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INTERVAL – VALUED $\breve{\mathbb{Q}}$ −HESITANT FUZZY NORMAL SUBNEARRINGS

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Abstract

In this paper, the concept of interval- valued hesitant fuzzy set is introduced in the abstract mathematical notion of nearring for further development of hesitant fuzzy set on a theoretical model. An attempt was made to study the algebraic nature of interval-valued $\breve{\mathbb{Q}}$**-** hesitant fuzzy subnearring of nearring under normality.

Keywords: Hesitant Fuzzy Set, Interval-Valued Hesitant Fuzzy Set, Interval-Valued Hesitant Fuzzy Subset, Interval-valued $\breve{\mathbb{Q}}$**-** hesitant fuzzy subnearring.

1. INTRODUCTION

Nearrings are the generalisation of rings. Nearrings arise naturally in studying functions on a group (G,+). M(G)=(f:G->G,+,°) is then a nearring which is not a ring. The systematic study and research on nearrings are continuous. In 1905, Dickson who defined near fields formalized the key idea behind nearrings. Wieland studied nearrings, which were not near fields in late 1930s and the extensive studies about the subject is found in two famous books on nearrings [14] and [16].

After the introduction of fuzzy sets [28] by L.A.Zadeh, Jun.Y.B and Kin.K.H [12] defined an interval-valued fuzzy R-subgroups of nearrings. The uncertainty are handling by fuzzy set theory which happens in day-to-day life problems. Fuzzy sets are extended to four main categories. The first one is Atanassov's Intuitionistic Fuzzy Sets (IFS)[2] which deals membership and the non – membership degree of each element. The second one is Type – 2 Fuzzy Sets (T2FS) [8] that model the uncertainty through the use of a fuzzy set. The third is the Interval–Valued Fuzzy Sets (IVFS) [4, 23], the membership degree of an element belongs to the closed subinterval of the unit interval in which the length of interval is measure the lack of certainty for building the precise membership degree of the element and the fourth main category is the fuzzy multisets [25], the membership degree of each element is given by a subset of [0,1]. The Hesitant Fuzzy Set (HFS) is introduced by Torra [22] which deals with the general complication that appears some possible values make to hesitate about selecting the right one. The literature review shows the application and the growth of HFS quantitative [6, 17, 26, 29] and qualitative [18] because the hesitation can arise modelling the uncertainty in both ways. Osman Kazanci, Sultan yamarkmand and Serife Yilmaz (2007) have introduced the notion of intuitionistic Q-fuzzification of N- subgroups (subnearrings) in a nearring and investigated some related properties.

The motivation of this paper is to develop the HFS with the abstract mathematical notion as subnearring. Hence it is necessary to develop the planar nearring which is applied in various fields such as group theory geometry and its branches, combination, design of statistical experiments, coding theory and cryptography and construction of balanced incomplete block design through Hesitant Fuzzy Set.

The following flow chart illustrates the motivation of this work.

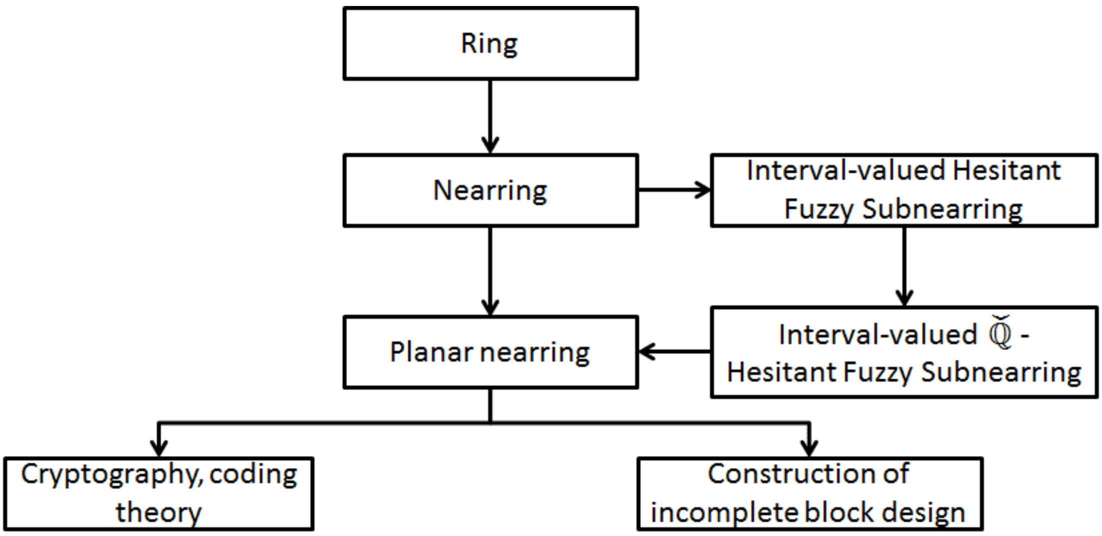

Fig1.1

The organisation of this paper is as follows. Section 2 introduces the basic concepts of HFS. In section 3 the definitions of $\breve{\mathbb{Q}}$ - hesitant fuzzy subset, union and intersection of $\breve{\mathbb{Q}}$ - hesitant fuzzy subset, interval-valued $\breve{\mathbb{Q}}$ - hesitant fuzzy subnearring, interval-valued $\breve{\mathbb{Q}}$ - hesitant fuzzy normal subnearring and the pre-image of interval-valued $\breve{\mathbb{Q}}$ -hesitant fuzzy sub subnearring and their product and relation are proposed. Section 4 presents the properties of interval-valued$\breve{\mathbb{Q}}$-hesitant fuzzy subnearring. The paper is finally concluded in section 5.

