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Bipolar-Valued Multi Fuzzy Subhemirings Of A Hemiring

V.K. Santhi

Department of Mathematics, Srimeenakshi Government Arts College, Madurai – 625 002, Tamilnadu, India.

K. Anbarasi

Department of Mathematics, Chellammal Womens College, Chennai-600 032, Tamilnadu, India.

Abstract

In this paper, we study some of the properties of bipolar-valued multi fuzzy subhemiring and prove some results on these.

Key Words: Bipolar-valued fuzzy subset, bipolar-valued multi fuzzy subset, bipolar-valued multi fuzzy subhemiring.

Introduction:

In 1965, Zadeh [13] introduced the notion of a fuzzy subset of a set, fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. Since then it has become a vigorous area of research in different domains, there have been a number of generalizations of this fundamental concept such as intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, soft sets etc [6]. Lee [8] introduced the notion of bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0, 1] to [−1, 1]. In a bipolar-valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degree (0, 1] indicates that elements somewhat satisfy the property and the membership degree [−1, 0) indicates that elements somewhat satisfy the implicit counter property. Bipolar-valued fuzzy sets and intuitionistic fuzzy sets look similar each other. However, they are different each other [8, 9]. Anitha.M.S., Muruganantha Prasad & K.Arjunan[1] defined as Bipolar-valued fuzzy subgroups of a group. We introduce the concept of bipolar-valued multi fuzzy subhemiring and established some results.

1.PRELIMINARIES:

1.1 Definition: A bipolar-valued fuzzy set (BVFS) A in X is defined as an object of the form A = { < x, $A^+(x)$, $A^-(x)$ >/ $x \in X$}, where $A^+ : X \to [0, 1]$ and $A^- : X \to [-1, 0]$. The positive membership degree $A^+(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set A and the negative membership degree $A^-(x)$ denotes the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar-valued fuzzy set A. If $A^+(x) \neq 0$ and $A^-(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for A and if $A^+(x) = 0$ and $A^-(x) \neq 0$, it is the situation that x does not satisfy the property of A, but somewhat satisfies the counter property of A. It is possible for an element x to be such that $A^+(x) \neq 0$ and $A^-(x) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of X.

1.2 Example: A = { < a, 0.5, –0.3 >, < b, 0.1, –0.7 >, < c, 0.5, –0.4 >} is a bipolar-valued fuzzy subset of X= {a, b, c }.

1.3 Definition: A bipolar-valued multi fuzzy set (BVMFS) A in X is defined as an object of the form A = { < x, $A_i^+(x)$, $A_i^-(x)$ >/ $x \in X$}, where $A_i^+ : X \to [0, 1]$ and $A_i^- : X \to [-1, 0]$. The positive membership degrees $A_i^+(x)$ denote the satisfaction degree of an element x to the property corresponding to a bipolar-valued multi fuzzy set A and the negative membership degrees $A_i^-(x)$ denote the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar-valued multi fuzzy set A. If $A_i^+(x) \neq 0$ and $A_i^-(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for A and if $A_i^+(x) = 0$ and $A_i^-(x) \neq 0$, it is the situation that x does not satisfy the property of A, but somewhat satisfies the counter property of A. It is possible for an element x to be such that $A_i^+(x) \neq 0$ and $A_i^-(x) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of X, where i = 1 to n.

1.4 Example: A = { < a, 0.5, 0,6, 0.3, –0.3, –0.6, –0.5 >, < b, 0.1, 0.4, 0.7, –0.7, –0.3, –0.6 >, < c, 0.5, 0.3, 0.8, –0.4, –0.5, –0.3 >} is a bipolar-valued multi fuzzy subset of X = { a, b, c }.

1.5 Definition: Let R be a hemiring. A bipolar-valued multi fuzzy subset A of R is said to be a bipolar-valued multi fuzzy subhemiring of R (BVMFSHR) if the following conditions are satisfied,

(i) $A_i^+(x+y) \geq \min\{ A_i^+(x), A_i^+(y) \}$
(ii) $A_i^+(xy) \geq \min\{ A_i^+(x), A_i^+(y) \}$
(iii) $A_i^-(x+y) \leq \max\{ A_i^-(x), A_i^-(y) \}$
(iv) $A_i^-(xy) \leq \max\{ A_i^-(x), A_i^-(y) \}$ for all x and y in R.

