Uniform Stability For Impulsive Differential Equation By Razumikhin Method

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Abstract

By using the Lyapunov functions and analysis technique along with Razumikhin technique, several results for the uniform stability criteria have been derived. The obtained findings extend and generalize several results existing in the literature.

Keywords: Impulsive delay systems, Lyapunov function, Razumikhin technique, Uniform stability.

I Introduction

Impulsive delay differential equations have enormous possible applications in the theory of control systems, power systems, time-varying nonlinear feedback systems, biology, pharmacokinetics, and so on [1, 4, 5, 8]. There are many applications in which there is need to find the qualitative properties of the solutions, because in many cases it is not possible to find the solutions of the differential equation. In those cases we depend on the qualitative properties of the equation.

In recent years many kinds of stability problems of impulsive delay differential equations have been studied and many interesting results have been obtained [2, 7]. An impulsive differential system generally consists of three components: a continuous differential equation, which directs the motion of the system between impulsive actions, a discrete system of difference equations, which direct the method of the system state change at once when a situation event occurs; and a condition for finding when the states of the system are to be rearranged. The important methods: Lyapunov functions and Razumikhin technique are very useful for the study of stability analysis of different kinds of delay differential equations, and it have been proved that these
techniques are a powerful instrument in the study and research of the properties of impulsive delay differential equations [4, 3]. There are a numerous research works appeared in the literature on impulsive delayed linear differential equations. In this paper we will study the uniform stability for impulsive differential equation by Razumikhin method. As a result criteria on uniform stability can be obtained.

This paper is organized as follows. In Section II, we introduce some basic definitions and notations. In Section III, we get some criteria for uniform stability for impulsive differential equation by Razumikhin method.

II Preliminaries
Consider the following impulsive differential:

\[ x'(t) = f(t, x_t), t \in [t_{k-1}, t_k), \]

\[ \Delta x(t_k) = I_k(t_k, x_{t-}), t = t_k, k \in N, \]

\[ x_{t_0} = \eta \]

(1)

We assume that function \( f, I_k: R_+ \times PC([-\tau, 0], R^n) \rightarrow R^n \) and \( \eta \in PC([-\tau, 0], R^n) \) satisfy all required conditions for existence and uniqueness of the solutions for all \( t \geq t_0 \). The time sequence \( \{t_k\}_{k=1}^{\infty} \) satisfy \( 0 = t_0 < t_1 < \cdots < t_k < \cdots \) and \( t_k = \lim_{k \to \infty} t_k = \infty \), \( \Delta x(t_k) = x(t^+) - x(t^-) \) and \( x_{t_0}, x_{t-} \in PC([-\tau, 0], R^n) \) are defined by \( x_t(r) = x(t + r) \) and \( x_{t-}(r) = x(t^- + r) \) for \( -\tau \leq r \leq 0 \) respectively. We shall assume that \( f(t, 0) = 0 \), and \( I_k(t, 0) = 0 \) for all \( t \in R_+ \) and \( k \in N \), so that system (1) admits the trivial solution.

Given a constant \( \tau > 0 \), we equip the linear space \( PC([-\tau, 0], R^n) \) with the norm \( \| \|_\tau = \sup_{-\tau \leq r \leq 0} \| \psi(r) \| \). Denote \( x(t) = x(t, t_0, \eta) \), be the solution of (1) s. t. \( x(\sigma + r) = \eta(r), r \in [-\tau, 0]. \)

We further assume that all the solutions \( x(t) \) of (1) are continuous except at \( t_k, k \in N \) at which \( x(t) \) is right continuous.

**Definition 1:** The function \( V: R_+ \times R^n \rightarrow R^+ \) is said to belong to the class \( \nu_0 \) if we have the following.

1) \( V \) is continuous in each of the sets \( [t_{k-1}, t_k) \times R^n \), and for each \( x \in R^n \), \( t \in [t_{k-1}, t_k) \), \( k \in N \), \( \lim_{(t,w) \rightarrow (t_k^-, x)} V(t, w) = V(t_k^-, x) \) exists.

2) \( V(t, x) \) is locally Lipschitz in all \( x \in R^n \), and for all \( t \geq t_0 \), \( V(t, 0) \equiv 0 \).

