Totally Regular Property of Cartesian product of Intuitionistic Fuzzy Graphs

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Abstract

In this paper, the Cartesian product of totally regular intuitionistic fuzzy graphs need not be a totally regular intuitionistic fuzzy graph is discussed. Also the conditions for the Cartesian product of totally regular intuitionistic fuzzy graphs to be totally regular under some restrictions are obtained.

Keywords: Total degree of a vertex, Cartesian product, Regular IFG, Totally regular IFG.

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1. INTRODUCTION

2. PRELIMINARIES

**Definition 2.1:** Let $G = (V, E)$ be an IFG. Then the degree of a vertex $v$ is defined by 
\[ d(v) = (d_\mu(v), d_\nu(v)), \]
where 
\[ d_\mu(v) = \sum_{u \neq v} \mu_2(u,v) \]
\[ \text{and} \]
\[ d_\nu(v) = \sum_{u \neq v} \nu_2(u,v). \]

**Definition 2.2:** Let $G = (V, E)$ be an IFG. If $(d_\mu(v), d_\nu(v)) = (k_1, k_2)$ for all $v \in V$ that is if each vertex has same membership degree $k_1$ and same nonmembership degree $k_2$ then $G$ is said to be a regular intuitionistic fuzzy graph.

**Definition 2.3:** Let $G = (V, E)$ be an IFG. Then the total degree of a vertex $u \in v$ is defined by 
\[ td(u) = (td_\mu(u), td_\nu(u)) = (\sum_{u \neq v} \mu_2(u,v) + \mu_1(u), \sum_{u \neq v} \nu_2(u,v) + \nu_1(u)) \]
if each vertex of $G$ has same membership total degree $k_1$ and same non membership total degree $k_2$, then said to be a total regular IFG.

3. TOTALLY REGULAR PROPERTY OF THE CARTESIAN PRODUCT

**Definition 3.1:** The Cartesian product of two intuitionistic fuzzy graphs $G_1$ and $G_2$ is defined as an intuitionistic fuzzy graph $G = G_1 \times G_2; (\mu \times \mu', v \times v')$ on $G^* = (V, E)$ where $V = V_1 \times V_2$ and $E = \{((u_1, u_2)(v_1, v_2) / u_1 = v_1, u_2 v_2 \in E_2 or u_2 = v_2, u_1 v_1 \in E_1\}$

\[(\mu_1 \times \mu'_1)((u_1, u_2)(u_2, v_2)) = \mu_1(u_1) \land \mu'_1(u_2), \text{for all } (u_1, u_2) \in V_1 \times V_2 \]
\[(v_1 \times v'_1)((u_1, u_2)(v_1, v_2)) = v_1(u_1) \lor v'_1(u_2), \text{for all } (u_1, u_2) \in V_1 \times V_2 \]
\[(\mu_2 \times \mu'_2)((u_1, u_2)(v_1, v_2)) = \begin{cases} \mu_1(u_1) \land \mu'_2(u_2v_2), & \text{if } u_1 = v_1, u_2v_2 \in E_2 \\ \mu'_1(u_2) \land \mu_2(u_1v_1), & \text{if } u_2 = v_2, u_1v_1 \in E_1 \end{cases} \]
\[(v_2 \times v'_2)((u_1, u_2)(v_1, v_2)) = \begin{cases} v_1(u_1) \lor v'_2(u_2v_2), & \text{if } u_1 = v_1, u_2v_2 \in E_2 \\ v'_1(u_2) \lor v_2(u_1v_1), & \text{if } u_2 = v_2, u_1v_1 \in E_1 \end{cases} \]

**Theorem 3.1:** Let $G : (\mu, \nu)$ be an intuitionistic fuzzy graph such that both $\mu, \nu$ are same constant functions. Then $G$ is a totally regular intuitionistic fuzzy graph if and only if $G$ is a partially regular intuitionistic fuzzy graph.