2. PRELIMINARIES

In this section some of the basic definitions of hesitant fuzzy set and interval-valued hesitant fuzzy set are discussed.

2.1. ***Definition*:*[22]*** Let X be a reference set, a HFS on X is a function $\mathfrak{h}$ that returns a subset of values in [0,1]: $\mathfrak{h}: X \rightarrow \wp([0,1])$. A HFS can be also constructed from a set of fuzzy sets.

2.2 ***Definition*:*[7]*** Let X be a reference set, and $I([0,1])$ be a set of all closed subintervals of $[0,1]$. An Interval –Valued Hesitant Fuzzy Set on X is, $\widetilde{A} = \{\langle x_i, \tilde{h}_A(x_i)\rangle / x_i \epsilon X, i = 1,2..n\}$ where $, \tilde{h}_A(x_i): X \rightarrow \wp([0,1])$ denotes all possible interval-valued membership degrees of the element $x_i \epsilon X$ to the set $\widetilde{A}$. For convenience , $\tilde{h}_A(x_i)$ is called an Interval- Valued Hesitant Fuzzy Element (IVHFE), where each $\tilde{\gamma}\ \epsilon\ \tilde{h}_A(x_i)$ is an interval and $\tilde{\gamma} = [\tilde{\gamma}^L, \tilde{\gamma}^U]$, being $\tilde{\gamma}^L\ and\ \tilde{\gamma}^U$ the lower and upper limits of $\tilde{\gamma}$, respectively.

2.3. ***Definition*:*[7]*** Let X be a fixed set. A mapping $\tilde{h}_A^{\sigma(\kappa)}: X \rightarrow Int[0,1]$ is called an Interval–Valued Hesitant Fuzzy Subset (briefly IVHFSS) of X, where Int[0,1] denotes the family of all closed subintervals of [0,1] and $\tilde{h}_A^{\sigma(\kappa)}(x) = \left\{\tilde{h}_A^{\sigma(\kappa)L}(x), \tilde{h}_A^{\sigma(\kappa)U}(x)\right\}$ for all x in X, where $\tilde{h}_A^{\sigma(\kappa)L} = \inf \tilde{h}_A^{\sigma(\kappa)}$ and $\tilde{h}_A^{\sigma(\kappa)U} = \sup \tilde{h}_A^{\sigma(\kappa)}$ are Hesitant Fuzzy Subset of X such that $\tilde{h}_A^{\sigma(\kappa)}$ stands for the kth largest interval number in the interval–valued hesitant fuzzy element $\mathfrak{h}$.

2.4.*Definition*:*[22,24]* Let $\mathfrak{h}$,$\mathfrak{h}_1$ and $\mathfrak{h}_2$ be three IVHFEs, then one has the following.

(i) $\mathfrak{h}^C = \{[\mathbf{1} - \boldsymbol{\gamma}^+, \mathbf{1} - \boldsymbol{\gamma}^-] / \boldsymbol{\gamma}\ \boldsymbol{\epsilon}\ \mathfrak{h}\}$,

(ii) $\mathfrak{h}_1 \cup \mathfrak{h}_2 = \{[\boldsymbol{\gamma}_1^- \vee \boldsymbol{\gamma}_2^-, \boldsymbol{\gamma}_1^+ \vee \boldsymbol{\gamma}_2^+] / \boldsymbol{\gamma}_1 \boldsymbol{\epsilon}\ \mathfrak{h}_1, \boldsymbol{\gamma}_2 \boldsymbol{\epsilon}\ \mathfrak{h}_2\}$,

(iii) $\mathfrak{h}_1 \cap \mathfrak{h}_2 = \{[\boldsymbol{\gamma}_1^- \wedge \boldsymbol{\gamma}_2^-, \boldsymbol{\gamma}_1^+ \vee \wedge] / \boldsymbol{\gamma}_1 \boldsymbol{\epsilon}\ \mathfrak{h}_1, \boldsymbol{\gamma}_2 \boldsymbol{\epsilon}\ \mathfrak{h}_2\}$,

(iv) $\mathfrak{h}^\lambda = \{[(\boldsymbol{\gamma}_1^-)^\lambda, (\boldsymbol{\gamma}_1^+)^\lambda] / \boldsymbol{\gamma} \boldsymbol{\epsilon}\ \mathfrak{h}\}, \boldsymbol{\lambda} > 0$,