**Definition 2:** Given a function \( V: R_+ \times R^n \rightarrow R^+ \), the upper right-hand derivative of \( V \) with respect to system (1) is defined by \( D^+ V(t, x(t)) = \limsup_{\delta \to 0^+} \frac{1}{\delta} [V(t+\delta, x(t+\delta)) - V(t, x(t))] \).

We shall make the following hypotheses.

(H1) \( f(t, \varphi) \) is composite-PC, i. e., if for each \( t_0 \in R_+ \) and \( \alpha > 0 \), where \( [t_0, t_0 + \alpha] \in R_+ \), if \( x \in PC([-\tau, t_0 + \alpha], R^n) \) and \( x \) is continuous at each \( t \neq t_k \) in \( (t_0, t_0 + \alpha] \), then the composite function \( g \) defined by \( g(t) = f(t, x_t) \) is an element o the function class \( PC([-t_0, t_0 + \alpha], R^n) \).
(H2) \(f(t, \varphi)\) is quasi-bounded, i.e., if for some \(t_0 \in R_+\) and \(\alpha > 0\), where \([t_0, t_0 + \alpha] \in R_+\), and for each compact set \(F \in R^n\) there exists some \(M > 0\) such that \(\|f(t, \varphi)\| \leq M\) for all \((t, \varphi) \in [t_0, t_0 + \alpha] \times PC([-\tau, 0], F)\). is

(H3) For each fixed \(t \in R_+, f(t, \varphi)\) is a continuous function of \(\varphi\) on \(PC([-\tau, 0], R^n)\).

It is shown in[3] that under assumptions (H1)-(H3), the initial value problem has a solution \(x(t, t_0, \eta) \equiv x(t)\) existing in a maximal interval \(I\). If, in addition, \(f(t, \varphi)\) is locally Lipschitz in \(\varphi\), then the solution \(x(t) \equiv 0\).

### III Main Results

In the following, we shall establish criteria on impulsive differential equation with any time delay for uniform stability. We have the followings results.

**Theorem 1:** Suppose that there exist a function \(V \in v_0\) and some positive constants \(p, c_1, c_2, \beta_k \geq 0, k \in N\) such that following conditions hold:

1. \(c_1|x|^p \leq V(t, x) \leq c_2|x|^p\), for any \(t \in R^+\) and \(x \in R^n\)
2. \(D^+V(t, \psi(0)) \leq -N(t)\psi(0), \) for all \(t \neq t_k\) in \(R_+\). Whenever \(V(t, \psi(0)) \geq V(t + r, \psi(r))\) for \(r \in [-\tau, 0], \) where \(N(t) \in PC([t_0 - \tau, \infty), R_+)\) and \(inf_{t \geq t_0 - \tau} N(t) \geq \delta\)
3. There exist a positive constant \(\xi_k\), where \(0 < \xi_k < 1, \forall k \in N\) s.t. \(V(t, \psi(0) + I(t_k, \psi)) \leq (1 + \xi_k) V(t, \psi(0))\), with \(\sum_{k=1}^{\infty} \xi_k < \infty\) and \(\psi(0) = \psi^(-)\)

Then the zero solution of (1) is uniformly stable.

**Proof:** Let \(x(t) = x(t, t_0, \eta)\) be any solution of the impulsive system (1) with the initial condition \(x(t_0) = \eta\) and \(V(t) = V(t, x(t))\). Let for any given \(\epsilon > 0\), there exist \(\delta > 0\) such that \(\frac{\epsilon^2 \mathcal{B}^1}{c_1} \delta < \epsilon\) and \(\|\eta\|_\tau < \delta\) where \(\mathcal{B} = \prod_{i=0}^{\infty} (1 + \xi_i)\), since \(\sum_{k=1}^{\infty} \xi_k < \infty\).

