Milovanović Bounds for Seidel Energy of a Graph

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Abstract

In this paper Seidel Energy of Cocktail Party graph and Crown graph are computed. Recently Milovanović et al. gave a sharper lower bounds for energy of a graph. Similar bounds for Siedel energy of a graph are established.

AMS Subject Classification: Primary 05C50, 05C69.
Keywords: Seidel set, Seidel matrix, Seidel eigenvalues, Seidel energy.

1. Introduction

The concept of energy of a graph was introduced by I. Gutman [5] in the year 1978. Let G be a graph with n vertices and m edges and let $A = (a_{ij})$ be the adjacency matrix of the graph. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A, assumed in non increasing order, are the eigenvalues of the graph G. As A is real symmetric, the eigenvalues of G are real with
sum equal to zero. The energy $E(G)$ of $G$ is defined to be the sum of the absolute values of the eigenvalues of $G$. i.e., $E(G) = \sum_{i=1}^{n} |\lambda_i|.$

For details on the mathematical aspects of the theory of graph energy see the reviews [6], papers [2, 3, 7] and the references cited there in. The basic properties including various upper and lower bounds for energy of a graph have been established in [9, 11] and it has found remarkable chemical applications in the molecular orbital theory of conjugated molecules [4, 8]. Further studies on covering energy and dominating energy can be found in [1, 13].

1.1. Seidel Energy

Let $G$ be a simple graph of order $n$ with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E$. The Seidel matrix of $G$ is the $n \times n$ matrix defined by $S(G) := (s_{ij})$, where

$$s_{ij} = \begin{cases} -1 & \text{if } v_iv_j \in E \\ 1 & \text{if } v_iv_j \notin E \\ 0 & \text{if } v_i = v_j \end{cases}$$

The characteristic polynomial of $S(G)$ is denoted by $f_n(G, \lambda) = \det(\lambda I - S(G))$. The Seidel eigenvalues of the graph $G$ are the eigenvalues of $S(G)$. Since $S(G)$ is real and symmetric, its eigenvalues are real numbers. The Seidel energy [14] of $G$ is defined as $SE(G) := \sum_{i=1}^{n} |\lambda_i|.$

2. Siedel Energy of Some Standard Graphs

**Definition 2.1.** The Cocktail party graph is denoted by $K_{n \times 2}$, is a graph having the vertex set $V = \bigcup_{i=1}^{n} \{u_i, v_i\}$ and the edge set $E = \{u_iu_j, v_iv_j : i \neq j\} \bigcup \{u_i, v_i : 1 \leq i < j \leq n\}$.

**Theorem 2.2.** For $n \geq 2$, the Siedel energy of Cocktail party graph $K_{n \times 2}$ is $6n - 6$. 
Proof. Let $K_{n \times 2}$ be the Cocktail party graph with vertex set $V = \bigcup_{i=1}^{n} \{u_i, v_i\}$. Then

$$S(K_{n \times 2}) = \begin{pmatrix}
0 & 1 & -1 & -1 & \cdots & -1 & -1 \\
1 & 0 & -1 & -1 & \cdots & -1 & -1 \\
-1 & -1 & 0 & 1 & \cdots & -1 & -1 \\
-1 & -1 & 1 & 0 & \cdots & -1 & -1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & -1 & -1 & -1 & \cdots & 0 & 1 \\
-1 & -1 & -1 & -1 & \cdots & 1 & 0
\end{pmatrix}$$

Characteristic equation is

$$(\lambda + 1)^n (\lambda - 3)^{n-1} [\lambda + (2n - 3)] = 0.$$ 

Siedel eigen values are

$$\lambda = -1 [n \text{times}], \lambda = 3 [(n - 1) \text{times}], \lambda = -(2n - 3)$$

Siedel energy,

$$SE(K_{n \times 2}) = | - 1 | n + | 3 | (n - 1) + | - (2n - 3) | = 6n - 6.$$ 

Definition 2.3. The Crown graph $S_{n}^0$ for an integer $n \geq 2$ is the graph with vertex set $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and edge set $\{u_i, v_j : 1 \leq i, j \leq n, i \neq j\}$. $S_{n}^0$ coincides with the Complete bipartite graph $K_{n,n}$ with horizontal edges removed.

Theorem 2.4. For $n \geq 2$, the siedel energy of the Crown graph $S_{n}^0$ is equal to $6n - 6$.

