Abstract

The concept of fuzzy graphs plays an important role in daily life situations which deals with uncertainty. In this paper we introduce a type of fuzzy digraph called fuzzy evidence graph and some of its properties. Fuzzy evidence graph becomes a best tool in evidence theory for calculating belief functions, plausibility functions etc.

Keywords: Fuzzy graph, Fuzzy evidence graph, Belief measure, Plausibility measure.

1. INTRODUCTION

Graphs are used to explain relations between objects. To include fuzziness in such relations fuzzy graphs were introduced. The first definition of fuzzy graph by Kaufmann was based on Zadeh’s fuzzy relations. After that Rosenfield considered fuzzy relation on fuzzy sets and developed the structure of fuzzy graphs. After the work of Rosenfield, Yeh and Bang introduced various connectedness concepts of graphs and digraphs into fuzzy graphs. In [3], Sunil Mathew and M.S Sunitha presented basic concepts in fuzzy graph connectivity, which plays a remarkable role in information networks and quality based clustering. In the book [2], John N Mordeson and Premchand S Nair presented many concepts and theoretical results of fuzzy graphs.

Fuzzy measure theory considers generalized measures in which the additive property is replaced by the weaker property of monotonicity. Fuzzy measure was introduced by Choquet in 1953 and independently defined by Sugeno in 1974. Evidence theory is a special branch of fuzzy measure theory which is based on belief measures and plausibility measures. In [4], Dempster–Shafer theory of belief functions is explained in detail.
In [1], George J klir and Bo Yuan provides a comprehensive coverage of theoretical foundations of fuzzy set theory as well as a broad overview of increasingly important applications of fuzzy set.

In this paper we introduce fuzzy evidence graph, a special type of fuzzy graph and study some of its properties. We also give applications of fuzzy evidence graph in evidence theory for calculating belief measure, plausibility measure and Dempster's rule of combination.

2. PRELIMINARIES

Definition 2.1
A fuzzy graph \( G = (V, \mu, \rho) \) is a non empty set \( V \) together with a pair of functions \( \mu : V \rightarrow [0,1] \) and \( \rho : V \times V \rightarrow [0,1] \) such that for all \( x, y \) in \( V \), \( \rho(x, y) \leq \mu(x) \wedge \mu(y) \). We call \( \mu \) the fuzzy vertex set of \( G \) and \( \rho \), the fuzzy edge set of \( G \) respectively.

We denote the underlying graph of the fuzzy graph \( G = (\mu, \rho) \) by \( G^* = (\mu^*, \rho^*) \), where \( \mu^* = \{ x \in V : \mu(x) > 0 \} \) and \( \rho^* = \{ (x,y) \in V \times V : \rho(x,y) > 0 \} \).

We will assume that unless otherwise specified, the underlying set is \( V \) and that is finite. So we can omit \( V \) in the sequel and use the notation \( G = (\mu, \rho) \).

Example 2.2
Let \( G = (\mu, \rho) \) be with \( \mu^* = \{ u, v, w, x \} \). Let \( \mu(u) = 0.7, \mu(v) = 0.8, \mu(w) = 1, \mu(x) = 0.5 \), and \( \rho(u, v) = 0.6, \rho(v, w) = 0.8, \rho(w, x) = 0.3, \rho(x, u) = 0.5 \) and \( \rho(u, w) = 0.4 \). Then \( G \) is a fuzzy graph since \( \rho(u, v) \leq \mu(u) \wedge \mu(v) \) for all \( u, v \) in \( \mu^* \).

Definition 2.3
The fuzzy graph \( H = (v, \tau) \) is called a partial fuzzy subgraph of \( G = (\mu, \rho) \) if \( v \subseteq \mu \) and \( \tau \subseteq \rho \).

Definition 2.4
The fuzzy graph \( H = (P, v, \tau) \) is called a fuzzy subgraph of \( G = (V, \mu, \rho) \) induced by \( P \) if \( P \subseteq V \), \( v(x) = \mu(x) \) for all \( x \in P \) and \( \tau(x, y) = \rho(x, y) \) for all \( x, y \in P \).

A single node is considered as a trivial path of length 0.