1.6 Example: Let R = Z_3 = { 0, 1, 2 } be a hemiring with respect to the ordinary addition and multiplication. Then A = { < 0, 0.5, 0.8, 0.6, – 0.6, – 0.5, – 0.7 >, < 1, 0.4, 0.7, 0.5, – 0.5, – 0.4, – 0.6 >, < 2, 0.4, 0.7, 0.5, – 0.5, – 0.4, – 0.6 > } is a bipolar-valued multi fuzzy subhemiring of R.

1.7 Definition: Let A = ⟨ A_i^+, A_i^- ⟩ and B = ⟨ B_i^+, B_i^- ⟩ be any two bipolar-valued multi fuzzy subsets of sets G and H, respectively. The product of A and B, denoted by A×B, is defined as A×B = { ⟨ (x, y), $(A_i\times B_i)^+$(x, y), $(A_i\times B_i)^-$(x, y) ⟩ / for all x in G and y in H } where $(A_i\times B_i)^+$(x, y) = min { A_i^+(x), B_i^+(y) } and $(A_i\times B_i)^-$(x, y) = max { A_i^-(x), B_i^-(y) } for all x in G and y in H.

1.8 Definition: Let A = ⟨ A_i^+, A_i^- ⟩ be a bipolar-valued multi fuzzy subset in a set S, the strongest bipolar-valued multi fuzzy relation on S, that is a bipolar-valued multi fuzzy relation on A is V = { ⟨ (x, y), V_i^+(x, y), V_i^-(x, y) ⟩ / x and y in S} given by V_i^+(x, y) = min {A_i^+(x), A_i^+(y)} and V_i^- (x, y) = max { A_i^-(x), A_i^-(y) } for all x and y in S.

2. PROPERTIES:

2.1 Theorem: A = ⟨ A_i^+, A_i^- ⟩ be a bipolar-valued multi fuzzy subhemiring of a hemiring R, then $A_i^+(x) \le A_i^+(e)$ and $A_i^-(x) \ge A_i^-(e)$ for x in R and the zero element e in R.

Proof: For x in R and e is zero element of R. Now $A_i^+(x) = A_i^+(x+e) \ge$ min { A_i^+(x), A_i^+(e) } and $A_i^+(e) = A_i^+(x.e) \ge$ min { A_i^+(x), A_i^+(e)}. If x+y = e then $A_i^+(e) = A_i^+(x+y) \ge$ min {A_i^+(x), A_i^+(y) }. Hence $A_i^+(e) \ge A_i^+(x)$ for all x in R. And $A_i^-(x) = A_i^-(x+e) \le$ max { A_i^-(x), A_i^-(e) } and $A_i^-(e) = A_i^-(x.e) \le$ max { A_i^-(x), A_i^-(e) }. If x+y = e then $A_i^-(e) = A_i^-(x+y) \le$ max {A_i^-(x), A_i^-(y)}. Hence $A_i^-(e) \le A_i^-(x)$ for all x in R.

2.2 Theorem: Let A = ⟨ A_i^+, A_i^- ⟩ be a bipolar-valued multi fuzzy subhemiring of a hemiring R.
(i) If $A_i^+(x+y) = 0$ then either $A_i^+(x) = 0$ or $A_i^+(y) = 0$ for x and y in R.
(ii) If A_i^+ (xy) = 0 then either $A_i^+(x) = 0$ or $A_i^+(y) = 0$ for x and y in R.
(iii) If A_i^- (x+y) = 0 then either A_i^- (x) = 0 or A_i^- (y) = 0 for x and y in R.
(iv) If A_i^- (xy) = 0 then either A_i^- (x) = 0 or A_i^- (y) = 0 for x and y in R.

Proof: Let x and y be in R. (i) By the definition $A_i^+(x+y) \ge$ min { A_i^+(x), A_i^+(y) } which implies that 0 ≥ min { A_i^+(x), A_i^+(y) }. Therefore either $A_i^+(x) = 0$ or $A_i^+(y) = 0$. (ii) By the definition $A_i^+(xy) \ge$ min { A_i^+(x), A_i^+(y) } which implies that 0 ≥ min { A_i^+(x), A_i^+(y) }. Therefore either $A_i^+(x) = 0$ or $A_i^+(y) = 0$. (iii) By the definition $A_i^-(x+y) \le$ max {A_i^-(x), A_i^-(y) } which implies that 0 ≤ max {A_i^-(x), A_i^-(y) }. Therefore either $A_i^-(x) = 0$ or $A_i^-(y) = 0$. (iv) By the definition $A_i^-(xy) \le$ max {A_i^-(x),

$A_i^-(y)$ } which implies that $0 \le \max\{A_i^-(x), A_i^-(y)\}$. Therefore either $A_i^-(x) = 0$ or $A_i^-(y) = 0$.