Proof. For the Crown graph $S_{n}^0$ with vertex set $V = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$. Then

$$S(S_{n}^0) = \begin{pmatrix}
0 & 1 & 1 & \cdots & 1 & 1 & -1 & -1 & \cdots & -1 \\
1 & 0 & 1 & \cdots & 1 & -1 & 1 & -1 & \cdots & -1 \\
1 & 1 & 0 & \cdots & 1 & -1 & -1 & 1 & \cdots & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & 0 & -1 & -1 & -1 & \cdots & 1 \\
1 & -1 & -1 & \cdots & -1 & 0 & 1 & 1 & \cdots & 1 \\
-1 & 0 & -1 & \cdots & -1 & 1 & 0 & 1 & \cdots & 1 \\
-1 & -1 & 0 & \cdots & -1 & 1 & 1 & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & -1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 0
\end{pmatrix}_{(2n \times 2n)}$$
Characteristic equation is
\[(\lambda - 1)^n(\lambda + 3)^{n-1}[\lambda - (2n - 3)] = 0.\]

Siedel eigen values are
\[\lambda = 1[n \text{ times}], \lambda = -3[(n - 1) \text{ times}], \lambda = 2n - 3\]

Siedel energy,
\[SE(S^0_n) = |1|n + |−3|(n − 1) + |(2n − 3)| = 6n − 6.\]

3. Properties of Seidel Eigenvalues

Lemma 3.1. Let G be a simple graph with vertex set \(V = \{v_1, v_2, \ldots, v_n\}\), edge set \(E\). If \(\lambda_1, \lambda_2, \ldots, \lambda_n\) are the eigenvalues of Seidel matrix \(S(G)\) then (i) \(\sum_{i=1}^{n} \lambda_i = 0\). (ii) \(\sum_{i=1}^{n} \lambda_i^2 = n(n - 1)\).

Proof.

i) We know that the sum of the eigenvalues of \(S(G)\) is the trace of \(S(G)\)
\[\therefore \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = 0.\]

(ii) Similarly the sum of squares of the eigenvalues of \(S(G)\) is trace of \([S(G)]^2\)
\[\therefore \sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}a_{ji}\]
\[= \sum_{i=1}^{n} (a_{ii})^2 + \sum_{i \neq j} a_{ij}a_{ji}\]
\[= \sum_{i=1}^{n} (a_{ii})^2 + 2\sum_{i < j} (a_{ij})^2\]
\[= 0 + 2\left[m(-1)^2 + \left(\frac{n^2 - n}{2} - m\right)(1)^2\right]\]
\[= n^2 - n.\]
4. Bounds for Seidel Energy

Similar to McClelland’s [11] bounds for energy of a graph, bounds for $SE(G)$ are given in the following theorem.

**Theorem 4.1.** Let $G$ be a simple graph with $n$ vertices and $m$ edges and $P = |\text{det} S(G)|$ then $\sqrt{(n^2 - n)} + n(n - 1)P^2 \leq SE(G) \leq \sqrt{n(n^2 - n)}$.

**Proof.**

Cauchy Schwarz inequality is

$$\left( \sum_{i=1}^{n} a_i b_i \right)^2 \leq \left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right)$$

If $a_i = 1, b_i = |\lambda_i|$ then

$$\left( \sum_{i=1}^{n} |\lambda_i| \right)^2 \leq \left( \sum_{i=1}^{n} 1 \right) \left( \sum_{i=1}^{n} \lambda_i^2 \right)$$

$$[SE(G)]^2 \leq n(n^2 - n) \quad \text{[Lemma 3.1]}$$

$$\implies SE(G) \leq \sqrt{n(n^2 - n)}$$

Since arithmetic mean is not smaller than geometric mean we have

$$\frac{1}{n(n - 1)} \sum_{i \neq j} |\lambda_i| \lambda_j \geq \left[ \prod_{i \neq j} |\lambda_i| \lambda_j \right] \left[ \prod_{i \neq j} |\lambda_i| \lambda_j \right] = \left[ \prod_{i=1}^{n} |\lambda_i| \lambda_j \right] = \left[ \prod_{i=1}^{n} |\lambda_i| \lambda_j \right] = \left[ \prod_{i=1}^{n} \lambda_i \right] = |\text{det} S(G)| \frac{2}{n} = P^2$$

$$\therefore \sum_{i \neq j} |\lambda_i| \lambda_j \geq n(n - 1)P^2$$
Now consider, 
\[ (SE(G))^2 = \left( \sum_{i=1}^{n} | \lambda_i | \right)^2 \]
\[ = \sum_{i=1}^{n} | \lambda_i |^2 + \sum_{i \neq j} | \lambda_i | | \lambda_j | \]
\[ \therefore (SE(G))^2 \geq (n^2 - n) + n(n - 1)P^2_n \] [From (4.1)]
\[ \text{i.e., } SE(G) \geq \sqrt{(n^2 - n) + n(n - 1)P^2_n} \]