Definition 2.5
The strength of a path is the weight of the weakest edge of the path.
Definition 2.6
A path $P$ in a fuzzy graph $(\mu, \rho)$ is a sequence of distinct vertices $x_0, x_1, x_2, \ldots, x_n$ (except possibly $x_0$ and $x_n$) such that $\rho(x_{i-1}, x_i) > 0$, $1 \leq i \leq n$. $n \geq 1$ is called the length of the path $P$.

The degree of membership of a weakest arc is defined as its strength.

If $x_0 = x_n$ and $n \geq 3$, then $P$ is called a cycle and a cycle $P$ is called a fuzzy cycle or f-cycle if it contains more than one weakest arc.

The consecutive pairs $(x_{i-1}, x_i)$ are called the edges of the path $P$.

The diameter of $x, y \in V$, written as $\text{diam}(x, y)$ is the length of the longest path joining $x$ to $y$.

Example 2.7
Let $G = (\mu, \rho)$ be with $\rho^* = \{u, v, w, x\}$. Let $\rho(u, v) = 0.2$, $\rho(v, w) = 0.2$, $\rho(w, x) = 0.3$, $\rho(x, u) = 0.5$ and $\rho(u, w) = 0.4$.

In $G$, $C_1 = u, v, w, x, u$ is a fuzzy cycle as it contains two weakest arcs namely arcs $(u, v)$ and $(v, w)$ whereas $C_2 = u, w, x, u$ is not a fuzzy cycle.

Definition 2.8
Let $G = (\mu, \rho)$ be a fuzzy graph. The strength of connectedness between two vertices $x$ and $y$ is defined as the maximum of the strengths of all paths between $x$ and $y$ and is denoted by $\text{CONN}_G(x, y)$. An $x$-$y$ path $P$ is called a strongest $x$-$y$ path if its strength equals $\text{CONN}_G(x, y)$.

Definition 2.9
An f-graph $G = (\mu, \rho)$ is connected if for every $x, y$ in $\rho^*$, $\text{CONN}_G(x, y) > 0$.

If $G$ is disconnected, maximal connected fuzzy graphs are called components.

If $G$ is connected, any two vertices are joined by a path.

Result 2.10
The strength of connectedness between any different pairs of vertices in a connected graph is 1, whereas in a connected fuzzy graph, it can be any real number in $(0, 1]$.

Result 2.11
An arc $(x, y)$ of a fuzzy graph $G$ is normal if and only if $\rho(x, y) = \text{CONN}_G(x, y)$. 

Fuzzy Evidence Graph
Definition 2.12
Consider the fuzzy matrix $M_G = (m_{ij})$ of the fuzzy graph $G = (\mu, \rho)$, where $m_{ij} = \begin{cases} \rho(v_i, v_j), & \text{if } i \neq j \\ \mu(v_i), & \text{if } i = j \end{cases}$. The matrix $M_G^k$ such that $M_G^k = M_G^{k+1}$, where $k$ is a positive integer is called the reachability matrix of $G$ denoted by $R_G = (r_{ij})$.

Definition 2.13
Complete fuzzy graph (CFG) is an f-graph $G = (\mu, \rho)$ such that $\rho(x, y) = \mu(x) \land \mu(y)$ for all $x$ and $y$.

Definition 2.14
Given a universal set $X$ and a nonempty family $\mathcal{C}$ of subsets of $X$, a fuzzy measure on $\langle X, \mathcal{C} \rangle$ is a function $g: \mathcal{C} \rightarrow [0,1]$ that satisfies the following requirements:

1. $g(\emptyset) = 0$ and $g(X) = 1$ (boundary requirements);
2. For all $A, B \in \mathcal{C}$, if $A \subseteq B$, then $g(A) \leq g(B)$ (monotonicity);
3. For any increasing sequence $A_1 \subseteq A_2 \subseteq A_3 \ldots$ in $\mathcal{C}$, if $\bigcup_{i=1}^{\infty} A_i \in \mathcal{C}$, then $\lim_{n \to \infty} g(A_i) = g(\bigcup_{i=1}^{\infty} A_i)$ (continuity from below);
4. For any decreasing sequence $A_1 \supseteq A_2 \supseteq A_3 \ldots$ in $\mathcal{C}$, if $\bigcap_{i=1}^{\infty} A_i \in \mathcal{C}$, then $\lim_{n \to \infty} g(A_i) = g(\bigcap_{i=1}^{\infty} A_i)$ (continuity from above).

