Product of Fuzzy Soft Groups

Anju S Mattam¹ and Sasi Gopalan²

¹(Assistant Professor, Department of Mathematics, Little Flower College, Calicut University, Kerala, India.

²(Associate Professor, Division of Science and Humanities, School of Engineering, Cochin University of Science & Technology, Kerala, India.

Abstract

In this paper the concept of fuzzy soft group is extended to product of fuzzy soft groups by introducing new definitions in the theory of fuzzy algebra. The properties and structural characteristics of a fuzzy soft group are discussed to strengthen the basic concepts of the results established. Internal and external product of fuzzy soft groups is defined. The definitions on abelian fuzzy soft group, conjugate fuzzy soft groups are defined in the space of product of fuzzy soft groups.

Keywords – soft group, Fuzzy Soft Group, abelian Fuzzy soft group, Product of Fuzzy soft Groups

1. INTRODUCTION

The theory of fuzzy sets first developed by Zadeh [1] has been applied to many branches of mathematics. Later the fuzzification of the group concept into fuzzy subgroup was made by Rosenfeld [2]. The notion of product of fuzzy sets in a group was introduced by Asok Ray[3]. Maji et al. and Molodtsov [4],[5] have studied the theory of soft sets and also introduced the concept of fuzzy soft sets, which is a combination of fuzzy sets and soft sets. Further Aktaş and Çağman [6] have introduced the notion of soft groups. Aygunoglu and Aygun [7] have generalized the concept of soft groups and introduced fuzzy soft groups using statistical function, idempotent t- norms. This theory is extended from the notion of a group to the algebraic structures of fuzzy soft sets. In this paper we use minimum function as the idempotent t-norm to define the fuzzy soft group. Let \( X_1, X_2, \ldots, X_n \) be \( n \) groups and we use multiplication notation for all group
operations. We can make \( \prod_{i=1}^{n} X_{i} \) into a group by means of a binary operation of multiplication by components. The purpose of this paper is to show a way to use known fuzzy soft group over \( X_{i} \) as building blocks to form a fuzzy soft group over \( \prod_{i=1}^{n} X_{i} \). The organization of this paper is as follows. In section 2 preliminary definitions and results are given which will be used in the paper to define product of fuzzy soft groups. In section 3 a definition of generalized external product of fuzzy soft set is defined and its characteristic properties are studied. We prove that if the product of fuzzy soft sets is a fuzzy soft group over \( \prod_{i=1}^{n} X_{i} \) then for at least one \( i \), the fuzzy soft set over \( X_{i} \) is a fuzzy soft group. Also we prove that if the product of fuzzy soft sets is a group over \( \prod_{i=1}^{n} X_{i} \) then its \( \alpha \)-level set is a soft group over \( \prod_{i=1}^{n} X_{i} \).

2. BASIC DEFINITIONS

Definition 2.1[5]
Let \( X \) be a group, \( A \) be the set of parameters and \( P(X) \) denote the power set of \( X \). Consider the mapping \( f \) from \( A \) into \( P(X) \). If \( f(a) \) is a sub group of \( X \), for \( a \in A \), then \( (f,A) \) is called a soft group over \( X \).

Definition 2.2[1]
Let \( X \) be a group and \( I^{X} \) denote set of all fuzzy sets in \( X \). Then the mapping \( f:A \rightarrow I^{X} \), denoted as \( (f,A) \) is said to be a fuzzy soft group over \( X \) iff for each \( a \in A \) and \( x,y \in X \)

1) \( f_{a}(xy) \geq \min(f_{a}(x), f_{a}(y)) \)

2) \( f_{a}(x^{-1}) = f_{a}(x) \)

An equivalent condition for the above definition is given in [1] ie. For each \( a \in A \) and \( x,y \in X \), \( f_{a}(xy^{-1}) \geq \min(f_{a}(x), f_{a}(y)) \)

Every Fuzzy soft group \( (f,A) \) over \( X \) satisfies the following inequality

\( f_{a}(e) \geq f_{a}(x) \)

Consider the following example

Example 2.3
Let \( N \), the set of natural numbers be the parameter set. We have \( Z \), the set of integers is a group under addition. For each \( a \in N \) and \( x \in z \) define
Product of Fuzzy Soft Groups

\[
f_a(x) = \begin{cases} 
\sqrt{(1 - 2^{-k})}, & \text{if } k \text{ is a positive integer and } x \in a^kZ \\
0, & \text{otherwise}
\end{cases}
\]

\[f_a(0) = 1, \forall a \in N.\] \(f_a\) measures the membership grade of \(x\) by the degree to which \(x\) is divisible by \(a\).