2.3 Theorem: If $A = \langle A_i^+, A_i^- \rangle$ is a bipolar-valued multi fuzzy subhemiring of a hemiring R then $H = \{ x \in R \mid A_i^+(x) = 1, A_i^-(x) = -1 \}$ is either empty or is a subhemiring of R.

proof: If no element satisfies this condition then H is empty. If x and y in H then $A_i^+(x+y) \ge \min\{ A_i^+(x), A_i^+(y) \} = \min\{1, 1\} = 1$. Therefore $A_i^+(x+y) = 1$. And $A_i^+(xy) \ge \min\{ A_i^+(x), A_i^+(y) \} = \min\{1, 1\} = 1$. Therefore $A_i^+(xy) = 1$. Also $A_i^-(x+y) \le \max\{ A_i^-(x), A_i^-(y) \} = \max\{-1, -1\} = -1$. Therefore $A_i^-(x+y) = -1$. And $A_i^-(xy) \le \max\{ A_i^-(x), A_i^-(y) \} = \max\{-1, -1\} = -1$. Therefore $A_i^-(xy) = -1$. That is $x+y \in H$ and $xy \in H$. Hence H is a subhemiring of R. Hence H is either empty or a subhemiring of R.

2.4 Theorem: If $A = \langle A_i^+, A_i^- \rangle$ is a bipolar-valued multi fuzzy subhemiring of R then $H = \{ x \in R \mid A_i^+(x) = A_i^+(e) \text{ and } A_i^-(x) = A_i^-(e) \}$ is a subhemiring of R.

proof: Here $H = \{x \in R \mid A_i^+(x) = A_i^+(e) \text{ and } A_i^-(x) = A_i^-(e) \}$ by Theorem 2.1. Now $A_i^+(x+y) \ge \min\{ A_i^+(x), A_i^+(y) \} = \min\{ A_i^+(e), A_i^+(e) \} = A_i^+(e)$. Hence $A_i^+(e) = A_i^+(x+y)$. And $A_i^+(xy) \ge \min\{ A_i^+(x), A_i^+(y) \} = \min\{ A_i^+(e), A_i^+(e) \} = A_i^+(e)$. Hence $A_i^+(e) = A_i^+(xy)$. Also $A_i^-(x+y) \le \max\{ A_i^-(x), A_i^-(y) \} = \max\{ A_i^-(e), A_i^-(e) \} = A_i^-(e)$. Therefore $A_i^-(e) = A_i^-(x+y)$. And $A_i^-(xy) \le \max\{ A_i^-(x), A_i^-(y) \} = \max\{ A_i^-(e), A_i^-(e)\} = A_i^-(e)$. Therefore $A_i^-(e) = A_i^-(xy)$. Therefore x+y and xy are in H. Hence H is a subhemiring of R.

2.5 Theorem: If $A = \langle A_i^+, A_i^- \rangle$ and $B = \langle B_i^+, B_i^- \rangle$ are two bipolar-valued multi fuzzy subhemirings of a hemiring R, then their intersection $A \cap B$ is a bipolar-valued multi fuzzy subhemiring of R.

Proof: Let $A = \{ < x, A_i^+(x), A_i^-(x) > / x \in G \}$, $B = \{ < x, B_i^+(x), B_i^-(x) > / x \in G \}$. Let $C = A \cap B$ and $C = \{ < x, C_i^+(x), C_i^-(x) > / x \in G \}$. Now $C_i^+(x+y) = \min\{ A_i^+(x+y), B_i^+(x+y) \} \ge \min\{ \min\{ A_i^+(x), A_i^+(y) \}, \min\{ B_i^+(x), B_i^+(y) \} \} \ge \min\{ \min\{ A_i^+(x), B_i^+(x) \}, \min\{A_i^+(y), B_i^+(y)\} \} = \min\{ C_i^+(x), C_i^+(y) \}$. Therefore $C_i^+(x+y) \ge \min\{ C_i^+(x), C_i^+(y) \}$. And $C_i^+(xy) = \min\{ A_i^+(xy), B_i^+(xy) \} \ge \min\{ \min\{A_i^+(x), A_i^+(y) \}, \min\{ B_i^+(x), B_i^+(y) \}\} \ge \min\{ \min\{ A_i^+(x), B_i^+(x) \}, \min\{A_i^+(y), B_i^+(y)\}\} = \min\{ C_i^+(x), C_i^+(y) \}$. Therefore $C_i^+(xy) \ge \min\{ C_i^+(x), C_i^+(y)\}$. Also $C_i^-(x+y) = \max\{ A_i^-(x+y), B_i^-(x+y) \} \le \max\{ \max\{A_i^-(x), A_i^-(y) \}, \max\{ B_i^-(x), B_i^-(y) \}\} \le \max\{ \max\{ A_i^-(x), B_i^-(x) \}, \max\{ A_i^-(y), B^-(y) \}\} = \max\{ C_i^-(x), C_i^-(y) \}$. Therefore $C_i^-(x+y) \le \max\{ C_i^-(x), C_i^-(y) \}$. And $C_i^-(xy) = \max\{A_i^-(xy), B_i^-(xy) \} \le \max\{ \max\{ A_i^-(x), A_i^-(y) \}, \max\{ B_i^-(x), B_i^-(y) \}\} \le \max\{ \max\{A_i^-(x), B_i^-(x)\}, \max\{A_i^-(y), B_i^-(y)\}\} = \max\{ C_i^-(x), C_i^-(y) \}$. Therefore $C_i^-(xy) \le \max\{C_i^-(x), C_i^-(y) \}$. Hence $A \cap B$ is a bipolar-valued multi fuzzy subhemiring of R.

