
E_{\omega_{ij}R_{ij}} f = \frac{1}{239} \left[ 114 E_{\omega_{ij}} f + 125 E_{R_{ij}} f \right]
\]
\[
= \frac{38}{10755} f^{(n)} \eta_2 - f^{(n)} \eta_1
\]
\[
= \frac{38}{10755} \int_{\eta_1}^{\eta_2} f^{(n)} x \, dx \text{ assuming } \eta_1 < \eta_2
\]
From this we obtain
\[
\left| E_{\omega_{ij}R_{ij}} f \right| \leq \frac{38}{10755} \int_{\eta_1}^{\eta_2} f^{(n)} x \, dx \leq \frac{38}{10755} \int_{\eta_1}^{\eta_2} f^{(n)} x \, dx
\]
or
\[
\left| E_{\omega_{ij}R_{ij}} f \right| \leq \frac{38M}{10755} \left| \eta_2 - \eta_1 \right|
\]
which gives only a theoretical error bound, as \( \eta_1, \eta_2 \) are unknown points in the interval \([-1,1]\). It shows that the error in the approximation will be less if the points \( \eta_1, \eta_2 \) are closed to each other.
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**Corollary**
The error bound for the truncation error $E_{aG_{3l4}} f$ is given by

$$\left| E_{aG_{3l4}} f \right| \leq \frac{76M}{10755}$$

**Proof**
We know from the Theorem 4.2 that

$$\left| E_{aG_{3l4}} f \right| \leq \frac{38M}{10755} \eta_2 - \eta_1$$

where $M = \max_{-1 \leq x \leq 1} f', \eta_1, \eta_2 \in -1,1$

Choosing $|\eta_2 - \eta_1| \leq 2$, we have

$$\left| E_{aG_{3l4}} f \right| \leq \frac{76M}{10755}$$

**V. ALGORITHM FOR ADAPTIVE QUADRATURE ROUTINE**
Applying the constituent rules $R_{aG_{3l4}} f$ and $R_{aG_{3l4}} f$, one can evaluate real definite integrals of the type $I f = \int_a^b f x \, dx$ in adaptive integration scheme. In the adaptive integration scheme, the desired accuracy is sought by progressively subdividing the interval of integration according to the computed behavior of the integrand, and applying the same formula over each subinterval. The algorithm for adaptive integration scheme is outlined using the mixed quadrature rule $R_{aG_{3l4}} f$ in the following four steps.

**Input:** Function $f : a,b \rightarrow R$ and the prescribed tolerance $\varepsilon$.

**Output:** An approximation $Q f$ to the integral $I f = \int_a^b f x \, dx$ such that $|Q f - I f| \leq \varepsilon$.

**Step-1:** The mixed quadrature rule $R_{aG_{3l4}} f$ is applied to approximate the integral $I f = \int_a^b f x \, dx$. The approximate value is denoted by $R_{aG_{3l4}} a,b$.

**Step-2:** The interval of integration $a,b$ is divided into two equal pieces, $a,c$ and $c,b$. The mixed quadrature rule $R_{aG_{3l4}} f$ is applied to approximate the integral $I_1 f = \int_a^c f x \, dx$ and the approximate value is denoted by $R_{aG_{3l4}} a,c$. Similarly, the mixed quadrature rule $R_{aG_{3l4}} f$ is applied to approximate the integral $I_2 f = \int_c^b f x \, dx$ and the approximate value is denoted by $R_{aG_{3l4}} c,b$.

**Step-3:** $R_{aG_{3l4}} a,c + R_{aG_{3l4}} c,b$ is compared with $R_{aG_{3l4}} a,b$ to estimate the error in $R_{aG_{3l4}} a,c + R_{aG_{3l4}} c,b$.

**Step-4:** If $|\text{estimated error}| \leq \varepsilon/2$ (termination criterion) then $R_{aG_{3l4}} a,c + R_{aG_{3l4}} c,b$ is
accepted as an approximation to $\int_a^b f(x) \, dx$. Otherwise the same procedure is applied to $a, c$ and $c, b$, allowing each pieces a tolerance of $ε/2$. If the termination criterion is not satisfied on one or more of the sub intervals, then those subintervals must be further subdivided and the entire process is repeated. When the process stops, the addition of all accepted values yields the desired approximate value $Q_f$ of the integral $\int f$. If $|Q_f - I_f| ≤ ε$.