We can prove that above fuzzy soft set \((f,N)\) is a fuzzy soft group over \(Z\).

Let \(x_1, x_2 \in Z\) be such that \(x_1 = a^kZ\) and \(x_2 = a^lZ\) then \(x_1 + x_2 \in a^{k+l}Z\)

\[
f_a(x_1 + x_2) \geq 1 - 2^{-k(k+l)} \geq 1 - 2^{-k} \land 1 - 2^{-l} \geq f_a(x_1) \land f_a(x_2)
\]

Since \(a^kZ\) is a group for all positive integers \(k\) and for all \(a\) in \(N\), it follows that \(x \in a^kZ\) iff \(x^{-1} \in a^kZ\)

**Theorem 2.4**

Let \((f,A)\) be a fuzzy soft group over \(X\) and \(H\) be the subgroup of \(X\). Then \((f,A)/H\) is a fuzzy soft group over \(H\)

**Theorem 2.5**

If \((f,A)\) is a fuzzy soft group over \(X\) and \(f^*(x) = \sup_{a \in A} f_a(x)\), then \(f^*\) is a fuzzy subgroup over \(X\).

**Definition 2.6**[1]

Let \((f, A)\) be a fuzzy soft group over \(X\) and \(\lambda \in (0,1]\). Then

(i) \((f, A)\) is said to be a \(\lambda\) - identity fuzzy soft group over \(X\) if

\[
f_a(x) = \begin{cases} 
\lambda, & \text{if } x = e \\
0, & \text{otherwise}
\end{cases}
\]

\(\forall x \in X, \forall a \in A\)

(ii) \((f, A)\) is said to be a \(\lambda\) - absolute fuzzy soft group over \(X\) if \(f_a(x) = \lambda, \forall x \in X, \forall a \in A\)

**Definition 2.7**[1]

The fuzzy soft group \((f, A)\) over \(X\) is called an abelian fuzzy soft group if for each \(a \in A, f_a(xy) = f_a(xy), \forall x,y \in X\)

Note that it is possible to construct an abelian fuzzy soft group over a non abelian group.
Let $X$ be the symmetric group on three letters. $X$ is a non abelian group under composition of permutations. $\lambda$ - absolute fuzzy soft group over $X$ is an abelian fuzzy soft group over $X$.

**Definition 2.8**

Let $X$ be a group, $(f, A)$ and $(g, A)$ are two fuzzy soft groups over $X$ then $(f, A)$ is said to be conjugate to $(g, A)$ if there exist $x \in X$ such that for all $y \in X$, $f_a(y) = g_a(xyx^{-1})$

**Note 2.9**

$f^*$ of an abelian fuzzy soft group($\lambda$ - identity fuzzy soft group, $\lambda$ - absolute fuzzy soft group) is an abelian fuzzy subgroup ($\lambda$ - identity fuzzy group, $\lambda$ - absolute fuzzy group) over the given group. Let $(f, A)$ and $(g, A)$ are two fuzzy soft groups over $X$ such that $(f, A)$ is conjugate to $(g, A)$ then the fuzzy subgroup $f^*$ over $X$ is conjugate to the fuzzy subgroup $g^*$ over $X$.