2.6 Theorem: The intersection of a family of bipolar-valued multi fuzzy subhemirings of a hemiring R is a bipolar-valued multi fuzzy subhemiring of R.

Proof: The theorem can easily prove by **Theorem 2.5.**

2.7 Theorem: If A = $\langle A_i^+, A_i^- \rangle$ and B = $\langle B_i^+, B_i^- \rangle$ are any two bipolar-valued multi fuzzy subhemirings of the hemirings R_1 and R_2 respectively, then A×B = $\langle (A_i \times B_i)^+, (A_i \times B_i)^- \rangle$ is a bipolar-valued multi fuzzy subhemiring of $R_1 \times R_2$.

Proof: Let A and B be two bipolar-valued multi fuzzy subhemirings of the hemirings R_1 and R_2 respectively. Let x_1, x_2 be in R_1, y_1 and y_2 be in R_2. Then (x_1, y_1) and (x_2, y_2) are in $R_1 \times R_2$. Now, $(A_i \times B_i)^+[(x_1, y_1) + (x_2, y_2)] = (A_i \times B_i)^+(x_1+x_2, y_1+y_2) = \min \{ A_i^+(x_1+x_2), B_i^+(y_1+y_2) \} \geq \min \{ \min\{ A_i^+(x_1), A_i^+(x_2) \}, \min \{ B_i^+(y_1), B_i^+(y_2) \}\} = \min \{ \min \{ A_i^+(x_1), B_i^+(y_1) \}, \min \{ A_i^+(x_2), B_i^+(y_2) \} \} = \min \{ (A_i \times B_i)^+(x_1, y_1), (A_i \times B_i)^+(x_2, y_2) \}$. Therefore $(A_i \times B_i)^+[(x_1, y_1) + (x_2, y_2)] \geq \min \{ (A_i \times B_i)^+(x_1, y_1), (A_i \times B_i)^+(x_2, y_2) \}$. And $(A_i \times B_i)^+[(x_1, y_1) (x_2, y_2)] = (A_i \times B_i)^+(x_1x_2, y_1y_2) = \min \{ A_i^+(x_1x_2), B_i^+(y_1y_2) \} \geq \min \{ \min \{ A_i^+(x_1), A_i^+(x_2) \}, \min \{ B_i^+(y_1), B_i^+(y_2) \} \} = \min \{ \min \{ A_i^+(x_1), B_i^+(y_1) \}, \min \{ A_i^+(x_2), B_i^+(y_2) \} \} = \min \{ (A_i \times B_i)^+(x_1, y_1), (A_i \times B_i)^+(x_2, y_2) \}$. Therefore, $(A_i \times B_i)^+[(x_1, y_1)(x_2, y_2)] \geq \min \{ (A_i \times B_i)^+(x_1, y_1), (A_i \times B_i)^+(x_2, y_2) \}$. Also $(A_i \times B_i)^-[(x_1, y_1) + (x_2, y_2)] = (A_i \times B_i)^-(x_1+x_2, y_1+y_2) = \max \{ A_i^-(x_1+x_2), B_i^-(y_1+y_2) \} \leq \max \{ \max \{ A_i^-(x_1), A_i^-(x_2) \}, \max \{ B_i^-(y_1), B_i^-(y_2) \} \} = \max \{ \max \{ A_i^-(x_1), B_i^-(y_1) \}, \max \{A_i^-(x_2), B_i^-(y_2) \}\} = \max \{(A_i \times B_i)^-(x_1, y_1), (A_i \times B_i)^-(x_2, y_2) \}$. Therefore $(A_i \times B_i)^-[(x_1, y_1)+(x_2, y_2)] \leq \max \{ (A_i \times B_i)^-(x_1, y_1), (A_i \times B_i)^- (x_2, y_2) \}$. And $(A_i \times B_i)^-[(x_1, y_1)(x_2, y_2)] = (A_i \times B_i)^-(x_1x_2, y_1y_2) = \max \{ A_i^-(x_1x_2), B_i^-(y_1y_2) \} \leq \max \{ \max \{ A_i^-(x_1), A_i^-(x_2) \}, \max \{ B_i^-(y_1), B_i^-(y_2) \} \} = \max \{ \max \{A_i^-(x_1), B_i^-(y_1) \}, \max \{ A_i^-(x_2), B_i^-(y_2) \} \} = \max \{ (A_i \times B_i)^-(x_1, y_1), (A_i \times B_i)^-(x_2, y_2) \}$. Therefore $(A_i \times B_i)^-[(x_1, y_1)(x_2, y_2)] \leq \max \{ (A_i \times B_i)^-(x_1, y_1), (A_i \times B_i)^-(x_2, y_2) \}$. Hence A×B is a bipolar-valued multi fuzzy subhemiring of $R_1 \times R_2$.