We shall show that

\[
V(t) \leq c_2 \sum_{k=1}^{\infty} (1 + \xi_i) \|\eta\|_\tau^p, t\epsilon[t_k-1, t_k), k\epsilon N, where \epsilon_0 = 0
\]

\[
V(t) - c_2 \sum_{k=1}^{\infty} (1 + \xi_i) \|\eta\|_\tau^p, t\epsilon[t_k-1, t_k), k\epsilon N
\]

Let \(M(t) = V(t) - c_2 \|\eta\|_\tau^p, t\epsilon[t_0 - \tau, t_0]\)

We need to show that \(M(t) \leq 0\) for all \(t \geq t_0\). It is clear that \(M(t) \leq 0\) for \(t\epsilon[t_0 - \tau, t_0]\)

Since \(M(t) \leq V(t) - c_2 \|\eta\|_\tau^p \leq 0\) by condition (i).

Take \(k = 1\), we shall show that \(M(t) \leq 0\) for \(t \in [t_0, t_1]\). Let \(\gamma > 0\) be arbitrary. Now we will show that \(M(t) \leq \gamma\) for \(t \in [t_0, t_1]\). Suppose not, then there exist some \(t \in [t_0, t_1]\) such that \(M(t) > \gamma\). Let \(t^* = \inf\{t \in [t_0, t_1]: M(t) > \gamma\}\), since \(M(t) \leq 0 < \gamma\) for \(t\epsilon[t_0 - \tau, t_0]\), we know \(t^* \in [t_0, t_1]\). Note that \(M(t)\) is continuous on \([t_0, t_1]\), then \(M(t^*) = \gamma\) and \(M(t) \leq \gamma\) for \(t\epsilon[t_0 - \tau, t^*]\).
Now \( V(t^*) = M(t^*) + c_2 \| \eta \|_r^p \) and for \( r \in [-\tau, 0) \), we have
\[
V(t^* + r) = M(t^* + r) + c_2 \| \eta \|_r^p \\
\leq \gamma + c_2 \| \eta \|_r^p
\]
Thus by condition (i),
\[
V(t^*) = V(t^* + r) - \gamma + c_2 \| \eta \|_r^p
\]
So by condition (ii), we have \( D^+V(t^*) \leq -\kappa(t^*)V(t^*) \), then we have
\[
D^+M(t^*) = D^+V(t^*) + \kappa(t^*)c_2 \| \eta \|_r^p \\
\leq -\kappa(t^*)(V(t^*) - c_2 \| \eta \|_r^p)
\]
which contradicts the definition of \( t^* \), so we get \( M(t) \leq \gamma \) for all \( t \in [t_0, t_1] \). Let \( \gamma \to 0^+ \), we have \( M(t) \leq 0 \) for \( t \in [t_0, t_1] \).
Now we assume that \( M(t) \leq 0 \) for \( t \in [t_0, t_m], m \geq 1 \). We shall show that \( M(t) \leq 0 \) for \( t \in [t_0, t_{m+1}] \).
By condition (iii), we have
\[
M(t_m) = V(t_m) - c_2 \prod_{i=0}^m (1 + \xi_i) \| \eta \|_r^p \\
\leq (1 + \xi_m)V(t_m) - c_2 \prod_{i=0}^m (1 + \xi_i) \| \eta \|_r^p
\]
Thus by condition (ii), \( D^+V(t^*) \leq -\kappa(t^*)V(t^*) \), and then we have
\[
D^+M(t^*) = D^+V(t^*) + \kappa(t^*)c_2 \prod_{i=0}^m (1 + \xi_i) \| \eta \|_r^p \\
\leq -\kappa(t^*)(V(t^*) - c_2 \prod_{i=0}^m (1 + \xi_i) \| \eta \|_r^p)
\]
Again this contradicts the definition of \( t^* \), which implies \( M(t) \leq \gamma \) for all \( t \in [t_m, t_{m+1}] \). Let \( \gamma \to 0^+ \), we have \( M(t) \leq 0 \) for all \( t \in [t_m, t_{m+1}] \). So \( M(t) \leq 0 \) for all \( t \in [t_0, t_{m+1}] \).
Thus by the method of induction, we get
\[
V(t) \leq c_2 \prod_{i=0}^{k-1} (1 + \xi_i) \| \eta \|_r^p, t \in [t_{k-1}, t_k], k \in N.
\]
By condition (i)-(iii), we have
Evaluating the integral of $\int_{t_0}^{t} f(s) \, ds$ yields $F(t) = F(t_0) + \int_{t_0}^{t} f(s) \, ds$. This expression represents the accumulation of the function $f(s)$ from $t_0$ to $t$. The initial condition $F(t_0) = C$ sets the baseline value for the integral. For instance, if $f(s) = s^2$ and $t_0 = 0$, then $F(t) = \int_{0}^{t} s^2 \, ds = \left[ \frac{s^3}{3} \right]_{0}^{t} = \frac{t^3}{3}$, where $C = 0$. This example illustrates how the integral relates the initial condition to the subsequent behavior of $F(t)$.
Suppose that $\mathcal{M}(t) \leq 0$ for $t \in [t_0, t_k), k \geq 1$. Now we shall show that $\mathcal{M}(t) \leq 0$ for $t \in [t_0, t_{k+1})$. For any $\gamma > 0$, we shall show that $\mathcal{M}(t) \leq \gamma$ for $t \in (t_k, t_{k+1})$. Suppose not, let $t^* = \inf \{ t \in [t_k, t_{k+1}); \mathcal{M}(t) > \gamma \}$. By condition (iii) we have