### 3. PRODUCT OF FUZZY SOFT GROUPS

**Definition 3.1**

If $(f, A)$ and $(g, B)$ are two fuzzy soft sets over $X$, then their internal product is denoted as $(f \times g, A \times B)$ where $(f \times g)_{(a,b)}(x) = f_a(x) \land g_b(x), \forall (a,b) \in A \times B$

**Definition 3.2**

Let $(f, A)$ be a fuzzy soft set over $X$. The soft set $(f, A)_\alpha = \{(f_a)_\alpha : a \in A\}$ for each $\alpha \in (0,1)$ is called $\alpha$ level soft set of the fuzzy soft set $(f, A)$ where $(f_a)_\alpha$ is an $\alpha$ level set of the fuzzy set $(f_a)$

**Definition 3.3**

If $(f, A)$ and $(g, B)$ are two fuzzy soft sets over $X_1 and X_2$ respectively, then their external product $(f \times g, A \times B)$ is defined as $(f \times g)_{(a,b)}(x_1, x_2) = f_a(x_1) \land g_b(x_2), \forall (x_1, x_2) \in X_1 \times X_2$ and $\forall (a,b) \in A \times B$

We can generalize the above definition to $n$ number of fuzzy soft groups as follows.

For each $i=1,2,3, \ldots$, let $(f_i, A_i)$ be fuzzy soft sets over $X_i$. Define a map $(f_1 \times f_2 \ldots \times f_n)_{(a_1,a_2,\ldots,a_n)} : X_1 \times X_2 \times \ldots \times X_n \to [0,1]$ by
Product of Fuzzy Soft Groups

\[(f_1 \times f_2 \times \ldots \times f_n)(a_1, a_2, \ldots, a_n) = (x_1, x_2, \ldots, x_n)\]

\[= \min \left\{ (f_1)_{a_1}(x), (f_2)_{a_2}(x), \ldots, (f_n)_{a_n}(x) \right\}\]

This mapping is called generalized product of fuzzy soft sets and is denoted as \(\prod_{i=1}^{n} (f_i, A_i)\).

**Theorem 3.4**

Let \((f_i, A_i)\) be fuzzy soft sets over \(X_i, i=1,2,3,\ldots\), and \(\alpha \in (0,1]\). Then

\[\prod_{i=1}^{n} (f_i, A_i)_{\alpha} = \prod_{i=1}^{n} (f_i, A_i)_{\alpha}\]

**Proof:**

\[\prod_{i=1}^{n} (f_i, A_i)_{\alpha}\]

is the soft set \(\{(f_1 \times f_2 \times \ldots \times f_n)(a_1, a_2, \ldots, a_n) : (a_1, a_2, \ldots, a_n) \in A_1 \times A_2 \times \ldots \times A_n\}\)

For a fixed \((a_1, a_2, \ldots, a_n) \in A_1 \times A_2 \times \ldots \times A_n\) let \((x_{j1}, x_{j2}, \ldots, x_{jn}) \in \prod_{i=1}^{n} (f_i, A_i)_{\alpha}\)

where \(j \in I\), an indexed set

\[\Leftrightarrow (f_1 \times f_2 \times \ldots \times f_n)(a_1, a_2, \ldots, a_n)(x_{j1}, x_{j2}, \ldots, x_{jn}) \geq \alpha, \text{ for } \forall j \in I\]

\[\Leftrightarrow \min \{ (f_1)_{a_1}(x_{j1}), (f_2)_{a_2}(x_{j2}), \ldots, (f_n)_{a_n}(x_{jn}) \} \geq \alpha, \text{ for } \forall j \in I\]

\[\Leftrightarrow (f_1)_{a_1}(x_{j1}) \geq \alpha, (f_2)_{a_2}(x_{j2}) \geq \alpha, \ldots, (f_n)_{a_n}(x_{jn}) \geq \alpha, \text{ for } \forall j \in I\]

\[\Leftrightarrow x_{ji} \in (f_i, A_i)_{\alpha}, \text{ for } \forall j \in I \text{ and } i=1,2,\ldots,n\]

\[\therefore \quad \prod_{i=1}^{n} (f_i, A_i)_{\alpha} = \prod_{i=1}^{n} (f_i, A_i)_{\alpha}\]