Definition 2.15
Evidence theory is based on two dual non additive measures: belief measures and plausibility measures.

Given a universal set $X$, assumed here to be finite, a belief measure is a function $Bel: \mathcal{P}(X) \rightarrow [0,1]$ such that $Bel(\emptyset) = 0$, $Bel(X) = 1$ and

$$Bel(A_1 \cup A_2 \cup A_3 \ldots \cup A_n) \geq \sum_j Bel(A_j) - \sum_{j<k} Bel(A_j \cap A_k) + \ldots$$

For each $A \in \mathcal{P}(X)$, $Bel(A)$ is interpreted as the degree of belief based on available evidence that a given element of $X$ belongs to the set $A$.

Result 2.16
Belief measures are superadditive and when $X$ is infinite, continuous from above.
Definition 2.17
A plausibility measure is a function $\Pi: \mathcal{P}(X) \to [0,1]$ such that $\Pi(\emptyset) = 0$, $\Pi(X) = 1$, and

$$\Pi(A_1 \cap A_2 \cap A_3 \ldots \cap A_n) \leq \sum_j \Pi(A_j) - \sum_{j<k} \Pi(A_j \cup A_k) + \ldots + (-1)^{n+1} \sum_j \Pi(A_1 \cup A_2 \cup \ldots \cup A_n)$$

for all possible families of subsets of $X$.

Result 2.18
Plausibility measures are subadditive and when $X$ is infinite, $\Pi$ is continuous from below.

Results 2.19
Bel($A$) + Bel($\overline{A}$) ≤ 1
$\Pi(A) = 1 - \text{Bel}(\overline{A})$ for all $A \in \mathcal{P}(X)$
Bel($A$) = $1 - \Pi(\overline{A})$
$\Pi(A) + \Pi(\overline{A})$ ≥ 1
Belief measures and plausibility measures are mutually dual. Plausibility measures are independent of belief measures.

Definition 2.20
Belief and plausibility measures are characterized by a function $m: \mathcal{P}(X) \to [0,1]$ such that
$m(\emptyset) = 0$ and $\sum_{A \in \mathcal{P}(X)} m(A) = 1$. This function is called basic probability assignment.

For each $A \in \mathcal{P}(X)$, the value $m(A)$ expresses the proportion to which all available and relevant evidence supports the claim that a particular element of $X$, whose characterization in terms of relevant attributes is deficient, belongs to the set $A$.

Result 2.21
Fundamental difference between probability distribution functions and basic probability assignments is that the former is defined on $X$, while the latter is defined on $\mathcal{P}(X)$.
3 MAIN RESULTS

Definition 3.1 (Fuzzy Evidence Graph)(FEG)

Let X be a crisp set. A fuzzy evidence graph is a non-empty set V = \mathcal{P}(X) \setminus \emptyset together with a pair of functions m : V \rightarrow [0,1] and \rho : V \times V \rightarrow [0,1] such that for all A, B \in V, (A, B) \in \rho, whenever A \subseteq B and \rho(A, B) = m(A) \land m(B). Also \sum_{A \in V} m(A) = 1.

FEG can be denoted by G = (V, m, \rho), where m is called the assignment function and \rho is called the edge function.

Theorem 3.2

Fuzzy evidence graph is a fuzzy graph.

Proof

Proof is evident from the definition of fuzzy evidence graph.

Theorem 3.3

Number of vertices of a fuzzy evidence graph corresponding to a crispest X with n elements is \(2^n - 1\).

Proof

Let G = (m, \rho) be the fuzzy evidence graph. Then by the definition, V = \mathcal{P}(X) \setminus \emptyset is the vertex set. So the number of vertices is \(2^n - 1\).

Theorem 3.4

Number of edges of a fuzzy evidence graph corresponding to a crisp set with n elements is

\[
\sum_{i=1}^{n-1} (n - 1)C_i nC_{n-1} + \sum_{i=1}^{n-2} (n - 2)C_i nC_{n-2} + \ldots + n.
\]

Proof

Proof is the consequence of set theory.