N:B: In this algorithm we can use any quadrature rule to evaluate real definite integrals in adaptive integration scheme.

VI. NUMERICAL VERIFICATION

Table 6. 1: Comparative study of the quadrature rules $R_{Gl_2} f$, $R_{Gl_3} f$ and $R_{Gl_4} f$ for approximation of some singular/non-singular integrals without using adaptive integration scheme

<table>
<thead>
<tr>
<th>Integrals</th>
<th>Exact Value $I_f$</th>
<th>Approximate Value $Q_f$</th>
<th>$R_{Gl_2} f$</th>
<th>$R_{Gl_3} f$</th>
<th>$R_{Gl_4} f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_1 f = \int_0^1 \sin^{-1} x , dx$</td>
<td>1.088793045</td>
<td>1.0798</td>
<td>1.0971</td>
<td>1.0802</td>
<td></td>
</tr>
<tr>
<td>$I_2 f = \int_0^1 e^{x^2} , dx$</td>
<td>2.92530349</td>
<td>2.5825</td>
<td>3.0201</td>
<td>2.5885</td>
<td></td>
</tr>
<tr>
<td>$I_3 f = \int_0^{\frac{\pi}{4}} \sin x , dx$</td>
<td>0.52840808</td>
<td>0.5310</td>
<td>0.5257</td>
<td>0.5309</td>
<td></td>
</tr>
<tr>
<td>$I_4 f = \int_1^\infty \frac{\ln x}{x - 1} , dx$</td>
<td>0.43296997</td>
<td>0.4356</td>
<td>0.4302</td>
<td>0.4355</td>
<td></td>
</tr>
<tr>
<td>$I_5 f = \int_0^{\frac{\pi}{4}} \cos 2x , dx$</td>
<td>0.88023061</td>
<td>0.7620</td>
<td>0.9395</td>
<td>0.7647</td>
<td></td>
</tr>
<tr>
<td>$I_6 f = \int_0^\infty \frac{e^{-x}}{\sqrt{1 - x}} , dx$</td>
<td>1.076159013</td>
<td>0.9500</td>
<td>1.1110</td>
<td>0.9522</td>
<td></td>
</tr>
<tr>
<td>$I_7 f = \int_{-1}^{1} \frac{dx}{x - 2} \left(1 - x^{3/4}\right)^{-1/2} \left(1 + x^{3/4}\right)$</td>
<td>-1.9490...</td>
<td>-1.2798</td>
<td>-1.8702</td>
<td>-1.2874</td>
<td></td>
</tr>
<tr>
<td>$I_8 f = \int_{-1}^{1} \frac{\cos \pi x}{1 - x^{1/3}} , dx$</td>
<td>-0.69049...</td>
<td>-0.5617</td>
<td>-0.4688</td>
<td>-0.5349</td>
<td></td>
</tr>
<tr>
<td>$I_9 f = \int_0^1 \ln x - \sin x , dx$</td>
<td>-4.804378845</td>
<td>-4.4960</td>
<td>-5.0503</td>
<td>-4.5066</td>
<td></td>
</tr>
<tr>
<td>$I_{10} f = \int_0^1 \ln x , dx$</td>
<td>-0.9159655942</td>
<td>-0.8171</td>
<td>-0.9912</td>
<td>-0.8202</td>
<td></td>
</tr>
<tr>
<td>$I_{11} f = \int_0^1 \frac{1}{\sqrt{x}} , dx$</td>
<td>2</td>
<td>1.6506</td>
<td>2.1009</td>
<td>1.6569</td>
<td></td>
</tr>
</tbody>
</table>
**Approximation of Singular Integrals by a Mixed Quadrature**

\[
I_{12} \ f = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \ dx \quad \pi \approx 3.1415926 \quad 2.4494 \quad 3.3373 \quad 2.4617
\]

\[
I_{13} \ f = \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}} \ dx \quad -1.488793045 \quad -0.9882 \quad -1.1661 \quad -0.9915
\]

\[
I_{14} \ f = \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}} \ dx \quad 0.777504634112 \quad 0.7775116 \quad 0.777497 \quad 0.7775110
\]

\[
I_{15} \ f = \int_{0}^{1} \frac{1}{x} \ dx \quad 0.69666666... \quad 0.6738 \quad 0.6598 \quad 0.6735
\]

\[
I_{16} \ f = \int_{0}^{1} e^{-x^2} \ dx \quad 0.7468241328 \quad 0.7465 \quad 0.7470 \quad 0.7466
\]

**Table 6.2:** Comparative study of quadrature/ mixed quadrature rules \( R_{a,t} \ f, R_{a,t,t} \ f, \) and \( R_{a,t,t} \ f \) for approximating some singular/non singular integrals (as given in Table-6.1) without using adaptive integration scheme.