(v) $\boldsymbol{\lambda}\,\mathfrak{h} = \{[1-(1-\gamma^{-})^{\lambda}, 1-(1-\gamma^{+})^{\lambda}]/\gamma\epsilon\,\mathfrak{h}\}, \boldsymbol{\lambda} > 0,$

(vi) $\mathfrak{h}_1 \oplus \mathfrak{h}_2 = \{[\gamma_1^{-}+\gamma_2^{-}-\gamma_1^{-}\gamma_2^{-}, \gamma_1^{+}+\gamma_2^{+}-\gamma_1^{+}\gamma_2^{+}]/\gamma_1\epsilon\,\mathfrak{h}_1, \gamma_2\epsilon\,\mathfrak{h}_2\},$

(vii) $\mathfrak{h}_1 \otimes \mathfrak{h}_2 = \{[\gamma_1^{-}\gamma_2^{-}, \gamma_1^{+}\gamma_2^{+}]/\gamma_1\epsilon\,\mathfrak{h}_1, \gamma_2\epsilon\,\mathfrak{h}_2\}$

3. INTERVAL – VALUED $\breve{\mathbb{Q}}$ −HESITANT FUZZY SUBNEARRING

In this section some of the new definitions of interval – valued $\breve{\mathbb{Q}}$ −Hesitant Fuzzy Subnearring are proposed.

Definition 3.1: Let X be a non empty set and $\breve{\mathbb{Q}}$ be a non empty set. A $\breve{\mathbb{Q}}$ - hesitant fuzzy subset $\tilde{h}_M^{\sigma(\kappa)}$ of X is a function $\tilde{h}_M^{\sigma(\kappa)}: X \times \breve{\mathbb{Q}} \to Int[0,1]$.

Definition 3.2: The union of two $\breve{\mathbb{Q}}$ - hesitant fuzzy subsets $\tilde{h}_M^{\sigma(\kappa)}$ and $\tilde{h}_N^{\sigma(\kappa)}$ of a set X is defined by $\tilde{h}_{M\cup N}^{\sigma(\kappa)}(x,q) = \rho\, max\left\{\tilde{h}_M^{\sigma(\kappa)}(x,q), \tilde{h}_N^{\sigma(\kappa)}(\mathrm{x,q})\right\}$ for all x in X and q in $\breve{\mathbb{Q}}$.

Definition 3.3 : The intersection of two $\breve{\mathbb{Q}}$ - hesitant fuzzy subsets $\tilde{h}_M^{\sigma(\kappa)}$ and $\tilde{h}_N^{\sigma(\kappa)}$ of a set X is defined by $\tilde{h}_{M\cap N}^{\sigma(\kappa)}(x,q) = \rho\, min\left\{\tilde{h}_M^{\sigma(\kappa)}(x,q), \tilde{h}_N^{\sigma(\kappa)}(\mathrm{x,q})\right\}$ for all x in X and q in $\breve{\mathbb{Q}}$.

Definition 3.4 : Let $(R,+,\cdot)$ be a nearring . A $\breve{\mathbb{Q}}$ - hesitant fuzzy subset $\tilde{h}_M^{\sigma(\kappa)}$ of R is said to be an Interval – Valued $\breve{\mathbb{Q}}$ - hesitant fuzzy subnearring of R if it satisfies the following conditions.

(i) $\tilde{h}_M^{\sigma(\kappa)}(x+y,q) \geq \rho \min\left\{\left[\tilde{h}_M^{\sigma(\kappa)L}(x,q), \tilde{h}_M^{\sigma(\kappa)U}(x,q)\right], \left[\tilde{h}_M^{\sigma(\kappa)L}(y,q), \tilde{h}_M^{\sigma(\kappa)U}(y,q)\right]\right\}$

(ii) $\tilde{h}_M^{\sigma(\kappa)}(-x,q) \geq \tilde{h}_M^{\sigma(\kappa)}(x,q)$

(iii) $\tilde{h}_M^{\sigma(\kappa)}(xy,q) \geq \rho \min\left\{\left[\tilde{h}_M^{\sigma(\kappa)L}(x,q), \tilde{h}_M^{\sigma(\kappa)U}(x,q)\right], \left[\tilde{h}_M^{\sigma(\kappa)L}(y,q), \tilde{h}_M^{\sigma(\kappa)U}(y,q)\right]\right\}$

Definition 3.5 : Let $(R,+,\cdot)$ be a nearring. An Interval – Valued $\breve{\mathbb{Q}}$ - hesitant fuzzy subnearring $\tilde{h}_M^{\sigma(\kappa)}$ of R is said to be an Interval – Valued $\breve{\mathbb{Q}}$ - hesitant fuzzy normal subnearring of R if $\tilde{h}_M^{\sigma(\kappa)}(xy,q) = \tilde{h}_M^{\sigma(\kappa)}(yx,q)$, for all x and y in R and q in $\breve{\mathbb{Q}}$.