$$\mathcal{M}(t_k) = V(t_k) - \omega_k\left(\omega_{k-1}(\ldots (\omega_1(\omega_0(V(t_0))) \ldots))\right) \leq \omega_k(V(t_k^-)) - \omega_k\left(\omega_{k-1}(\ldots (\omega_1(\omega_0(V(t_0))) \ldots))\right) \leq \omega_k\left(\omega_{k-1}(\ldots (\omega_1(\omega_0(V(t_0))) \ldots))\right) - \omega_k\left(\omega_{k-1}(\ldots (\omega_1(\omega_0(V(t_0))) \ldots))\right) = 0$$

As $\mathcal{M}(t_k) \leq 0 < \gamma$, so by the continuity of $\mathcal{M}(t)$, we have $t^* = t_k, \mathcal{M}(t^*) = \gamma$ and $\mathcal{M}(t) \leq \gamma$ for $t \in [t_0 - \tau, t^*)$.

Since $V(t^*) = \mathcal{M}(t^*) + \omega_k\left(\omega_{k-1}(\ldots (\omega_1(\omega_0(V(t_0))) \ldots))\right)$, when $t^* + r \geq t_k, \forall r \in [-\tau, 0]$, then for $r \in [-\tau, 0]$ we have

$$V(t^* + r) = \mathcal{M}(t^* + r) + \omega_k\left(\omega_{k-1}(\ldots (\omega_1(\omega_0(V(t_0))) \ldots))\right) \leq \gamma + \omega_k\left(\omega_{k-1}(\ldots (\omega_1(\omega_0(V(t_0))) \ldots))\right) \leq V(t^*)$$

When $t^* + r < t_k$ for some $r \in [-\tau, 0]$, then for $0 \leq \omega_k(jr) \leq j\omega_k(r)$ and $\omega_k(r) \geq r$ holds for any $j \geq 0$ and $r \geq 0$, we have, for any $r \in [-\tau, 0]$ and $q < k, q, k \in N$,

$$\omega_q\left(\omega_{q-1}(\ldots (\omega_1(\omega_0(V(t_0))) \ldots))\right) \leq \omega_k\left(\omega_{k-1}(\ldots (\omega_1(\omega_0(V(t_0))) \ldots))\right)$$

So in this case we can also get $V(t^* + r) \leq V(t^*)$ holds for all $r \in [-\tau, 0]$. Thus by condition (ii), we have $D^+V(t^*) \leq -\mathcal{M}(t^*)$, and then we have

$$D^+ \mathcal{M}(t^*) = D^+V(t^*) + \omega_k\left(\omega_{k-1}(\ldots (\omega_1(\omega_0(V(t_0))) \ldots))\right) \leq -\mathcal{M}(t^*) + \omega_k\left(\omega_{k-1}(\ldots (\omega_1(\omega_0(V(t_0))) \ldots))\right) \leq -(V(t^*) - \omega_k\left(\omega_{k-1}(\ldots (\omega_1(\omega_0(V(t_0))) \ldots))\right)) \leq -\gamma < 0$$