2.8 Theorem: Let A = $\langle A_i^+, A_i^- \rangle$ and B = $\langle B_i^+, B_i^- \rangle$ be any two bipolar-valued multi fuzzy subsets of the hemirings R and H respectively. Suppose that e and e^l are the zero elements of R and H respectively. If A×B is a bipolar-valued multi fuzzy subhemiring of G×H, then at least one of the following two statements must hold.

(i) $B_i^+(e^l) \geq A_i^+(x)$ for all x in R and $B_i^-(e^l) \leq A_i^-(x)$ for all x in R,

(ii) $A_i^+(e) \geq B_i^+(y)$ for all y in H and $A_i^-(e) \leq B_i^-(y)$ for all y in H.

Proof: Let A×B is a bipolar-valued multi fuzzy subhemiring of R×H. By contraposition, suppose that none of the statements (i) and (ii) holds. Then we can find a in R and b in H such that $A_i^+(a) > B_i^+(e^l)$, $A_i^-(a) < B_i^-(e^l)$ and $B_i^+(b) > A_i^+(e)$, $B_i^-(b) < A_i^-(e)$. We have $(A_i \times B_i)^+(a, b) = \min \{A_i^+(a), B_i^+(b) \} > \min \{ A_i^+(e), B_i^+(e^l) \} = (A_i \times B_i)^+(e, e^l)$. Also $(A_i \times B_i)^-(a, b) = \max \{ A_i^-(a), B_i^-(b) \} < \max \{ A_i^-(e), B_i^-(e^l) \} = (A_i \times B_i)^- (e, e^l)$. Thus A×B is not a bipolar-valued multi fuzzy subhemiring of

$R\times H$. Hence either $B_i^+(e^l) \geq A_i^+(x)$, for all x in R and $B_i^-(e^l) \leq A_i^-(x)$ for all x in R or $A_i^+(e) \geq B_i^+(y)$ for all y in H and $A_i^-(e) \leq B_i^-(y)$ for all y in H.

2.9 Theorem: Let $A = \langle A_i^+, A_i^- \rangle$ and $B = \langle B_i^+, B_i^- \rangle$ be any two bipolar-valued multi fuzzy subsets of the hemirings R and H, respectively and $A\times B$ is a bipolar-valued multi fuzzy subhemiring of $R\times H$. Then the following are true:
(i) if $A_i^+(x) \leq B_i^+(e^l)$ for all x in R and $A_i^-(x) \geq B_i^-(e^l)$ for all x in R then A is a bipolar-valued multi fuzzy subhemiring of R where e^l is zero element of H.
(ii) if $B_i^+(x) \leq A_i^+(e)$ for all x in H and $B_i^-(x) \geq A_i^-(e)$ for all x in H then B is a bipolar-valued multi fuzzy subhemiring of H where e is zero element of R.
(iii) either A is a bipolar-valued multi fuzzy subhemiring of R or B is a bipolar-valued multi fuzzy subhemiring of H where e and e^l are the zero elements of R and H respectively.