Definition 3.6: Let $\tilde{h}_M^{\sigma(\kappa)}$ and $\tilde{h}_N^{\sigma(\kappa)}$ be an Interval – Valued $\breve{\mathbb{Q}}$ - hesitant fuzzy subsets of sets G and H respectively. The product of $\tilde{h}_M^{\sigma(\kappa)}$ and $\tilde{h}_N^{\sigma(\kappa)}$ is defined by $\tilde{h}_{M\times N}^{\sigma(\kappa)}[(x,y),q] = \rho\, min\left\{\tilde{h}_M^{\sigma(\kappa)}(x,q), \tilde{h}_N^{\sigma(\kappa)}(\mathrm{y,q})\right\}$.

Definition 3.7: Let $\tilde{h}_M^{\sigma(\kappa)}$ be an Interval – Valued $\breve{\mathbb{Q}}$ - hesitant fuzzy subset in a set S , the strongest Interval – Valued $\breve{\mathbb{Q}}$ - hesitant fuzzy relation on S , that is an Interval – valued $\breve{\mathbb{Q}}$ - hesitant fuzzy relation on $\tilde{h}_M^{\sigma(\kappa)}$ is $\tilde{h}_V^{\sigma(\kappa)}$ given by $\tilde{h}_V^{\sigma(\kappa)}((x,y),q) = \rho \min\left\{\left[\tilde{h}_M^{\sigma(\kappa)L}(x,q), \tilde{h}_M^{\sigma(\kappa)U}(y,q)\right]\right\}$ for all x and y in S and q in $\breve{\mathbb{Q}}$.

Definition 3.8: Let $(R,+,\cdot)$ and $(R',+,\cdot)$ be any two nearrings. Let f : R→ R' be any function and $\tilde{h}_M^{\sigma(\kappa)}$ be an interval-valued $\breve{\mathbb{Q}}$ - hesitant fuzzy subnearring in R, $\tilde{h}_V^{\sigma(\kappa)}$ be an interval-valued $\breve{\mathbb{Q}}$ - hesitant fuzzy subnearring in $f(R) = R'$, defined by $\tilde{h}_V^{\sigma(\kappa)}(y,q) = \operatorname{Sup}_{x \epsilon f^{-1}(y)} \tilde{h}_M^{\sigma(\kappa)}(x,q)$ for all x in R and y in R' and q in $\breve{\mathbb{Q}}$. Then $\tilde{h}_M^{\sigma(\kappa)}$is called a preimage of $\tilde{h}_V^{\sigma(\kappa)}$ under f and is denoted by $f^{-1}\left(\tilde{h}_V^{\sigma(\kappa)}\right)$.

4. PROPERTIES OF INTERVAL – VALUED $\breve{\mathbb{Q}}$ - HESITANT FUZZY SUBNEARRINGS

In this section the properties of interval – valued $\breve{\mathbb{Q}} -$ hesitant fuzzy subnearring are discussed . The following properties of normality are studied to lay the theoretical frame work for further studies in this area.

Theorem 4.1:

Let $(R,+,\cdot)$ be a nearing. If $\tilde{h}_M^{\sigma(\kappa)}$and $\tilde{h}_N^{\sigma(\kappa)}$are two interval - valued $\breve{\mathbb{Q}}$ - hesitant fuzzy normal subnearrings of R, then their intersection $\tilde{h}_{M\cap N}^{\sigma(\kappa)}$ is a interval – valued $\breve{\mathbb{Q}}$ - hesitant fuzzy normal subnearring of R.

Proof:

Let x and y in R and q in $\breve{\mathbb{Q}}$

Let $\tilde{h}_M^{\sigma(\kappa)} = \{< (x,q), \tilde{h}_M^{\sigma(\kappa)}(x,q) >/\ x$ in R and q in $\breve{\mathbb{Q}}\}$

and $\tilde{h}_N^{\sigma(\kappa)} = \{< (x,q), \tilde{h}_N^{\sigma(\kappa)}(x,q) >/\ x$ in R and q in $\breve{\mathbb{Q}}\}$

be an internal-valued $\breve{\mathbb{Q}}$-hesitant fuzzy normal subnearring of a nearing R. Let $\tilde{h}_{M\cap N}^{\sigma(\kappa)}$ and

$K = \{< (x,q), \tilde{h}_k^{\sigma(\kappa)}(x,q) >/\ x$ in R and q in $\breve{\mathbb{Q}}\}$

Where $\tilde{h}_K^{\sigma(\kappa)}(x,q) = \rho \min\{\tilde{h}_M^{\sigma(\kappa)}(x,q), \tilde{h}_N^{\sigma(\kappa)}(x,q)\}$

Then clearly K is an interval-valued $\breve{\mathbb{Q}}$-hesitant fuzzy subnearring of a nearing R, since $\tilde{h}_M^{\sigma(\kappa)}$ and $\tilde{h}_N^{\sigma(\kappa)}$ are two internal-valued $\breve{\mathbb{Q}}$-hesitant fuzzy subnearring of a nearing R. And $\tilde{h}_K^{\sigma(\kappa)}(x,q) = \rho \min\{\tilde{h}_M^{\sigma(\kappa)}(x,q), \tilde{h}_N^{\sigma(\kappa)}(x,q)\} = \rho \min\{\tilde{h}_M^{\sigma(\kappa)}(yx,q), \tilde{h}_N^{\sigma(\kappa)}(yx,q)\}$

$= \tilde{h}_K^{\sigma(\kappa)}(yx,q)$

For all x and y in R and q in $\breve{\mathbb{Q}}$.