Let us start with singleton sets. Every vertex corresponding to singleton sets is adjacent to all the vertices corresponding to their supersets - 2-element sets, 3-element sets etc.

\{a\} \rightarrow \{a, b\}, \{a, b, c\}, \{a, b, c, d\} \ldots

This can be done in \(\sum_{i=1}^{n-1} (n - 1)C_i\) ways.

The number of singleton sets for a set with n elements is n or nC_{n-1}.

So the total cases corresponding to singleton sets is \(\sum_{i=1}^{n-1} (n - 1)C_i nC_{n-1}\).
Fuzzy Evidence Graph

The vertices corresponding to 2-element sets is adjacent to all vertices corresponding to all vertices corresponding to their super sets - 3-element sets, 4-element sets etc.
\{a,b\}→ \{a,b,-\}, \{a,b,-,-\}, \{a,b,-,-,-\} etc.
This can be done \(\sum_{i=1}^{n-2}(n - 2)C_i\) ways and the number of 2-elements sets is \(nC_{n-2}\).
So the total number of cases corresponding to 2-element sets is \((\sum_{i=1}^{n-2}(n - 2)C_i)nC_{n-2}\).

Similarly proceeding we get the cases as
\[
(\sum_{i=1}^{n-1}(n - 1)C_i)nC_{n-1} + \sum_{i=1}^{n-2}(n - 2)C_i)nC_{n-2} + \ldots + \sum_{i=1}^{n-(n-1)}(n - (n - 1))C_i)nC_{n-(n-1)}
\]
\[
= (\sum_{i=1}^{n-1}(n - 1)C_i)nC_{n-1} + \sum_{i=1}^{n-2}(n - 2)C_i)nC_{n-2} + \ldots + n
\]

**Theorem 3.5**
Fuzzy evidence graph is complete

**Proof**
A complete fuzzy graph is a fuzzy graph \(G = (V, m, \rho)\) such that \(\rho(x,y) = m(x) \land m(y)\) for all \(x\) and \(y\). So by definition every fuzzy evidence graph is complete.

**Theorem 3.6**
There does not exist an edge \((A,B)\) such that \(\rho(A,B) = 1\) in a fuzzy evidence graph \(G=(m, \rho)\)

**Proof**
If possible, suppose that there exist an edge \((A,B)\) such that \(\rho(A,B) = 1\)
\(\Rightarrow m(A)=m(B)=1\) by the definition of FEG.
\(\Rightarrow \sum_i m(A_i)\neq1\), a contradiction.

**Definition 3.7**
The fuzzy graph \(H=(n, \tau)\) is called a partial fuzzy evidence subgraph of \(G=(m, \rho)\) if \(n(A) \leq m(A)\) for all \(A\) and \(\tau(A,B) \leq \rho(A,B)\) for all \(A, B\) such that \(A \subseteq B\)

The fuzzy graph \(H=(P, n, \tau)\) is called a fuzzy evidence subgraph of \(G = (V,m,\rho)\) induced by \(P\) if \(P \subseteq V\), \(n(A) = m(A)\), \(\tau(A,B) = \rho(A,B)\) for all \(A, B\) such that \(A \subseteq B\).
Proposition 3.8
The partial fuzzy evidence subgraph and fuzzy evidence subgraph of an FEG need not be a FEG.

Proof
For a partial fuzzy evidence graph and fuzzy evidence graph, \( \sum_i m(A_i) \) need not be equal to 1. But \( \sum_i m(A_i) \leq 1 \).

Theorem 3.9
The partial fuzzy evidence subgraph of an FEG is an FEG if and only if \( m(A) = n(A) \) for all \( A \) and \( \tau(A, B) = \rho(A, B) \) for all \( A, B \).

Proof
Let \( G \) be a FEG. By definition \( \sum_i m(A_i) = 1 \).
Let \( H=(n, \tau) \) be a partial fuzzy evidence subgraph of \( G \).
For a partial fuzzy evidence subgraph \( H=(n, \tau) \) of \( G=(m, \rho) \), \( n(A) \leq m(A) \) for all \( A \) and \( \tau(A, B) \leq \rho(A, B) \).
But \( \sum_i n(A_i) \neq 1 \) if \( n(A) < m(A) \). So \( n(A) = m(A) \) for all \( A \) which implies \( \tau(A, B) = \rho(A, B) \) for all \( A, B \).
Converse is obvious.