**Theorem 3.5**

If \((f_i, A_i)\) is the fuzzy soft group over \(X_i, i=1,2,3,\ldots\) then \(\prod_{i=1}^{n} (f_i, A_i)\) is a fuzzy soft group over \(\prod_{i=1}^{n} X_i\).
Proof
Let \((x_1, x_2, \ldots, x_n)\) and \((y_1, y_2, \ldots, y_n)\) \(\in X_1 \times X_2 \times \ldots \times X_n\)

\[
(f_1 \times f_2 \times \ldots \times f_n)_{(i_1, i_2, \ldots, i_n)}((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n))^{-1}
\]

\[
= (f_1 \times f_2 \times \ldots \times f_n)_{(i_1, i_2, \ldots, i_n)}((x_1 y_1^{-1}, x_2 y_2^{-1}, \ldots, x_n y_n^{-1}))
\]

\[
= \min \left\{ (f_{i_1})_{a_i} (x_1 y_1^{-1}), (f_{i_2})_{a_i} (x_2 y_2^{-1}), \ldots, (f_{i_n})_{a_i} (x_n y_n^{-1}) \right\}
\]

\[
\geq \min \left\{ \min\left\{ (f_{i_1})_{a_i} (x_1), (f_{i_2})_{a_i} (x_2), \ldots, (f_{i_n})_{a_i} (x_n) \right\}, \min\left\{ (f_{i_1})_{a_i} (y_1), (f_{i_2})_{a_i} (y_2), \ldots, (f_{i_n})_{a_i} (y_n) \right\} \right\}
\]

\[
\geq \min\left\{ \min\left\{ (f_{i_1})_{a_i} (x_1), (f_{i_2})_{a_i} (x_2), \ldots, (f_{i_n})_{a_i} (x_n) \right\}, \min\left\{ (f_{i_1})_{a_i} (y_1), (f_{i_2})_{a_i} (y_2), \ldots, (f_{i_n})_{a_i} (y_n) \right\} \right\}
\]

\[
\geq \min\left\{ (f_{i_1})_{a_i} (x_1), (f_{i_2})_{a_i} (x_2), \ldots, (f_{i_n})_{a_i} (x_n) \right\}, \min\left\{ (f_{i_1})_{a_i} (y_1), (f_{i_2})_{a_i} (y_2), \ldots, (f_{i_n})_{a_i} (y_n) \right\}
\]

Using definition 2.2 we have \(\prod_{i=1}^n (f_i, A_i)\) is a fuzzy soft group over \(X_1 \times X_2 \times \ldots \times X_n\).

Theorem 3.6
Generalized external product of abelian fuzzy soft group is an abelian fuzzy soft group over \(\prod_{i=1}^n X_i\).

Proof
By theorem 3.5, if \((f_i, A_i)\) is the fuzzy soft group over \(X_i, i=1,2,3,\ldots\), then

\(\prod_{i=1}^n (f_i, A_i)\) is a fuzzy soft group over \(\prod_{i=1}^n X_i\).

If \((f_i, A_i)\) is an abelian fuzzy soft group over \(X_i, i=1,2,3,\ldots\), then

\[(f_i)_{a_i} (x_i y_i) = (f_i)_{a_i} (y_i x_i), \forall a_i \in A_i, i=1,2,\ldots\]

\[(f_1 \times f_2 \times \ldots \times f_n)_{(i_1, i_2, \ldots, i_n)}((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n))
\]

\[
= (f_1 \times f_2 \times \ldots \times f_n)_{(i_1, i_2, \ldots, i_n)}((x_1 y_1^{-1}, x_2 y_2^{-1}, \ldots, x_n y_n^{-1}))
\]

\[
= \min \left\{ (f_{i_1})_{a_i} (x_1 y_1^{-1}), (f_{i_2})_{a_i} (x_2 y_2^{-1}), \ldots, (f_{i_n})_{a_i} (x_n y_n^{-1}) \right\}
\]

\[
= \min \left\{ \min\left\{ (f_{i_1})_{a_i} (x_1), (f_{i_2})_{a_i} (x_2), \ldots, (f_{i_n})_{a_i} (x_n) \right\}, \min\left\{ (f_{i_1})_{a_i} (y_1), (f_{i_2})_{a_i} (y_2), \ldots, (f_{i_n})_{a_i} (y_n) \right\} \right\}
\]

\[
= \min\left\{ \min\left\{ (f_{i_1})_{a_i} (x_1), (f_{i_2})_{a_i} (x_2), \ldots, (f_{i_n})_{a_i} (x_n) \right\}, \min\left\{ (f_{i_1})_{a_i} (y_1), (f_{i_2})_{a_i} (y_2), \ldots, (f_{i_n})_{a_i} (y_n) \right\} \right\}
\]

\[
= \min\left\{ (f_{i_1})_{a_i} (x_1), (f_{i_2})_{a_i} (x_2), \ldots, (f_{i_n})_{a_i} (x_n) \right\}, \min\left\{ (f_{i_1})_{a_i} (y_1), (f_{i_2})_{a_i} (y_2), \ldots, (f_{i_n})_{a_i} (y_n) \right\}
\]