Therefore $\tilde{h}_K^{\sigma(\kappa)}(xy,q) = \tilde{h}_K^{\sigma(\kappa)}(yx,q)$, for all x and and y in R and q in $\breve{\mathbb{Q}}$.

Hence $\tilde{h}_{M\cap N}^{\sigma(\kappa)}$ is an internal-valued $\breve{\mathbb{Q}}$-hesitant fuzzy normal subnearring of a nearring R.

Theorem 4.2:
Let $(R, +,\cdot)$ be a nearing. The intersection of a family of interval-valued $\breve{\mathbb{Q}}$-hesitant fuzzy normal subnearring of R is an interval-valued $\breve{\mathbb{Q}}$-hesitant fuzzy normal subnearring of R.

Proof:
Let be a family of interval-valued $\breve{\mathbb{Q}}$-hesitant fuzzy normal subnearring of a nearing R and let
Then for x and y in R and q in $\breve{\mathbb{Q}}$ clearly intersection of a family of internal-valued $\breve{\mathbb{Q}}$-hesitant fuzzy subnearrings of the nearing R is an interval-valued $\breve{\mathbb{Q}}$-hesitant fuzzy subnearring of the nearring R and $\tilde{h}_M^{\sigma(\kappa)}(xy, q) = \inf \tilde{h}_{Mi}^{\sigma(\kappa)}(xy, q) = \tilde{h}_M^{\sigma(\kappa)}(yx, q)$ for all x and y in R and q in $\breve{\mathbb{Q}}$**.**
Therefore $\tilde{h}_M^{\sigma(\kappa)}(xy, q) = \tilde{h}_M^{\sigma(\kappa)}(yx, q)$ for all x and y in R and q in $\breve{\mathbb{Q}}$**.** Hence the intersection of a family of interval-valued $\breve{\mathbb{Q}}$-hesitant fuzzy normal subnearring of a nearring R is an interval-valued $\breve{\mathbb{Q}}$-hesitant fuzzy normal subnearring of a nearring R.

Theorem 4.3:
Let $\tilde{h}_M^{\sigma(\kappa)}$ and $\tilde{h}_N^{\sigma(\kappa)}$ be an interval-valued $\breve{\mathbb{Q}}$-hesitant fuzzy normal subnearring of the nearrings G and H respectively. If $\tilde{h}_M^{\sigma(\kappa)}$ and $\tilde{h}_N^{\sigma(\kappa)}$ are interval-valued $\breve{\mathbb{Q}}$-hesitant fuzzy normal subnearring then $\tilde{h}_{MxN}^{\sigma(\kappa)}$ is an interval-valued $\breve{\mathbb{Q}}$-hesitant fuzzy normal subnearring G and H.

Proof:
Let $\tilde{h}_M^{\sigma(\kappa)}$ and $\tilde{h}_N^{\sigma(\kappa)}$ be an interval-valued $\breve{\mathbb{Q}}$-hesitant fuzzy normal subnearring of the nearrings G and H respectively. Clearly $\tilde{h}_{MxN}^{\sigma(\kappa)}$ is an interval-valued $\breve{\mathbb{Q}}$-hesitant fuzzy normal subnearring G x H.
Let $x_1$and x_2 be in G, $y_1$and y_2 be in H.
Then (x_1, y_1) and (x_2, y_2) are in G x H and q in $\breve{\mathbb{Q}}$**.**
Now $\tilde{h}_{MxN}^{\sigma(\kappa)}[(x_1, y_1)(x_2, y_2), q] = \tilde{h}_{MxN}^{\sigma(\kappa)}[(x_1, x_2, y_1, y_2), q]$
$= \rho \min\left\{\tilde{h}_M^{\sigma(\kappa)}(x_1, x_2, q), \tilde{h}_N^{\sigma(\kappa)}(y_1, y_2, q)\right\}$
$= \rho \min\left\{\tilde{h}_M^{\sigma(\kappa)}(x_2, x_1, q), \tilde{h}_N^{\sigma(\kappa)}(y_2, y_1, q)\right\}$
$= \tilde{h}_{MxN}^{\sigma(\kappa)}[(x_2x_1, y_2y_1), q]$
$= \tilde{h}_{MxN}^{\sigma(\kappa)}[(x_2, y_2)(x_1, y_1), q]$
Therefore $\tilde{h}_{MxN}^{\sigma(\kappa)}[(x_1, y_1)(x_2, y_2), q] = \tilde{h}_{MxN}^{\sigma(\kappa)}[(x_2, y_2)(x_1, y_1), q]$ (x_1, y_1) and (x_2, y_2) are in G x H and q in $\breve{\mathbb{Q}}$**.**
Hence $\tilde{h}_{MxN}^{\sigma(\kappa)}$ is an interval-valued $\breve{\mathbb{Q}}$-hesitant fuzzy normal subnearring of G x H.