Proof: Let $A\times B$ be a bipolar-valued multi fuzzy subhemiring of $R\times H$ and x and y in R. Then (x, e^l) and (y, e^l) are in $R\times H$. Now using the property if $A_i^+(x) \leq B_i^+(e^l)$ for all x in R and $A_i^-(x) \geq B_i^-(e^l)$ for all $x\in R$ where $e^l$is zero element of H, we get $A_i^+(x+y) = \min\{ A_i^+(x+y), B_i^+(e^l+e^l) \} = (A_i\times B_i)^+((x+y), (e^l+e^l)) = (A_i\times B_i)^+[(x, e^l)+(y, e^l)] \geq \min\{ (A_i\times B_i)^+(x, e^l), (A_i\times B_i)^+(y, e^l) \} = \min\{\min\{A_i^+(x), B_i^+(e^l) \}, \min\{ A_i^+(y), B_i^+(e^l) \}\} = \min\{ A_i^+(x), A_i^+(y)\} \geq \min\{A_i^+(x), A_i^+(y) \}$. Therefore $A_i^+(x+y) \geq \min\{ A_i^+(x), A_i^+(y) \}$ for all x and y in R. And $A_i^+(xy) = \min\{ A_i^+(xy), B_i^+(e^le^l) \} = (A_i\times B_i)^+((xy), (e^le^l)) = (A_i\times B_i)^+[(x, e^l)(y, e^l)] \geq \min\{(A_i\times B_i)^+(x, e^l), (A_i\times B_i)^+(y^{-1}, e^l) \} = \min\{ \min\{ A_i^+(x), B_i^+(e^l) \}, \min\{ A_i^+(y), B_i^+(e^l) \} \} = \min\{ A_i^+(x), A_i^+(y^{-1}) \} \geq \min\{ A_i^+(x), A_i^+(y) \}$. Therefore $A_i^+(xy) \geq \min\{ A_i^+(x), A_i^+(y) \}$ for all x and y in R. Also $A_i^-(x+y) = \max\{ A_i^-(x+y), B_i^-(e^l+e^l) \} = (A_i\times B_i)^-((x+y), (e^l+e^l)) = (A_i\times B_i)^-[(x, e^l)+(y, e^l)] \leq \max\{ (A_i\times B_i)^-(x, e^l), (A_i\times B_i)^-(y, e^l) \} = \max\{ A_i^-(x), B_i^-(e^l) \}, \max\{A_i^-(y^{-1}), B_i^-(e^l) \} \} = \max\{ A_i^-(x), A_i^-(y) \}\leq \max\{ A_i^-(x), A_i^-(y) \}$. Therefore $A_i^-(x+y) \leq \max\{ A_i^-(x), A_i^-(y) \}$ for all x and y in R. And $A_i^-(xy) = \max\{ A_i^-(xy), B_i^-(e^le^l) \} = (A_i\times B_i)^-((xy), (e^le^l)) = (A_i\times B_i)^-[(x, e^l)(y, e^l)] \leq \max\{ (A_i\times B_i)^-(x, e^l), (A_i\times B_i)^-(y, e^l) \} = \max\{ A_i^-(x), B_i^-(e^l) \}, \max\{ A_i^-(y), B_i^-(e^l) \} \} = \max\{ A_i^-(x), A_i^-(y) \} \leq \max\{ A_i^-(x), A_i^-(y) \}$. Therefore $A_i^-(xy) \leq \max\{ A_i^-(x), A_i^-(y) \}$ for all x and y in R. Hence A is a bipolar-valued multi fuzzy subhemiring of R. Thus (i) is proved. Now using the property $B_i^+(x) \leq A_i^+(e)$ for all x in H and $B_i^-(x) \geq A_i^-(e)$ for all x in H, we get $B_i^+(x+y) = \min\{ B_i^+(x+y), A_i^+(e+e) \} = (A_i\times B_i)^+((e+e), (x+y)) = (A_i\times B_i)^+[(e, x)+(e, y)] \geq \min\{ (A_i\times B_i)^+(e, x), (A_i\times B_i)^+(e, y) \} = \min\{ \min\{ A_i^+(e), B_i^+(x) \}, \min\{ A_i^+(e), B_i^+(y) \}\} = \min\{ B_i^+(x), B_i^+(y) \} \geq \min\{B_i^+(x), B_i^+(y) \}$. Therefore $B_i^+(x+y) \geq \min\{ B_i^+(x), B_i^+(y) \}$ for all x and y in H. And $B_i^+(xy) = \min\{ B_i^+(xy), A_i^+(e.e) \} = (A_i\times B_i)^+((e.e), (xy)) = (A_i\times B_i)^+[(e, x)(e, y)] \geq \min\{(A_i\times B_i)^+(e, x), (A_i\times B_i)^+(e, y) \} = \min\{ \min\{ A_i^+(e), B_i^+(x) \}, \min\{ A_i^+(e), B_i^+(y) \} \} = \min\{B_i^+(x), B_i^+(y) \} \geq \min\{ B_i^+(x), B_i^+(y) \}$. Therefore $B_i^+(xy) \geq \min\{ B_i^+(x), B_i^+(y) \}$ for all x and y in H. Also $B_i^-(x+y) = \max\{ B_i^-(x+y), A_i^-(e+e) \} = (A_i\times B_i)^-((e+e), (x+y)) = (A_i\times B_i)^-[(e, x)+(e, y)] \leq \max\{ (A_i\times B_i)^-(e, x), (A_i\times B_i)^-(e, y)\} = \max\{ \max\{A_i^-(e), B_i^-(x) \}, \max\{ A_i^-(e), B_i^-(y) \} \} = \max\{B_i^-(x), B_i^-(y) \} \leq \max\{ B_i^-(x),$