**Proof:** Assume that $G$ is a $k$-totally regular intuitionistic fuzzy graph.
Let \( \mu_2(u, v) = c_1 \) for all \( uv \in V \) and \( \mu_1(u) = c_2 \) for all \( u \in V \), where \( c_1 \) and \( c_2 \) are constants. Then \( td_{\mu(G)}(u) = d_{\mu(G)}(u) + \mu_1(u) = \sum_{u \neq v} \mu_2(u, v) + \mu_1(u) \)
\[
\Rightarrow k = c_1 d_G^*(u) + c_2
\]
\[
\Rightarrow d_G^*(u) = \frac{k-c_2}{c_1} \text{ for all } u \in V.
\]
Let \( v_2(u, v) = c_3 \) for all \( uv \in V \) and \( v_1(u) = c_4 \) for all \( u \in V \), where \( c_3 \) and \( c_4 \) are constants. Then \( td_{v(G)}(u) = d_{v(G)}(u) + v_1(u) = \sum_{u \neq v} v_2(u, v) + v_1(u) \)
\[
\Rightarrow k = c_3 d_G^*(u) + c_4
\]
\[
\Rightarrow d_G^*(u) = \frac{k-c_4}{c_3} \text{ for all } u \in V.
\]
So, \( G^* \) is regular and hence \( G \) is a partially regular intuitionistic fuzzy graph.

Conversely, assume that \( G \) is a partially regular intuitionistic fuzzy graph.

Let \( G^* \) be a \( r \)-regular intuitionistic fuzzy graph.

Then \( td_{\mu(G)}(u) = d_{\mu(G)}(u) + \mu_1(u), \quad td_{v(G)}(u) = d_{v(G)}(u) + v_1(u) \).
\[
\Rightarrow td_{\mu(G)}(u) = c_1 d_G^*(u) + c_2 = c_1 r + c_2 \text{ for all } u \in V
\]
\[
\Rightarrow td_{v(G)}(u) = c_3 d_G^*(u) + c_4 = c_1 r + c_2 \text{ for all } u \in V
\]
So, \( G \) is totally regular intuitionistic fuzzy graph.

**Theorem 3.2:** Let \( G_1: (\mu_1, \mu_2) \) and \( G_2: (\mu_1', \mu_2') \) be two intuitionistic fuzzy graphs such that \( \mu_1 \leq \mu_1' \) and \( v_1 \leq v_1' \). Then \( \mu_1 \leq \mu_1' \) and \( v_1 \leq v_1' \).

**Proof:** Since by the definition of an intuitionistic fuzzy graph
\[
\mu_2'(uv) \leq \mu_1'(u) \wedge \mu_1'(v) \text{ for all } u, v \in V_2, \text{ we have } \min \mu_2' \leq \mu_1'.
\]
Now \( \mu_1 \leq \mu_1' \Rightarrow \mu_1 \leq \min \mu_2' \Rightarrow \mu_1 \leq \min \mu_2' \leq \mu_1' \Rightarrow \mu_1 \leq \mu_1'.
\]
\[
v_2'(uv) \leq v_1'(u) \vee v_1'(v) \text{ for all } u, v \in V_2, \text{ we have } \max v_2' \leq v_1'.
\]
Now \( v_1 \leq v_2' \Rightarrow v_1 \leq \max v_2' \Rightarrow v_1 \leq \max v_2' \leq v_1' \Rightarrow v_1 \leq v_1'.
\]

**Example 3.3:** Consider the graphs \( G_1, G_2 \) are totally regular IFGs, but \( G_1 \times G_2 \) is not a totally regular IFG.
Example 3.4: In the following figures $G_1$ is a totally regular IFG but $G_2$ is not a totally regular IFG and $G_1 \times G_2$ is a totally regular IFG.
Remark 3.5: The above examples show that the Cartesian product of totally regular intuitionistic fuzzy graphs need not be a totally regular intuitionistic fuzzy graph and if $G_1 \times G_2$ is totally regular intuitionistic fuzzy graph, both $G_1$ and $G_2$ need not be totally regular intuitionistic fuzzy graphs.

