On Information Measure for Fuzzy Set

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Abstract:
A Measure is developed for measuring the amount of information given when the membership function of a fuzzy set is partly specified. Also discussion is given for a new class of information measure (measure of entropy) for fuzzy sets.

Keywords: Fuzzy sets, Information measure, membership function

INTRODUCTION:
Let A is a fuzzy set with n support points \(x_1, x_2, x_3, \ldots, x_{n-1}, x_n\) and \(\mu_A(x)\) as its membership function. If we know, \(\mu_A(x_1), \mu_A(x_2), \mu_A(x_3), \ldots, \mu_A(x_{n-1}), \mu_A(x_n)\) we have complete knowledge or full information about the fuzzy set. If we have, some knowledge about \(\mu_A(x_1), \mu_A(x_2), \mu_A(x_3), \ldots, \mu_A(x_{n-1}), \mu_A(x_n)\), this however does not enable us to determine them uniquely. we say we have only partial information about the fuzzy set.

Thus, we have partial information about the fuzzy set if

1. We know only some of the values \(\mu_A(x_i)\)’s.
2. We know some relations between these values, which however do not determine these values uniquely.

Obviously, Information given by this incomplete knowledge is less than full information. Now question arises how much less it is. Now we have to find a quantitative measure for the information given by partial knowledge of the values.
To solve this problem, we use the concept of fuzzy entropy. For the probability
distribution: \( P = (p_1, p_2, \ldots, p_n) \) Shannon [1] obtained in 1948 the measure of
entropy
\[
H(P) = - \sum_{i=1}^{n} p_i \ln p_i
\]
Kullback and Leibler [2] obtained in 1951 the measure of directed divergence of
probability distribution \( P = (p_1, p_2, \ldots, p_n) \) from the probability distribution \( Q = (q_1, q_2, \ldots, q_n) \) as
\[
D(P:Q) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i}
\]
In 1959, Kullback [3] suggested the use of measure of symmetric divergence
\[
J(P:Q) = D(P:Q) + D(Q:P)
\]
\[
= \sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i} + \sum_{i=1}^{n} q_i \ln \frac{q_i}{p_i} = \sum_{i=1}^{n} (p_i - q_i) \ln \frac{p_i}{q_i}
\]
In 1961, Kerridge [4] gave his measure of inaccuracy of \( P \) relative to \( Q \) as
\[
I(P:Q) = - \sum_{i=1}^{n} p_i \ln q_i
\]
For measuring the uncertainty of a probability distribution, concept of entropy was
developed. Zadeh [5] developed his theory of fuzzy sets to enable him to measure the
ambiguity of fuzzy set. A fuzzy set \( A \) is characterized by a membership function
\( \mu_A(x) \) where to each \( x \) in the universe of discourse, there is associated a membership
value in \([0,1]\) i.e. \( \mu_A(x) \) represents the grade of membership of \( x \) of the set \( A \). Thus
if \( x_1, x_2, x_3, \ldots, x_n \) are members of the universe of discourse
\( \mu_A(x_1), \mu_A(x_2), \mu_A(x_3), \ldots, \mu_A(x_{n-1}), \mu_A(x_n) \) all lie between 0 and 1, but
these are not probabilities because their sum is not unity.

However
\[
\phi_A(x_i) = \frac{\mu_A(x_i)}{\sum_{i=1}^{n} \mu_A(x_i)}, i = 1, 2, 3, \ldots, n (\because \sum_{i=1}^{n} \phi_A(x_i) = 1)
\]
gives a probability distribution. Fuzziness represents uncertainty of a certain type. If $\mu_A(x) = 0$, $x$ does not belong to $A$ and there is no ambiguity, if $\mu_A(x) = 1$, $x$ definitely belongs to $A$ and again there is no ambiguity and if $\mu_A(x) = \frac{1}{2}$ there is maximum ambiguity.

For measuring the uncertainty of a probability distribution, concept of entropy was developed and therefore it was natural for researchers in fuzzy set theory to make use of entropy concept for measuring fuzziness.

The Kanfman [6] defined entropy of a fuzzy set $A$ having $n$ support points by

$$H_K(A) = -\frac{1}{\ln n} \sum_{i=1}^{n} \phi_A(x_i) \ln \phi_A(x_i)$$

............... (1)

Since $\mu_A(x)$ and $1 - \mu_A(x)$ gave same degree of fuzziness and as such Deluca and Termini [7] suggested the measure

$$H_D(A) = -\sum_{i=1}^{n} [\mu_A(x_i) \ln \mu_A(x_i) + (1 - \mu_A(x_i)) \ln (1 - \mu_A(x_i))]$$

.... (2)

Many authors propose different types of measures of entropy.