Which is a contradiction of the definition of $t^*$, so we get $\mathcal{M}(t) \leq \gamma$ for all $t \in [t_0, t_{k+1})$. Let $\gamma \to 0^+$, we have $\mathcal{M}(t) \leq 0$ for $t \in [t_k, t_{k+1})$. Thus $\mathcal{M}(t) \leq 0$ for all $t \in [t_0, t_{k+1})$ by principle of mathematical induction proves that $V(t) \leq \omega_k\left(\omega_{k-1}(\ldots (\omega_1(\omega_0(V(t_0))) \ldots))\right)$, for $t \in [t_{k-1}, t_k), k \in N$. By condition (iii), we have

$$\omega_k\left(\omega_{k-1}(\ldots (\omega_1(\omega_0(V(t_0))) \ldots))\right) \leq \omega_k\left(\omega_{k-1}(\ldots (\omega_1(\omega_0(V(t_0))) \ldots))\right) \leq \mathcal{R}V(t_0), t \geq t_0$$

Thus by condition (i), we have

$$c_1\|x\|^p \leq V(t) \leq c_2\mathcal{R}\|\eta\|_\tau^P, t \geq t_0$$

$$\|x\| \leq \frac{c_2\mathcal{R}}{c_1}t^{\frac{1}{p}}\|\eta\|_\tau, t \geq t_0$$

Hence the proof is complete.

Corollary 1: Let us consider that the hypothesis (H1-H3) are satisfied, also the condition (i), (ii) of the theorem 2 hold and the condition (iii) is changed by
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(iii) \( V(t_k, \psi(0) + I_k(t_k, \psi)) \leq \omega_k \left( V(t_k^-, \psi(0)) \right) \), where \( \psi(0^-) = \psi(0) \), and \( \omega_k(r) = \left( 1 + \frac{1}{k^3 + r^2} \right) r, \forall k \in N \)

Then the zero solution of (1) is uniformly stable.

**Proof:** Note that \( \omega_k(r) = \left( 1 + \frac{1}{k^3 + r^2} \right) r \leq |r| \left( 1 + \frac{1}{k^3} \right), k \in N \), then by Theorem 2 result is true.

IV. Example

Assume the impulsive delay differential equation

\[
\begin{align*}
x' (t) &= -q(t)x(t) + \frac{r(t)}{1 + (x(t))}x(t - \tau), t \geq t_0 = 0, \\
x(t_k^-) &= (1 + b_k)x(t_k^-), t_k = k, k \in N, \\
x_{t_0} &= \eta,
\end{align*}
\]

Where constants \( \tau, b_k > 0 \) with \( \sum_{k=1}^{\infty} b_k < \infty \), the functions \( q, r \in C(R, R^+), \psi \in PC([-\tau, 0], R^n). \) If \( q(t) \geq r(t) + a \) where \( a \) is a constant then zero solution of this system is uniformly stable.

Proof: Let \( V(x) = V(t, x) = |x|, N(t) = a, \forall t \geq t_0 - \tau, \) then

\[
D^+ V(t, \psi(0)) \leq sgn(\psi(0))[-q(t)\psi(0) + \frac{r(t)}{1 + (\psi(0))} \psi(-\tau)]
\]

\[
\leq -q(t)|\psi(0)| + r(t)|\psi(-\tau)|
\]

\[
\leq -q(t)V(\psi(0)) + r(t)V(\psi(-\tau))
\]

For any result of the equation (2), which hold \( V(t, \psi(0)) \geq V(t + r, \psi(r)) \) for \( r \in [-\tau, 0] \), we know that \( V(\psi(\tau)) \leq V(\psi(0)) \), so

\[
D^+ V(t, \psi(0)) \leq (-q(t) + r(t))V(\psi(0))
\]

Also \( q(t) \geq r(t) + a \), then

\[
D^+ V(t, \psi(0)) \leq -aV(\psi(0)) \leq N(t)V(\psi(0))
\]

When \( V(t, \psi(0)) \geq V(t + r, \psi(r)) \) for \( r \in [-\tau, 0] \), and also \( V(t_k, \psi(0) + I(t_k, \psi)) \leq (1 + \xi_k)V(t_k^-, \psi(0)) \). Then by Theorem 1 the zero solution of (2) is uniformly stable.

V. References