Hence \(\prod_{i=1}^n (f_i, A_i)\) is an abelian fuzzy soft group over \(X_1 \times X_2 \times \ldots \times X_n\).
**Product of Fuzzy Soft Groups**

**Theorem 3.7**
If the fuzzy soft group \((f_i, A_i)\) over \(X_i\) is conjugate to the fuzzy soft group \((g_i, A_i)\) over \(X_i\) for \(i = 1, 2, 3, \ldots\), then \(\prod_{i=1}^{n} (f_i, A_i)\) is conjugate to \(\prod_{i=1}^{n} (g_i, A_i)\).

**Proof**
The proof is obvious as the definition 2.8

**Theorem 3.8**
Generalized external product of \(\lambda\) identity fuzzy soft group (\(\lambda\) absolute fuzzy soft group) is a \(\lambda\) identity fuzzy soft group (\(\lambda\) absolute fuzzy soft group) over \(\prod_{i=1}^{n} X_i\).

**Proof**
Proof of the theorem follows from definition 2.6

**Theorem 3.9**
For each \(i = 1, 2, 3, \ldots\), let \((f_i, A_i)\) be fuzzy soft sets over the group \(X_i\) and \(e_i \in X_i\) be the identity element. If \(\prod_{i=1}^{n} (f_i, A_i)\) is a fuzzy soft group over \(\prod_{i=1}^{n} X_i\), then for at least one \(i = 1, 2, 3, \ldots\) \((f_i, A_i)\) is a fuzzy soft group over \(X_i\).

**Proof**
First to prove that there exist at least one \(i\) such that
\[
(f_1, x_1) \leq (f_1, f_2, \ldots, f_{i-1}, f_i, e_{i+1}, \ldots, f_n)(e_1, e_2, \ldots, e_{i-1}, 1, \ldots, e_n),
\]
for all \(a_i \in A_i\) then \((f_i, A_i)\) is a fuzzy soft group over \(X_i\).

On contrary, suppose that there doesn’t exist an \(i\) such that statement (3) holds. i.e., we can find \(x_i \in X_i\) such that
\[
(f_1, x_1) \leq (f_1, f_2, \ldots, f_{i-1}, f_i, e_{i+1}, \ldots, f_n)(e_1, e_2, \ldots, e_{i-1}, 1, \ldots, e_n) < (f_i, a)(x_i), \forall i = 1, 2, \ldots
\]
Then for particular \((x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \cdots \times X_n\),
\[(f_1 \times f_2 \ldots \times f_n)(a_1, a_2, \ldots, a_n)(x_1, x_2, \ldots, x_n)\]
\[= \min \{ (f_1)_a(x_1), (f_2)_a(x_2), \ldots, (f_n)_a(x_n) \} \]
\[> \min \{ (f_1 \times f_2 \times \ldots \times f_n)(a_1, a_2, \ldots, a_n)(e_1, e_2, \ldots, e_n) \}
\[\geq \min \{ \min \{ (f_1)_a(e_1), (f_2)_a(e_2), \ldots, (f_n)_a(e_n) \} \},
\[\geq \min \{(f_1)_a(x_1), (f_2)_a(x_2), \ldots, (f_n)_a(x_n)\}\]
\[= (f_1 \times f_2 \times \ldots \times f_n)(a_1, a_2, \ldots, a_n)(e_1, e_2, \ldots, e_n)\]

Hence by definition 2.2, \( \prod_{i=1}^{n}(f_i, A_i) \) is not a fuzzy soft group over \( \prod_{i=1}^{n}X_i \).

Hence statement (3) holds.