Theorem 4.4:

Let $\tilde{h}_M^{\sigma(\kappa)}$ be an interval-valued $\breve{\mathbb{Q}}$-hesitant fuzzy subset in a nearring R and $\tilde{h}_V^{\sigma(\kappa)}$ be the strongest interval-valued $\breve{\mathbb{Q}}$-hesitant fuzzy relation on R. Then $\tilde{h}_M^{\sigma(\kappa)}$ is an interval-valued $\breve{\mathbb{Q}}$-hesitant fuzzy normal subnearring of R if and only if $\tilde{h}_V^{\sigma(\kappa)}$ is an interval-valued $\breve{\mathbb{Q}}$-hesitant fuzzy normal subnearring of $R \times R$.

Proof:

Suppose that $\tilde{h}_M^{\sigma(\kappa)}$ is an interval-valued $\breve{\mathbb{Q}}$-hesitant fuzzy normal subnearring of R. Then for any $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in $R \times R$ and q in $\breve{\mathbb{Q}}$ clearly $\tilde{h}_V^{\sigma(\kappa)}$ is an interval-valued $\breve{\mathbb{Q}}$-hesitant fuzzy normal subnearring of nearring R. We have

$$\tilde{h}_V^{\sigma(\kappa)}(xy,q) = \tilde{h}_V^{\sigma(\kappa)}\,[(x_1,x_2)(y_1,y_2),q]$$
$$= \tilde{h}_V^{\sigma(\kappa)}\,[(x_1y_1,x_2y_2),q]$$
$$= \rho\min\{[\tilde{h}_M^{\sigma(\kappa)L}(x_1y_1,q),\tilde{h}_M^{\sigma(\kappa)U}(x_1y_1,q)],[\tilde{h}_M^{\sigma(\kappa)L}(x_2y_2,q),\tilde{h}_M^{\sigma(\kappa)U}(x_2y_2,q)]\}$$
$$\geq \rho\min\{[\tilde{h}_M^{\sigma(\kappa)L}(x_1y_1,q),\tilde{h}_M^{\sigma(\kappa)L}(x_2y_2,q)],[\tilde{h}_M^{\sigma(\kappa)U}(x_1y_1,q),\tilde{h}_M^{\sigma(\kappa)U}(x_2y_2,q)]\}$$
$$\geq \rho\min\{inf[\tilde{h}_M^{\sigma(\kappa)}(x_1y_1,q),\tilde{h}_M^{\sigma(\kappa)}(x_1y_1,q)],sup[\tilde{h}_M^{\sigma(\kappa)}(x_2y_2,q),\tilde{h}_M^{\sigma(\kappa)}(x_2y_2,q)]\}$$
$$= \rho\min\{\tilde{h}_M^{\sigma(\kappa)}(x_1y_1,q),\tilde{h}_M^{\sigma(\kappa)}(x_2y_2,q)\}$$
$$= \rho\min\{\tilde{h}_M^{\sigma(\kappa)}(y_1x_1,q),\tilde{h}_M^{\sigma(\kappa)}(y_2x_2,q)\}$$
$$= \tilde{h}_V^{\sigma(\kappa)}\,[(y_1x_1,y_2x_2),q]$$
$$= \tilde{h}_V^{\sigma(\kappa)}\,[(y_1,y_2)(x_1,x_2),q]$$
$$= \tilde{h}_V^{\sigma(\kappa)}(yx,q)$$

Therefore $\tilde{h}_V^{\sigma(\kappa)}(xy,q) = \tilde{h}_V^{\sigma(\kappa)}(yx,q)$, for all x and y in $R \times R$ and q in $\breve{\mathbb{Q}}$.

This proves that $\tilde{h}_V^{\sigma(\kappa)}$ is an interval-valued $\breve{\mathbb{Q}}$-hesitant fuzzy normal subnearring of $R \times R$.