<table>
<thead>
<tr>
<th>Integrals</th>
<th>Approximate Value</th>
<th>( Q \ f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I_1 \ f = \int_{0}^{1} \frac{\sin^{-1}x}{x} \ dx )</td>
<td>1.0854</td>
<td>1.0844</td>
</tr>
<tr>
<td>( I_2 \ f = \int_{0}^{1} \frac{e^x}{\sqrt{x}} \ dx )</td>
<td>2.6785</td>
<td>2.6506</td>
</tr>
<tr>
<td>( I_3 \ f = \int_{0}^{1} \frac{\sin x}{x^{1/4}} \ dx )</td>
<td>0.5292</td>
<td>0.5295</td>
</tr>
<tr>
<td>( I_4 \ f = \int_{0}^{1} \frac{\ln x}{x^{-1/4}} \ dx )</td>
<td>0.4336</td>
<td>0.4338</td>
</tr>
<tr>
<td>( I_5 \ f = \int_{0}^{1} \frac{\cos 2x}{x^{1/3}} \ dx )</td>
<td>0.8045</td>
<td>0.7930</td>
</tr>
<tr>
<td>( I_6 \ f = \int_{0}^{1} \frac{e^{-x}}{\sqrt{1-x}} \ dx )</td>
<td>0.9853</td>
<td>0.9751</td>
</tr>
<tr>
<td>( I_7 \ f = \int_{-1}^{1} \frac{dx}{x^{2} - 1 - x^{1/4} + x^{3/4}} )</td>
<td>-1.4026</td>
<td>-1.3662</td>
</tr>
<tr>
<td>( I_8 \ f = \int_{0}^{1} \frac{\cos \pi x}{x^{1/2}} \ dx )</td>
<td>-0.3167</td>
<td>-0.2571</td>
</tr>
<tr>
<td>( I_9 \ f = \int_{0}^{1} \frac{\ln x - \sin x}{x} \ dx )</td>
<td>-4.6514</td>
<td>-4.6159</td>
</tr>
<tr>
<td>( I_{10} \ f = \int_{0}^{1} \frac{\ln x}{x^{1/2}} \ dx )</td>
<td>-0.8638</td>
<td>-0.8520</td>
</tr>
<tr>
<td>( I_{11} \ f = \int_{0}^{1} \frac{1}{\sqrt{x}} \ dx )</td>
<td>1.7508</td>
<td>1.7221</td>
</tr>
</tbody>
</table>
\[ I_{12} f = \int_{0}^{1} \frac{1}{\sqrt{1 + x}} dx \quad 2.6457 \quad 2.5890 \quad 2.8794 \]

\[ I_{13} f = \int_{0}^{1} \frac{\ln x}{\sqrt{1 - x^2}} dx \quad -1.0380 \quad -1.0266 \quad -1.0748 \]

\[ I_{14} f = \int_{0}^{1} \frac{x}{e^x - 1} dx \quad 0.777504624 \quad 0.777504616 \quad 0.777504633 \]

\[ I_{15} f = \int_{0}^{1} \sqrt{x} dx \quad 0.6691 \quad 0.6700 \quad 0.66700 \]

\[ I_{16} f = \int_{0}^{1} e^{-x^2} dx \quad 0.746814 \quad 0.746806 \quad 0.7468235 \]

\textit{N. B:} \( R_{St,Gl,2} f = \text{mixed quadrature of Steffensen’s 4-point rule and Gauss-Legendre 2-point rule(constructed in[16])} \)

\textbf{Table 6.3:} Comparison of the results following from the Gauss-Legendre 2-point rule, anti-Gauss 3-point rule and Steffensen’s 4-point rule for approximating some singular/non singular integrals (as given in Table-6.1) using the adaptive integration scheme