Bhandari and Pal [8] surveyed the literature on information measures on fuzzy sets and gave some new measures.

Thus corresponding to Shannon[1] measure, we have the measure (1).

Corresponding to Renyi’s [9] measure

$$\frac{1}{1-\alpha} \ln \sum_{i=1}^{n} p_i^\alpha, \alpha > 0, \alpha \neq 1$$

They suggested the measure

$$\frac{1}{1-\alpha} \sum_{i=1}^{n} \ln [\mu_A^\alpha(x_i) + (1 - \mu_A(x_i))^\alpha], \alpha > 0, \alpha \neq 1$$

...(3)

Now we try to develop a measure when we have only partial information about the fuzzy set. To solve this problem, we use entropy of a fuzzy set $A$.

Measure of entropy given by Delucia & Termini is
\[ H_D(A) = -\sum_{i=1}^{n} \left[ \mu_A(x_i) \ln \mu_A(x_i) + \left(1 - \mu_A(x_i) \right) \ln \left(1 - \mu_A(x_i) \right) \right] \]

This measures the fuzziness of the set. This is maximum when each \( \mu_A(x_i) = 1/2 \) i.e. when A is the most fuzzy set. This is minimum when \( \mu_A(x_i) = 1 \) or 0 i.e. when the set is a crisp set.

If we do not know, \( \mu_A(x_1), \mu_A(x_2), \mu_A(x_3), \ldots \ldots , \mu_A(x_{n-1}), \mu_A(x_n) \) each of these can take any value between 0 and 1. In this case, the maximum value of S is given by

\[ H_D(A)_{\text{max}} = -\sum_{i=1}^{n} \left[ \frac{1}{2} \ln \frac{1}{2} + \left(1 - \frac{1}{2} \right) \ln \left(1 - \frac{1}{2} \right) \right] = -\sum_{i=1}^{n} \ln \frac{1}{2} \]

and it arises for the most fuzzy set.

Its minimum value is \( H_D(A)_{\text{min}} = 0 \) and it arises when the set is a crisp set.

So that \( H_D(A)_{\text{max}} = n \ln 2, \ H_D(A)_{\text{min}} = 0 \) \hfill \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (5) \]

The fuzziness gap, which is the difference between \( H_D(A)_{\text{max}} \) and \( H_D(A)_{\text{min}} \) is \( n \ln 2 \).

Measure of entropy given by Kanfman is

\[ H_K(A) = -\frac{1}{\ln n} \sum_{i=1}^{n} \phi_A(x_i) \ln \phi_A(x_i) \]

\[ = -\frac{1}{\ln n} \sum_{i=1}^{n} \frac{\mu_A(x_i)}{\sum_{i=1}^{n} \mu_A(x_i)} \ln \frac{\mu_A(x_i)}{\sum_{i=1}^{n} \mu_A(x_i)} \]

\[ \because \ H_K(A)_{\text{max}} = -\frac{1}{\ln n} \sum_{i=1}^{n} \frac{1/2}{\sum_{n=1}^{n} 1/2} \ln \frac{1/2}{\sum_{n=1}^{n} 1/2} = -\frac{1}{\ln n} \sum_{i=1}^{n} \frac{1}{n} \ln \frac{1}{n} \]

\[ = -\frac{1}{\ln n} \left( n \ln \frac{1}{n} \right) = -\frac{n \ln n}{\ln n} = 1 \]

\[ \because \ H_K(A)_{\text{min}} = 0 \]

So that \( H_K(A)_{\text{max}} = 1 \) & \( H_K(A)_{\text{min}} = 0 \) \hfill \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (6) \]
The fuzziness gap, which is the difference between $H_K(A)_{\text{max}}$ and $H_K(A)_{\text{min}}$ is 1

Measure of entropy given by Bhandari and Pal is

$$S(A) = \frac{1}{1-\alpha} \sum_{i=1}^{n} \ln \left[ \mu_A^\alpha(x_i) + (1 - \mu_A(x_i))^{\alpha} \right], \alpha > 0, \alpha \neq 1$$

$$S(A)_{\text{max}} = \frac{1}{1-\alpha} \sum_{i=1}^{n} \ln \left[ \frac{1}{2^\alpha} + \frac{1}{2^\alpha} \right] = \frac{1}{1-\alpha} \sum_{i=1}^{n} \ln \frac{2}{2\alpha} = \frac{n}{1-\alpha} \ln \frac{1}{2^{1-\alpha}}$$

$$= \frac{n}{1-\alpha} \ln 2^{1-\alpha} = n \ln 2, \text{ Which is free from } \alpha$$

$$S(A)_{\text{min}} = 0$$

So that $S(A)_{\text{max}} = 1$ & $S(A)_{\text{min}} = 0$

The fuzziness gap, which is the difference between $S(A)_{\text{max}}$ and $S(A)_{\text{min}}$ is $n \ln 2$.