Conversely assume that $\tilde{h}_V^{\sigma(\kappa)}$ is an interval-valued $\breve{\mathbb{Q}}$-hesitant fuzzy normal subnearring of $R \times R$, then for any $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in $R \times R$, we know that $\tilde{h}_M^{\sigma(\kappa)}$ is an interval-valued $\breve{\mathbb{Q}}$-hesitant fuzzy normal subnearring of R , then

$$\tilde{h}_M^{\sigma(\kappa)}(x_1y_1,q) = \rho\min\{[\tilde{h}_M^{\sigma(\kappa)}(x_1y_1,q),\tilde{h}_M^{\sigma(\kappa)}(x_2y_2,q)]\}$$
$$= \tilde{h}_V^{\sigma(\kappa)}\,[(x_1y_1,x_2y_2),q]$$
$$= \tilde{h}_V^{\sigma(\kappa)}\,[(x_1,x_2)(y_1,y_2),q]$$
$$= \tilde{h}_V^{\sigma(\kappa)}(xy,q)$$
$$= \tilde{h}_V^{\sigma(\kappa)}(yx,q)$$
$$= \tilde{h}_V^{\sigma(\kappa)}\,[(y_1,y_2)(x_1,x_2),q]$$
$$= \tilde{h}_V^{\sigma(\kappa)}\,[(y_1x_1,y_2x_2),q]$$

$= \rho \min\{\tilde{h}_M^{\sigma(\kappa)}(y_1x_1, q), \tilde{h}_M^{\sigma(\kappa)}(y_2x_2, q)\}$

$= \tilde{h}_M^{\sigma(\kappa)}(y_1x_1, q)$. If $x_2 = 0, y_2 = 0$, we get
$\tilde{h}_M^{\sigma(\kappa)}(x_1y_1, q) = \tilde{h}_M^{\sigma(\kappa)}(y_1x_1, q)$ for all x_1 *and* y_1 in R and q in $\breve{\mathbb{Q}}$**.**
Therefore $\tilde{h}_M^{\sigma(\kappa)}$ is an interval-valued $\breve{\mathbb{Q}}$-hesitant fuzzy normal subnearring of R.

Theorem 4.5:
Let $(R, +,\cdot)$ and $(R', +,\cdot)$ be any two nearrings .The interval – valued $\widetilde{\mathbb{Q}}$ - hesitant fuzzy normal subnearring of R under the homomorphic image is a an interval – valued $\widetilde{\mathbb{Q}}$ - hesitant fuzzy normal subnearring of $f(R) = R'$.

Proof:
Let $(R, +,\cdot)$ and $(R', +,\cdot)$ be any two nearrings and $f: R \to R'$ be a homomorphism, then

$f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all x and y in R. Let $\tilde{h}_M{}^{\sigma(k)}$ be an interval – valued $\widetilde{\mathbb{Q}}$ - hesitant fuzzy normal subnearring of a nearring R and $\tilde{h}_V{}^{\sigma(k)}$ be the homomorphic image of $\tilde{h}_M{}^{\sigma(k)}$ under f. It is to be proved that $\tilde{h}_V{}^{\sigma(k)}$ is an interval – valued $\widetilde{\mathbb{Q}} -$ hesitant fuzzy normal subnearring of a nearring $f(R) = R'$. Now , for $f(x)$ and $f(y)$ in R', and q in $\widetilde{\mathbb{Q}}$, clearly $\tilde{h}_V{}^{\sigma(k)}$ is an interval – valued $\widetilde{\mathbb{Q}}$ - hesitant fuzzy subnearring of the nearring R' , since $\tilde{h}_M{}^{\sigma(k)}$ is an interval – valued $\widetilde{\mathbb{Q}}$ - hesitant fuzzy subnearring of a nearring R. Now, $\tilde{h}_V{}^{\sigma(k)}[f(x)f(y), q] = \tilde{h}_V{}^{\sigma(k)}[f(xy), q]$ as f is a homomorphism.

$\geq \tilde{h}_M{}^{\sigma(k)}[xy, q]$

$= \tilde{h}_M{}^{\sigma(k)}[yx, q]$

$\leq \tilde{h}_V{}^{\sigma(k)}[f(yx), q]$

$= \tilde{h}_V{}^{\sigma(k)}[f(y)f(x), q]$, as f is a homomorphism.

Which implies that,

$\tilde{h}_V{}^{\sigma(k)}[f(x)f(y), q] = \tilde{h}_V{}^{\sigma(k)}[f(y)f(x), q]$ for all f(x) and f(y) in R' and q in $\widetilde{\mathbb{Q}}$.

Hence $\tilde{h}_V{}^{\sigma(k)}$ is an interval – valued $\widetilde{\mathbb{Q}}$ - hesitant fuzzy normal subnearring of the nearring R'.

Theorem 4.6:

Let $(R,+,\cdot)$ and $(R',+,\cdot)$ be any two nearrings .The interval – valued $\tilde{\mathbb{Q}}$ - hesitant fuzzy normal subnearring of $f(R) = R'$ under homomorphic preimage is an interval – valued $\tilde{\mathbb{Q}}$ - hesitant fuzzy normal subnearring of R.

Proof:

Let $(R,+,\cdot)$ and $(R',+,\cdot)$ be any two nearrings and $f: R \to R'$ be a homomorphism, then

$f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all x and y in R. Let $\tilde{h}_V{}^{\sigma(k)}$ be an

interval – valued $\tilde{\mathbb{Q}}$ - hesitant fuzzy normal subnearring of a nearring R' and $\tilde{h}_M{}^{\sigma(k)}$ be a homomorphic pre-image of $\tilde{h}_V{}^{\sigma(k)}$ under f. It is to be proved that $\tilde{h}_M{}^{\sigma(k)}$ is an interval – valued $\tilde{\mathbb{Q}}$ - hesitant fuzzy subnearring of the nearring R. Let x and y in R and q in $\tilde{\mathbb{Q}}$.