Theorem 3.6: Let $G_1: (\mu_1, \mu_2)$ and $G_2: (\mu_1', \mu_2')$ be two intuitionistic fuzzy graphs. If $\mu_1 \geq \mu_2'$, $\mu_1' \geq \mu_2$ and $\mu_1 \land \mu_1'$ is a constant function, $v_1 \geq v_2'$, $v_1' \geq v_2$ and $v_1 \lor v_1'$ is a constant function then $G_1 \times G_2$ is a totally regular intuitionistic fuzzy graph if and only if $G_1$ and $G_2$ are regular intuitionistic fuzzy graphs.

Proof: Let $\mu_1(u) \land \mu_1'(v) = c_1$ and $v_1(u) \lor v_1'(v) = c_2$, for all $u \in V_1, v \in V_2$, where $c_1, c_2$ are constants. Suppose that $G_1$ and $G_2$ are regular intuitionistic fuzzy graphs of degree $k_1$ and $k_2$ respectively. By definition, for any $(u_1, u_2) \in V_1 \times V_2$. The total degree of a vertex in Cartesian product is

$$td_{\mu(G_1 \times G_2)}(u_1, u_2) = \sum_{(u_1, v_1)(u_2, v_2) \in E} (\mu_2 \times \mu_2')(u_1, u_2)(v_1, v_2) + \mu_1(u_1) \land \mu_1'(u_2)$$

$$\Rightarrow td_{\mu(G_1 \times G_2)}(u_1, u_2) = \sum_{u_1v_1 \in E_1} \mu_1(u_1) \land \mu_2'(u_2) + \sum_{u_2v_2 \in E_2} \mu_2(u_1, u_2) + \mu_1(u_1) \land \mu_1'(u_2)$$

$$= \sum_{u_2v_2 \in E_2} \mu_2'(u_2v_2) + \sum_{u_1v_1 \in E_1} \mu_2(u_1, u_1v_1) + \mu_1(u_1) \land \mu_1'(u_2)$$

$$\Rightarrow td_{\mu(G_1 \times G_2)}(u_1, u_2) = d_{\mu(G_1)}(u_1) + d_{\mu(G_2)}(u_2) + \mu_1(u_1) \land \mu_1'(u_2) \quad \text{-------- (3.1)}$$

$$\Rightarrow td_{\mu(G_1 \times G_2)}(u_1, u_2) = k_1 + k_2 + c_1$$

$$td_{\nu(G_1 \times G_2)}(u_1, u_2) = \sum_{(u_1, v_1)(u_2, v_2) \in E} (v_2 \times v_2')(u_1, u_2)(v_1, v_2) + v_1(u_1) \lor v_1'(u_2)$$

$$\Rightarrow td_{\nu(G_1 \times G_2)}(u_1, u_2) = \sum_{u_1v_1 \in E_1} v_1(u_1) \lor v_2'(u_2) v_2(u_1v_1) + v_1(u_1) \lor v_1'(u_2) v_2(u_1v_1)$$

$$= \sum_{u_2v_2 \in E_2} v_2'(u_2v_2) + \sum_{u_1v_1 \in E_1} v_2(u_1v_1) + v_1(u_1) \lor v_1'(u_2)$$

$$\Rightarrow td_{\nu(G_1 \times G_2)}(u_1, u_2) = d_{\nu(G_1)}(u_1) + d_{\nu(G_2)}(u_2) + v_1(u_1) \lor v_1'(u_2) \quad \text{-------- (3.2)}$$

$$\Rightarrow td_{\nu(G_1 \times G_2)}(u_1, u_2) = k_1 + k_2 + c_2$$
Hence $G_1 \times G_2$ is a totally regular intuitionistic fuzzy graph. Conversely assume that $G_1 \times G_2$ is a totally regular intuitionistic fuzzy graph then for any two points $(u_1, u_2)$ and $(v_1, v_2)$ in $V_1 \times V_2$.