<table>
<thead>
<tr>
<th>Integrals</th>
<th>Approximate Value ( Q f )</th>
<th>Gauss-Legendre 2-point rule ( R_{Gl,2} )</th>
<th>No. of steps</th>
<th>anti-Gauss 3-point rule ( R_{ag,3} )</th>
<th>No. of steps</th>
<th>Steffensen’s 4-point rule ( R_{St,4} )</th>
<th>No. of steps</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I_1 f = \int_{0}^{1} \frac{1}{x} \sin^{-1}x dx )</td>
<td>1.0887927</td>
<td>31</td>
<td>1.0887932</td>
<td>31</td>
<td>1.0887928</td>
<td>31</td>
<td></td>
</tr>
<tr>
<td>( I_2 f = \int_{0}^{1} \frac{1}{\sqrt{x}} e^{x} dx )</td>
<td>2.92530203</td>
<td>139</td>
<td>2.92530509</td>
<td>131</td>
<td>2.92530206</td>
<td>137</td>
<td></td>
</tr>
<tr>
<td>( I_3 f = \int_{0}^{1} \frac{1}{x} \sin x dx )</td>
<td>0.52840833</td>
<td>21</td>
<td>0.5284078</td>
<td>21</td>
<td>0.52840831</td>
<td>21</td>
<td></td>
</tr>
<tr>
<td>( I_4 f = \int_{0}^{1} \frac{1}{x - 1} \ln x dx )</td>
<td>0.43297013</td>
<td>23</td>
<td>0.4329698</td>
<td>23</td>
<td>0.43297012</td>
<td>23</td>
<td></td>
</tr>
<tr>
<td>( I_5 f = \int_{0}^{1} \frac{1}{x^{1/3}} \cos 2x dx )</td>
<td>0.88022962</td>
<td>91</td>
<td>0.8802314</td>
<td>89</td>
<td>0.88022963</td>
<td>89</td>
<td></td>
</tr>
<tr>
<td>( I_6 f = \int_{0}^{1} \frac{1}{x^{1/4}} e^{-x} dx )</td>
<td>1.07615765</td>
<td>115</td>
<td>1.07616052</td>
<td>107</td>
<td>1.07615763</td>
<td>111</td>
<td></td>
</tr>
<tr>
<td>( I_7 f = \int_{-1}^{1} \frac{dx}{x^{3/4} - 1 - x^{1/4}} )</td>
<td>-1.94901581</td>
<td>197</td>
<td>-1.94904174</td>
<td>177</td>
<td>-1.94901654</td>
<td>197</td>
<td></td>
</tr>
</tbody>
</table>
Approximation of Singular Integrals by a Mixed Quadrature

\[ I_8 \ f = \int_{-1}^{1} \frac{\cos \pi x}{1-x^2} dx \quad -0.6904813 \quad 101 \quad -0.6905059 \quad 95 \quad -0.6904817 \quad 101 \]

\[ I_9 \ f = \int_{0}^{1} \ln (x - \sin x) \ dx \quad -4.808437299 \quad 83 \quad -4.80843859 \quad 83 \quad -4.80843724 \quad 83 \]

\[ I_{10} \ f = \int_{0}^{1} \frac{\ln x}{1+x^2} dx \quad -0.91596499 \quad 69 \quad -0.91596610 \quad 69 \quad -0.9159649 \quad 63 \]

\[ I_{11} \ f = \int_{0}^{1} \frac{1}{\sqrt{x}} dx \quad 1.9999985 \quad 139 \quad 2.0000016 \quad 131 \quad 1.999998 \quad 137 \]

\[ I_{12} \ f = \int_{-1}^{1} \frac{1-x}{1+x} dx \quad 3 \quad 141591373 \quad 179 \quad 3 \quad 141594091 \quad 171 \quad 3.141591392 \quad 177 \]

\[ I_{13} \ f = \int_{0}^{1} \frac{\ln x}{\sqrt{1-x^2}} dx \quad -1.08879265 \quad 85 \quad -1.088793352 \quad 85 \quad -1.088792603 \quad 81 \]

\[ I_{14} \ f = \int_{0}^{1} \frac{x}{e^x-1} dx \quad 0 \quad 777504660 \quad 3 \quad 0 \quad 777504607 \quad 3 \quad 0.777504658 \quad 3 \]

\[ I_{15} \ f = \int_{0}^{1} \sqrt{x} dx \quad 0.6666670 \quad 29 \quad 0.6666634 \quad 29 \quad 0.6666698 \quad 29 \]

\[ I_{16} \ f = \int_{0}^{1} e^{-x^2} dx \quad 0 \quad 746824127 \quad 15 \quad 0 \quad 746824138 \quad 15 \quad 0 \quad 7468241280 \quad 15 \]