Then clearly $\tilde{h}_M{}^{\sigma(k)}$ is an interval – valued $\tilde{\mathbb{Q}}$ - hesitant fuzzy subnearring of the nearring R, since $\tilde{h}_V{}^{\sigma(k)}$ is an interval – valued $\tilde{\mathbb{Q}}$ - hesitant fuzzy subnearring of a nearring R' . Now

$\tilde{h}_M{}^{\sigma(k)}[xy,q] = \tilde{h}_V{}^{\sigma(k)}[f(xy),q]$.

Since $\tilde{h}_M{}^{\sigma(k)}(x,q) = \tilde{h}_V{}^{\sigma(k)}[f(x),q]$

$= \tilde{h}_V{}^{\sigma(k)}[f(x)f(y),q]$, as f is a homomorphism

$= \tilde{h}_M{}^{\sigma(k)}[yx,q]$.

Since $\tilde{h}_M{}^{\sigma(k)}(x,q) = \tilde{h}_V{}^{\sigma(k)}[f(x),q]$

Which implies that $\tilde{h}_M{}^{\sigma(k)}[xy,q] = \tilde{h}_M{}^{\sigma(k)}[yx,q]$ for all x and y in R and q in $\tilde{\mathbb{Q}}$.

Hence $\tilde{h}_M{}^{\sigma(k)}$ is an interval – valued $\tilde{\mathbb{Q}}$ - hesitant fuzzy normal subnearring of R.

Theorem 4.7:

Let $(R,+,\cdot)$ and $(R',+,\cdot)$ be any two nearrings .The interval – valued $\tilde{\mathbb{Q}}$ - hesitant fuzzy normal subnearring of R under the anti-homomorphic image is an interval – valued $\tilde{\mathbb{Q}}$ - hesitant fuzzy normal subnearring of $f(R) = R'$.

Proof:

Let $(R,+,\cdot)$ and $(R',+,\cdot)$ be any two nearrings and $f: R \to R'$ be an anti-homomorphism. Then $f(x+y) = f(y) + f(x)$ and $f(xy) = f(y)f(x)$ for all x and y in R. Let $\tilde{h}_M{}^{\sigma(k)}$ be an interval-valued $\tilde{\mathbb{Q}}$ - hesitant fuzzy normal subnearring of R and ${\tilde{h}_V}^{\sigma(k)}$ be an anti-homomorphic image of ${\tilde{h}_M}^{\sigma(k)}$ under f. It is to be proved that ${\tilde{h}_V}^{\sigma(k)}$ is an interval-valued $\tilde{\mathbb{Q}}$ - hesitant fuzzy normal subnearring of $f(R) = R'$. Now, for f(x) and f(y) in R' and q in $\tilde{\mathbb{Q}}$, clearly ${\tilde{h}_V}^{\sigma(k)}$ is an interval-valued $\tilde{\mathbb{Q}}$ - hesitant fuzzy subnearring R' , since ${\tilde{h}_M}^{\sigma(k)}$ is an interval-valued $\tilde{\mathbb{Q}}$ - hesitant fuzzy subnearring of R.

${\tilde{h}_V}^{\sigma(k)}[f(x)f(y), q] = {\tilde{h}_V}^{\sigma(k)}[f(yx), q]$, as f is an anti-homomorphism

$\geq {\tilde{h}_V}^{\sigma(k)}(yx, q)$

$= {\tilde{h}_M}^{\sigma(k)}(xy, q)$

$\leq {\tilde{h}_V}^{\sigma(k)}[f(xy), q]$

$= {\tilde{h}_V}^{\sigma(k)}[f(y)f(x), q]$ as f is an anti-homomorphism which implies that ${\tilde{h}_V}^{\sigma(k)}[f(x)f(y), q] = {\tilde{h}_V}^{\sigma(k)}[f(y)f(x), q]$ for all f(x) and f(y) in R' and q in $\tilde{\mathbb{Q}}$.

Hence ${\tilde{h}_V}^{\sigma(k)}$ is an interval-valued $\tilde{\mathbb{Q}}$ - hesitant fuzzy normal subnearring of R'.

5. CONCLUSION

This paper is concluded that, the concept of interval-valued $\tilde{\mathbb{Q}}$ - hesitant fuzzy subnearrings are defined. Based on these definitions, the properties of interval-valued $\tilde{\mathbb{Q}}$ - hesitant fuzzy subnearrings under the concept of normality are discussed. In future the work will be extended to interval-valued hesitant fuzzy Planar near-ring because planar near-rings are applied in various fields like Software testing, Cryptography, Digital Computing, Automata Theory, Finite Geometry and Construction of balanced incomplete block design etc.

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