$$td_{\mu(G_1 \times G_2)}(u_1, u_2) = td_{\nu(G_1 \times G_2)}(v_1, v_2)$$

From (3.1),

$$d_{\mu(G_1)}(u_1) + d_{\mu(G_2)}(u_2) + \mu_1(u_1) \land \mu'_1(u_2) = d_{\mu(G_1)}(v_1) + d_{\mu(G_2)}(v_2) + \mu_1(v_1) \land \mu'_1(v_2)$$

$$d_{\mu(G_1)}(u_1) + d_{\mu(G_2)}(u_2) + c_1 = d_{\mu(G_1)}(v_1) + d_{\mu(G_2)}(v_2) + c_1$$

$$\Rightarrow d_{\mu(G_1)}(u_1) + d_{\mu(G_2)}(u_2) = d_{\mu(G_1)}(v_1) + d_{\mu(G_2)}(v_2) \quad (3.3)$$

$$td_{\nu(G_1 \times G_2)}(u_1, u_2) = td_{\nu(G_1 \times G_2)}(v_1, v_2)$$

From (3.2),

$$d_{\nu(G_1)}(u_1) + d_{\nu(G_2)}(u_2) + \nu_1(u_1) \lor \nu'_1(u_2) = d_{\nu(G_1)}(v_1) + d_{\nu(G_2)}(v_2) + \nu_1(v_1) \lor \nu'_1(v_2)$$

$$d_{\nu(G_1)}(u_1) + d_{\nu(G_2)}(u_2) + c_2 = d_{\nu(G_1)}(v_1) + d_{\nu(G_2)}(v_2) + c_2$$

$$\Rightarrow d_{\nu(G_1)}(u_1) + d_{\nu(G_2)}(u_2) = d_{\nu(G_1)}(v_1) + d_{\nu(G_2)}(v_2) \quad (3.4)$$

Fix $u \in V_1$ and consider $(u, u_2)$ and $(u, v_2)$ in $V_1 \times V_2$, where $u_2, v_2 \in V_2$ are arbitrary.

From (3.3) $d_{\mu(G_1)}(u) + d_{\mu(G_2)}(u_2) = d_{\mu(G_1)}(u) + d_{\mu(G_2)}(v_2) \Rightarrow d_{\mu(G_2)}(u_2) = d_{\mu(G_2)}(v_2)$.

From (3.4) $d_{\nu(G_1)}(u) + d_{\nu(G_2)}(u_2) = d_{\nu(G_1)}(u) + d_{\nu(G_2)}(v_2) \Rightarrow d_{\nu(G_2)}(u_2) = d_{\nu(G_2)}(v_2)$.

This is true for all $u_2, v_2 \in V_2$. Thus $G_2$ is regular intuitionistic fuzzy graph.

Fix $v \in V_2$ and consider $(u_1, v)$ and $(v_1, v)$ in $V_1 \times V_2$, where $u_1, v_1 \in V_1$ are arbitrary.

From (3.3) $d_{\mu(G_1)}(u_1) + d_{\mu(G_2)}(v) = d_{\mu(G_1)}(v_1) + d_{\mu(G_2)}(v) \Rightarrow d_{\nu(G_1)}(u_1) = d_{\nu(G_1)}(v_1)$.

From (3.4) $d_{\nu(G_1)}(u_1) + d_{\nu(G_2)}(v) = d_{\nu(G_1)}(v_1) + d_{\nu(G_2)}(v) \Rightarrow d_{\nu(G_1)}(u_1) = d_{\nu(G_1)}(v_1)$.

This is true for all $u_1, v_1 \in V_1$. Thus $G_1$ is a regular intuitionistic fuzzy graph.

**Theorem 3.7:** Let $G_1: (\mu_1, \mu_2)$ and $G_2: (\mu'_1, \mu'_2)$ be two intuitionistic fuzzy graphs. If $\mu_1 \geq \mu'_2, \mu'_1 \geq \mu_2$ and $\mu_1 \lor \mu'_1$ is a constant function, $v_1 \geq v'_2, v'_1 \geq v_2$ and $v_1 \land v'_1$
is a constant function then $G_1 \times G_2$ is totally regular intuitionistic fuzzy graph if and only if $G_1$ and $G_2$ are totally regular intuitionistic fuzzy graphs.