For a particular \( i \) which satisfies inequality (3),
\[(f_i)_a(x_i, y_i) = \min \{ (f_i)_a(x_i), (f_i \times f_2 \times \ldots \times f_n)(a_1, a_2, \ldots, a_n)(e_1, e_2, \ldots, e_n) \}
\[= (f_i \times f_2 \times \ldots \times f_n)(a_1, a_2, \ldots, a_n)(e_1, e_2, \ldots, e_n)\]
\[\geq \min \{ (f_i \times f_2 \times \ldots \times f_n)(a_1, a_2, \ldots, a_n)(e_1, e_2, \ldots, e_n) \}
\[\geq \min \{ \min \{ (f_i)_a(x_i, y_i), (f_i \times f_2 \times \ldots \times f_n)(a_1, a_2, \ldots, a_n)(e_1, e_2, \ldots, e_n) \} \},
\[\geq \min \{(f_i)_a(x_i), (f_i)_a(y_i)\}\]

Similarly we can prove that \( f_i)_a(x_i)^{-1} = f_i)_a(x_i)\)

Therefore \( f_i, A_i \) is a fuzzy soft group over \( X_i \)

**Theorem 3.10**

Let \( f_i, A_i \) is a fuzzy soft group over \( X_i \) with \( (f_i)_a(e_i) \geq \alpha, \forall a_i \epsilon A_i \)

where \( i=1, 2, 3 \ldots \), and \( \alpha \epsilon [0,1] \) Then \( \alpha \) level subset of \( \prod_{i=1}^{n}(f_i, A_i) \) is a soft group over \( \prod_{i=1}^{n}X_i \).

**Proof**

Let \( H(a_1, a_2, \ldots, a_n) \) be the \( \alpha \) level subset of \( \prod_{i=1}^{n}(f_i, A_i) \) corresponding to the parameter \( (a_1, a_2, \ldots, a_n) \).
Product of Fuzzy Soft Groups

\[ H_{(a_1, a_2, \ldots, a_n)} = \{(x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \cdots \times X_n : \min \{f_{i_1}(x_{j_1}), f_{i_2}(x_{j_2}), \ldots, f_{i_n}(x_{j_n})\} \geq \alpha \} \]

Since \( f_{i_k}(a_i) \geq \alpha, \forall i \in A_i, i = 1, 2, \ldots \)

We get \((e_1, e_2, \ldots, e_n) \in H_{(a_1, a_2, \ldots, a_n)}\)

Let \(x = (x_1, x_2, \ldots, x_n)\) and \(y = (y_1, y_2, \ldots, y_n) \in H_{(a_1, a_2, \ldots, a_n)}\)

Then we have
\[ \min \{f_{i_1}(x_{j_1}), f_{i_2}(x_{j_2}), \ldots, f_{i_n}(x_{j_n})\} \geq \alpha \]

\[ \min \{f_{i_1}(y_{j_1}), f_{i_2}(y_{j_2}), \ldots, f_{i_n}(y_{j_n})\} \geq \alpha \]

\[ \min \{f_{i_1}(x_{j_1}), f_{i_2}(x_{j_2}), \ldots, f_{i_n}(x_{j_n})\} \wedge \min \{f_{i_1}(y_{j_1}), f_{i_2}(y_{j_2}), \ldots, f_{i_n}(y_{j_n})\} \geq \alpha \]

\[ \min \{f_{i_1}(x_{j_1}), f_{i_2}(x_{j_2}), \ldots, f_{i_n}(x_{j_n})\} \wedge \min \{f_{i_1}(y_{j_1}), f_{i_2}(y_{j_2}), \ldots, f_{i_n}(y_{j_n})\} \geq \alpha \]

Hence we have \(x y^{-1} \in H_{(a_1, a_2, \ldots, a_n)}\)

\(H_{(a_1, a_2, \ldots, a_n)}\) is a subgroup of \(X_1 \times X_2 \times \cdots \times X_n\)

i.e. \(\alpha\)-level subset of \(\prod_{i=1}^{n}(f_i, A_i)\) is a soft group over \(X_1 \times X_2 \times \cdots \times X_n\).

4. CONCLUSION

We summarized the basic concepts of fuzzy soft groups and then presented a detailed theoretical study of generalized external product of a fuzzy soft sets and fuzzy soft groups. This work can be extended to the properties of different notions of product of fuzzy soft groups.

REFERENCES