**N. B:** The prescribed tolerance \( \varepsilon \) = 0.00001

**Table 6.4:** Comparison of the results following from the Gauss-Legendre 3-point rule, mixed quadrature rule \( R_{St_{GL_3}} f \) and mixed quadrature rule \( R_{St_{GL_2}} f \) for approximating some singular/non-singular integrals (as given in Table 6.1) using the adaptive integration scheme.
\[
\begin{array}{|c|c|c|c|c|}
\hline
I_4 f &= \frac{1}{4} \ln x \quad 0.43297007 \\
I_5 f &= \frac{1}{2} \cos 2x \quad 0.88022986 \\
I_6 f &= \frac{1}{\sqrt{1-x^2}} d\theta \quad 1.07615798 \\
I_7 f &= \frac{1}{x - 2} \sqrt{1 - x^2} \quad -1.94902185 \\
I_8 f &= \frac{1}{x + 1} d\theta \quad -0.6904835 \\
I_9 f &= \ln x \quad -4.80843752 \\
I_{10} f &= \ln \frac{1}{x} \quad -0.91596517 \\
I_{11} f &= \frac{1}{\sqrt{x}} d\theta \quad 1.9999990 \\
I_{12} f &= \frac{1}{\sqrt{1 + x}} \quad 3.141591812 \\
I_{13} f &= \frac{1}{\sqrt{1 - x^2}} d\theta \quad -1.088792781 \\
I_{14} f &= \frac{1}{\sqrt{1 - e^x}} d\theta \quad 0.7775046339 \\
I_{15} f &= \sqrt{x} d\theta \quad 0.66666689 \\
I_{16} f &= \frac{1}{\sqrt{e^x - 1}} d\theta \quad 0.74682413241 \\
\hline
\end{array}
\]

\(N:B:\) The prescribed tolerance \(\varepsilon = 0.00001\)

All the computations are done using ‘C’ Program [15].

VII. RESULTS ANALYSIS OF TEST INTEGRALS

(i) From table-6. 1, we observe that, though Gauss-Legendre 2-point rule, Steffensen’s 4-point rule and anti-Gauss 3-point rule are of same precision, the anti-Gauss 3-point rule is giving better result than the other two when integrals \(I_6 f, I_7 f, I_{10} f, I_{15} f\) are evaluated without using adaptive integration scheme.

(ii) From tables 6. 1 and 6. 2, we see that when integrals \(I_4 f - I_{16} f\) are evaluated without using adaptive integration scheme the mixed quadrature rule \(R_{\text{anti}}\) \(f\) is giving more accurate result than their corresponding constituent rules. Also it gives better results than Gauss-Legendre 3-point rule and the already established mixed quadrature of Steffensen’s 4-point rule and Gauss-Legendre 2-point rule.
(iii) From table-6. 3, we see that the anti-Gauss 3-point rule involves significantly less number of steps in approximating the integrals \( I_i f - I_{ik} f \) in the adaptive integration scheme than those in Gauss-Legendre 2-point rule and Steffensen’s 4-point rule.

(iv) Tables 6. 3 and 6. 4 reveal that the mixed quadrature rule \( R_{\alpha, \beta} f \) involves significantly less number of steps in approximating the integrals \( I_i f - I_{ik} f \) in the adaptive integration scheme and yields better result than their constituent rules, Gauss-Legendre 3-point rule \( R_{\alpha} f \) and the mixed quadrature of Steffensen’s 4-point rule and Gauss-Legendre 2-point rule \( R_{\alpha, \beta} f \).

VIII. CONCLUSION

From above observations we conclude that both in adaptive and non adaptive environments, the mixed quadrature of anti-Gauss 3-point rule and Steffensen’s 4-point rule is more efficient than its constituent rules such as

(i) anti-Gauss 3-point rule and Steffensen’s 4-point rule;
(ii) Gauss-Legendre 3-point rule,
(iii) the mixed quadrature of Steffensen’s 4-point rule and Gauss-Legendre 2-point rule which has been established earlier in a paper.

REFERENCES