Recently Milovanović [12] et al. gave a sharper lower bounds for energy of a graph. In this paper similar bounds for minimum dominating Seidel energy of a graph are established. 

**Theorem 4.2.** Let \( G \) be a graph with \( n \) vertices and \( m \) edges. Let \( |\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n| \) be a non-increasing order of Seidel eigenvalues of \( S(G) \) then \( SE(G) \geq \sqrt{n(n^2 - n) - \alpha(n)(|\lambda_1| - |\lambda_n|)^2} \) where \( \alpha(n) = n \left[ \frac{n}{2} \right] \left( 1 - \frac{1}{n} \left[ \frac{n}{2} \right] \right) \) and \( [x] \) denotes the integral part of a real number.

**Proof.** Let \( a, a_1, a_2, \ldots a_n, A \) and \( b, b_1, b_2, \ldots b_n, B \) be real numbers such that \( a \leq a_i \leq A \) and \( b \leq b_i \leq B \forall i = 1, 2, \ldots n \) then the following inequality is valid.

\[
\left| \sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \right| \\
\leq \alpha(n)(A - a)(B - b)
\]

where

\[
\alpha(n) = n \left[ \frac{n}{2} \right] \left( 1 - \frac{1}{n} \left[ \frac{n}{2} \right] \right)
\]

and equality holds if and only if \( a_1 = a_2 = \ldots = a_n \) and \( b_1 = b_2 = \ldots = b_n \). If \( a_i = |\lambda_i|, b_i = |\lambda_i|, a = b = |\lambda_n| \) and \( A = B = |\lambda_1| \), then

\[
\left| n \sum_{i=1}^{n} |\lambda_i|^2 - \left( \sum_{i=1}^{n} |\lambda_i| \right)^2 \right| \leq \alpha(n)(|\lambda_1| - |\lambda_n|)^2
\]

But

\[
\sum_{i=1}^{n} |\lambda_i|^2 = n^2 - n
\]
and $SE(G) \leq \sqrt{n(n^2 - n)}$ [13] then the above inequality becomes
\[
n(n^2 - n) - (SE(G))^2 \leq \alpha(n)(|\lambda_1| - |\lambda_n|)^2
\]
i.e., $SE(G) \geq \sqrt{n(n^2 - n) - \alpha(n)(|\lambda_1| - |\lambda_n|)^2}$

\[\square\]

**Theorem 4.3.** Let $G$ be a graph with $n$ vertices and $m$ edges. Let $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n| > 0$ be a non-increasing order of eigenvalues of $S(G)$ then

$$SE(G) \geq \frac{n^2 - n + n|\lambda_1||\lambda_n|}{(|\lambda_1| + |\lambda_n|)}.$$  

*Proof.* Let $a_i \neq 0$, $b_i$, $r$ and $R$ be real numbers satisfying $ra_i \leq b_i \leq Ra_i$, then the following inequality holds. [Theorem 2, [12]]

$$\sum_{i=1}^{n} b_i^2 + rR \sum_{i=1}^{n} a_i \leq (r + R) \sum_{i=1}^{n} a_i b_i$$

Put $b_i = |\lambda_i|$, $a_i = 1$, $r = |\lambda_n|$ and $R = |\lambda_1|$ then

$$\sum_{i=1}^{n} |\lambda_i|^2 + |\lambda_1||\lambda_n| \sum_{i=1}^{n} 1 \leq (|\lambda_1| + |\lambda_n|) \sum_{i=1}^{n} |\lambda_i|$$

i.e., $n^2 - n + |\lambda_1||\lambda_n| n \leq (|\lambda_1| + |\lambda_n|)SE(G)$

$$\therefore SE(G) \geq \frac{n^2 - n + n|\lambda_1||\lambda_n|}{(|\lambda_1| + |\lambda_n|)}.$$  

\[\square\]

**References**