This is similar as obtained by taking measure of entropy given by Delucia & Termini.

Now if we take measure of entropy

$$S(A) = -\frac{1}{n \ln 2} \sum_{i=1}^{n} \left[ \mu_A(x_i) \ln \mu_A(x_i) + (1 - \mu_A(x_i)) \ln (1 - \mu_A(x_i)) \right] \quad \ldots \ldots \ (7)$$

or

$$S(A) = \frac{1}{n \ln 2} \sum_{i=1}^{n} \ln \left[ \mu_A^\alpha(x_i) + (1 - \mu_A(x_i))^{\alpha} \right], \alpha > 0, \alpha \neq 1 \quad \ldots \ldots \ (8)$$

Then fuzziness gap will be 1; which is similar as obtained in case (ii).

Now if we are given some knowledge about

$\mu_A(x_1), \mu_A(x_2), \mu_A(x_3), \ldots \ldots, \mu_A(x_n)$; it can only decrease maximum value of measures i.e. $H_D(A)_{\text{max}}$ or $H_K(A)_{\text{max}}$ or $S(A)_{\text{max}}$, and increase $H_D(A)_{\text{min}}$ or $H_K(A)_{\text{min}}$ or $S(A)_{\text{min}}$ So that this knowledge reduces the fuzziness gap.
The reduction in fuzziness gap is due to the information given by the partial knowledge given. Thus

Information given by partial knowledge = fuzziness gap before we use the knowledge - fuzziness gap after we use this knowledge

We can thus calculate the information given by the partial knowledge.

Now we discuss reduction in all above three cases when one values of $\mu_A(x_i)$'s is given:

**Case (i)** - When we use measure of entropy given by Delucia & Termini

\[
H_D(A)_{\text{max}} = -\mu_A(x_1) \ln \mu_A(x_1) + (1 - \mu_A(x_1)) \ln (1 - \mu_A(x_1)) + (n - 1) \ln 2
\]

\[
H_D(A)_{\text{min}} = -\mu_A(x_1) \ln \mu_A(x_1) + (1 - \mu_A(x_1)) \ln (1 - \mu_A(x_1)) + 0
\]

So that $H_D(A)_{\text{max}} - H_D(A)_{\text{min}} = (n - 1) \ln 2$

Fuzziness gap before this value given is $n \ln 2$.

Reduction in fuzziness gap= $n \ln 2 - (n - 1) \ln 2 = \ln 2$

\[ \therefore \text{Information given by each of } \mu_A(x_1), \mu_A(x_2), \mu_A(x_3), \ldots, \mu_A(x_{n-1}), \mu_A(x_n) \text{ is } \ln 2 \text{ and if we use logarithms to base 2, the information given by each is unity.} \]

**Case (ii)** - When we use measure of entropy given by Kanfman

\[
H_K(A)_{\text{max}} = -\frac{1}{\ln n} \frac{\mu_A(x_1)}{\sum_{i=1}^{n} \mu_A(x_i)} \ln \frac{\mu_A(x_1)}{\sum_{i=1}^{n} \mu_A(x_i)} + 1
\]

\[
H_K(A)_{\text{min}} = -\frac{1}{\ln n} \frac{\mu_A(x_1)}{\sum_{i=1}^{n} \mu_A(x_i)} \ln \frac{\mu_A(x_1)}{\sum_{i=1}^{n} \mu_A(x_i)} + 0
\]

So that $H_K(A)_{\text{max}} - H_K(A)_{\text{min}} = 1$.

Fuzziness gap before this value given is 1.

Reduction in fuzziness gap = 0
Case (iii)- When we use measure of entropy given by Bhandari and Pal

\[
S(A)_{max} = \frac{1}{1-\alpha} \ln \left[ \mu_A(x_1)^\alpha + (1 - \mu_A(x_1))^\alpha \right] + (n - 1) \ln 2
\]

\[
S(A)_{min} = \frac{1}{1-\alpha} \ln \left[ \mu_A(x_1)^\alpha + (1 - \mu_A(x_1))^\alpha \right] + 0
\]

\[
\therefore S(A)_{max} - S(A)_{min} = (n - 1) \ln 2
\]

Fuzziness gap before this value given is \( n \ln 2 \).

Reduction in fuzziness gap = \( n \ln 2 - (n - 1) \ln 2 = \ln 2 \), which is similar as obtained in case (i).

REFERENCES