$B_i^-(y)$ }. Therefore, $B_i^-(x+y) \leq \max\{ B_i^-(x), B_i^-(y) \}$ for all x and y in H. And $B_i^-(xy) = \max\{ B_i^-(xy), A_i^-(ee) \} = (A_i\times B_i)^-((ee), (xy)) = (A_i\times B_i)^-[(e, x)(e, y)] \leq \max\{ (A_i\times B_i)^-(e, x), (A_i\times B_i)^-(e, y) \} = \max\{ \max\{ A_i^-(e), B_i^-(x) \}, \max\{ A_i^-(e), B_i^-(y) \} \} = \max\{ B_i^-(x), B_i^-(y) \} \leq \max\{ B_i^-(x), B_i^-(y) \}$. Therefore $B_i^-(xy) \leq \max\{ B_i^-(x), B_i^-(y) \}$ for all x and y in H. Hence B is a bipolar-valued multi fuzzy subhemiring of H. Thus (ii) is proved. Hence (iii) is clear.

2.10 Theorem: Let $A = \langle A_i^+, A_i^- \rangle$ be a bipolar-valued multi fuzzy subset of a hemiring R and $V = \langle V_i^+, V_i^- \rangle$ be the strongest bipolar-valued multi fuzzy relation of R. Then A is a bipolar-valued multi fuzzy subhemiring of R if and only if V is a bipolar-valued multi fuzzy subhemiring of R×R.