Definition 3.10
The vertex in a fuzzy evidence graph which is adjacent from every other vertex is called complete vertex.

Definition 3.11 (Path)
In an FEG \( G=(m, \rho) \), a path \( P \) of length \( n \) is a sequence of distinct vertices \( A_0, A_1, \ldots, A_n \) such that \( \rho(A_{i-1}, A_i) > 0 \), \( A_{i-1} \subseteq A_i \), \( i=1,2,\ldots,n \) and the degree of membership of the weakest arc is its strength.

Theorem 3.12
Maximum length of a path \( P \) in a FEG with \( n \) vertices is \( n-1 \).

Proof
Consider a FEG with \( n \) vertices \( A_0, A_1, \ldots, A_n \).
Start from an arbitrary vertex \(A_i\). Since there are only \(n-1\) vertices remaining, choose a vertex \(A_j\) such that \(\rho(A_i,A_j) > 0\), \(A_i \subseteq A_j\), \(i \neq j\). Similarly choose a vertex \(A_r\) from the remaining \(n-2\) such that \(\rho(A_j,A_r) > 0\), \(A_j \subseteq A_r\), and so on. Since there are only \(n\) distinct vertices the process must terminate at a vertex \(A_p\) such that \(\rho(A_{p-1},A_p) > 0\), \(A_{p-1} \subseteq A_p\), \(p < n\).

We get the sequence \(A_i,A_j,\ldots,A_p\), which is of length less than \(n\) and equal to \(n-1\) only if every vertex is ordered by the relation \(\subseteq\).

**Theorem 3.13**

FEG does not contain cycles and so fuzzy cycles.

**Proof**

Since in a FEG \(G = (V, m, \rho)\), for all \(A,B \in V\), \((A,B) \in \rho\) whenever \(A \subseteq B\) there will not be an edge \((B,A)\) and so a cycle.

**Definition 3.14**

The vertex \(A\) such that \(m(A) \leq m(B)\) for any \(B\) is called arc is called weakest vertex.

The arc \((A,B)\) determined by the weakest vertex is called weakest arc.

An FEG can have more than one weakest vertices and weakest arcs

**Definition 3.15**

A fuzzy evidence graph \(G = (V, m, \rho)\) is evidently connected if for every \(A, B\) in \(V\) with \(A \subseteq B\), \(\text{CONN}_G(A,B) > 0\).

Two vertices \(A\) and \(B\) such that \(A \subseteq B\) or \(B \subseteq A\) are evidently connected if there exist an edge \((A,B)\) or \((B,A)\).

**Proposition 3.16**

FEG is always disconnected

**Proof**

For \(X = \{x,y\}\) there does not exist a path between \(x\) and \(y\).

**Definition 3.17**

Consider the fuzzy evidence matrix \(M_G = (m_{AB})\) of \(G = (m, \rho)\), where \(m_{AB} = m(A,B)\) if \(A \subseteq B\) and 0 otherwise.
The matrix $M^k_G$ such that $M^k_G = M^{k+1}_G$, where $k$ is a positive integer is called the evidence reachability matrix of $G$ denoted by $R_G=(r_{AB})$.

**Definition 3.18**

Two vertices $A$ and $B$ of a fuzzy evidence graph is said to be mutually disconnected if there is neither an edge $(A,B)$ nor $(B,A)$.

The vertices $A$ and $B$ in fig.4 are mutually disconnected.

**Proposition 3.19**

Belief and plausibility measures of a complete vertex are always one.

**CONCLUSION**

In this paper we introduce new type of fuzzy graph called fuzzy evidence graph and its subgraph. We find some of its properties like completeness, paths, connectivity etc. We also present fuzzy evidence graph’s application in evidence theory for finding belief measure plausibility measure etc in uncertain situations. We can calculate belief measure using FEG as $\text{Bel}(A) = m(A) + \sum_B m(B)$, $(B, A)$ is an edge; where $m(A)$ is the degree of the vertex $A$ in FEG.

**REFERENCES**