Proof: Suppose that A is a bipolar-valued multi fuzzy subhemiring of R. Then for any $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in R×R. We have, $V_i^+(x+y) = V_i^+[(x_1, x_2)+(y_1, y_2)] = V_i^+(x_1+y_1, x_2+y_2) = \min\{ A_i^+(x_1+y_1), A_i^+(x_2+y_2) \} \geq \min\{ \min\{ A_i^+(x_1), A_i^+(y_1) \}, \min\{A_i^+(x_2), A_i^+(y_2) \} \} = \min\{ \min\{ A_i^+(x_1), A_i^+(x_2) \}, \min\{ A_i^+(y_1), A_i^+(y_2) \} \} = \min\{ V_i^+(x_1, x_2), V_i^+(y_1, y_2) \} = \min\{ V_i^+(x), V_i^+(y) \}$. Therefore $V_i^+(x+y) \geq \min\{ V_i^+(x), V_i^+(y)\}$ for all x and y in R×R. And $V_i^+(xy) = V_i^+[(x_1, x_2)(y_1, y_2)] = V_i^+(x_1y_1, x_2y_2) = \min\{A_i^+(x_1y_1), A_i^+(x_2y_2) \} \geq \min\{\min\{A_i^+(x_1), A_i^+(y_1) \}, \min\{A_i^+(x_2), A_i^+(y_2) \}\} = \min\{ \min\{A_i^+(x_1), A_i^+(x_2) \}, \min\{A_i^+(y_1), A_i^+(y_2) \} \} = \min\{ V_i^+(x_1, x_2), V_i^+(y_1, y_2) \}= \min\{V_i^+(x), V_i^+(y) \}$. Therefore $V_i^+(xy) \geq \min\{ V_i^+(x), V_i^+(y) \}$ for all x and y in R×R. Also we have $V_i^-(x+y) = V_i^-[(x_1, x_2)+(y_1, y_2)] = V_i^-(x_1+y_1, x_2+y_2) = \max\{ A_i^-(x_1+y_1), A_i^-(x_2+y_2) \} \leq \max\{ \max\{ A_i^-(x_1), A_i^-(y_1) \}, \max\{A_i^-(x_2), A_i^-(y_2) \} \} = \max\{ \max\{A_i^-(x_1), A_i^-(x_2) \}, \max\{ A_i^-(y_1), A_i^-(y_2) \}\}= \max\{ V_i^-(x_1, x_2), V_i^-(y_1, y_2) \} = \max\{ V_i^-(x), V_i^-(y) \}$. Therefore $V_i^-(x+y) \leq \max\{ V_i^-(x), V_i^-(y) \}$ for all x, y in R×R. And $V_i^-(xy) = V_i^-[(x_1, x_2)(y_1, y_2)] = V_i^-(x_1y_1, x_2y_2) = \max\{ A_i^-(x_1y_1), A_i^-(x_2y_2) \} \leq \max\{ \max\{ A_i^-(x_1), A_i^-(y_1) \}, \max\{A_i^-(x_2), A_i^-(y_2) \} \} = \max\{ \max\{A_i^-(x_1), A_i^-(x_2) \}, \max\{ A_i^-(y_1), A_i^-(y_2) \} \} = \max\{ V_i^-(x_1, x_2), V_i^-(y_1, y_2) \} = \max\{ V_i^-(x), V_i^-(y) \}$. Therefore $V_i^-(xy) \leq \max\{ V_i^-(x), V_i^-(y) \}$ for all x, y in R×R. This proves that V is a bipolar-valued multi fuzzy subhemiring of R×R. Conversely assume that V is a bipolar-valued multi fuzzy subhemiring of R×R, then for any $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in R×R, we have $\min\{ A_i^+(x_1+y_1), A_i^+(x_2+y_2) \} = V_i^+(x_1+y_1, x_2+y_2) = V_i^+[(x_1, x_2)+(y_1, y_2)] = V_i^+(x+y) \geq \min\{ V_i^+(x), V_i^+(y) \}= \min\{ V_i^+(x_1, x_2), V_i^+(y_1, y_2) \} = \min\{\min\{A_i^+(x_1), A_i^+(x_2) \}, \min\{ A_i^+(y_1), A_i^+(y_2) \} \}$. Put $x_2 = y_2 = e$, we get, $A_i^+(x_1+y_1) \geq \min\{ A_i^+(x_1), A_i^+(y_1) \}$ for all x_1 and y_1 in R. And $\min\{ A_i^+(x_1y_1), A_i^+(x_2y_2) \}=V_i^+(x_1y_1, x_2y_2) = V_i^+[(x_1, x_2)(y_1, y_2)] = V_i^+(xy) \geq \min\{ V_i^+(x), V_i^+(y) \} = \min\{ V_i^+(x_1, x_2), V_i^+(y_1, y_2) \} = \min\{\min\{ A_i^+(x_1), A_i^+(x_2) \}, \min\{ A_i^+(y_1), A_i^+(y_2) \} \}$. Put $x_2 = y_2 = e$ we get $A_i^+(x_1y_1) \geq \min\{ A_i^+(x_1), A_i^+(y_1) \}$ for all x_1 and y_1 in R. Also we have $\max\{ A_i^-(x_1+y_1), A_i^-(x_2+y_2) \} = V_i^-(x_1+y_1, x_2+y_2) = V_i^-[(x_1, x_2)+(y_1, y_2)] =V_i^-(x+y) \leq \max\{V_i^-(x), V_i^-(y) \} = \max\{ V_i^-(x_1, x_2), V_i^-(y_1, y_2) \}= \max\{ \max\{ A_i^-(x_1), A_i^-(x_2) \}, \max\{ A_i^-(y_1), A_i^-(y_2) \} \}$. If we put $x_2 = y_2 = e$ we get $A_i^-(x_1+y_1) \leq \max\{A_i^-(x_1), A_i^-(y_1) \}$ for all x_1 and y_1 in R. And $\max\{A_i^-(x_1y_1), A_i^-(x_2y_2) \} = V_i^-(x_1y_1, x_2y_2) = V_i^-[(x_1, x_2)(y_1, y_2)] = V_i^-(xy) \leq \max\{ V_i^-(x), V_i^-(y) \}$

$= \max \{ V_i^-(x_1, x_2), V_i^-(y_1, y_2) \} = \max \{ \max \{ A_i^-(x_1), A_i^-(x_2) \}, \max \{ A_i^-(y_1), A_i^-(y_2) \} \}$. If we put $x_2 = y_2 = e$ we get $A_i^-(x_1y_1) \leq \max \{A_i^-(x_1), A_i^-(y_1) \}$ for all x_1 and y_1 in R. Hence A is a bipolar-valued multi fuzzy subhemiring of R.

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