**Proof:** Let $\mu_1(u) \lor \mu_1'(v) = c_1$, $v_1(u) \land v_1'(v) = c_2$, for all $u \in V_1, v \in V_2$, where $c_1, c_2$ are constants. Suppose that $G_1$ and $G_2$ are totally regular intuitionistic fuzzy graphs of degree $k_1$ and $k_2$ respectively. By definition, for any $(u_1, u_2) \in V_1 \times V_2$. The total degree of a vertex in Cartesian product is

$$td_{\mu(G_1 \times G_2)}(u_1, u_2) = \sum_{u_1=v_1, u_2=v_2} \mu_1(u_1)\land \mu_2'(u_2) + \sum_{u_1=v_1, u_2=v_2} \mu_2(u_1)\land \mu_1'(u_2) - \mu_1(u_1)\lor \mu_1'(u_2)$$

$$= \sum_{u_2v_2 \in E_2} \mu_2'(u_2v_2) + \sum_{u_1v_1 \in E_1} \mu_2(u_1v_1) + \mu_1(u_1)\lor \mu_1'(u_2) - \mu_1(u_1)\lor \mu_1'(u_2)$$

$$= \sum_{u_2v_2 \in E_2} \mu_2'(u_2v_2) + \sum_{u_1v_1 \in E_1} \mu_2(u_1v_1) + \mu_1(u_1)\lor \mu_1'(u_2)$$

$$\Rightarrow td_{\mu(G_1 \times G_2)}(u_1, u_2)$$

$$= d_{\mu(G_1)}(u_1) + \mu_1(u_1) + d_{\mu(G_2)}(u_2) + \mu_1'(u_2) - \mu_1(u_1)\lor \mu_1'(u_2)$$

$$= td_{\mu(G_1 \times G_2)}(u_1, u_2) = td_{\mu(G_1)}(u_1) + td_{\mu(G_2)}(u_2) - \mu_1(u_1)\lor \mu_1'(u_2)$$

$$\Rightarrow td_{\mu(G_1 \times G_2)}(u_1, u_2) = k_1 + k_2 - c_1$$

$$= \sum_{u_1v_1 \in E_1} v_1(u_1)\lor v_1'(u_2) + \sum_{u_1v_1 \in E_1} v_1(u_1)\lor v_1'(u_2)$$

$$\Rightarrow td_{\mu(G_1 \times G_2)}(u_1, u_2)$$

$$= \sum_{u_2v_2 \in E_2} v_2(u_2v_2) + \sum_{u_1v_1 \in E_1} v_1(u_1)\lor v_1'(u_2)$$

$$= \sum_{u_2v_2 \in E_2} v_2(u_2v_2) + \sum_{u_1v_1 \in E_1} v_2(u_1v_1) + \sum_{u_1v_1 \in E_1} v_2(u_1v_1) + \sum_{u_1v_1 \in E_1} v_2(u_1v_1)$$

$$\Rightarrow td_{\mu(G_1 \times G_2)}(u_1, u_2)$$

$$= \sum_{u_2v_2 \in E_2} v_2(u_2v_2) + \sum_{u_1v_1 \in E_1} v_1(u_1)\lor v_1'(u_2)$$

$$= d_{\mu(G_1)}(u_1) + v_1(u_1)\lor d_{\mu(G_2)}(u_2) + v_1'(u_2) - v_1(u_1)\lor v_1'(u_2)$$

$$\Rightarrow td_{v(G_1 \times G_2)}(u_1, u_2) = td_{v(G_1)}(u_1) + td_{v(G_2)}(u_2) - v_1(u_1)\lor v_1'(u_2)$$

$$\Rightarrow td_{v(G_1 \times G_2)}(u_1, u_2) = k_1 + k_2 - c_2$$
Hence $G_1 \times G_2$ is a totally regular intuitionistic fuzzy graph. Conversely, assume that $G_1 \times G_2$ is a totally regular intuitionistic fuzzy graph then for any two points $(u_1, u_2)$ and $(v_1, v_2)$ in $V_1 \times V_2$.

$$td_{\mu(G_1 \times G_2)}(u_1, u_2) = td_{\mu(G_1 \times G_2)}(v_1, v_2)$$

$$td_{\mu(G_1)}(u_1) + td_{\mu(G_2)}(u_2) - \mu_1(u_1) \lor \mu'_1(u_2) = td_{\mu(G_1)}(v_1) + td_{\mu(G_2)}(v_2) - \mu_1(v_1) \lor \mu'_1(v_2)$$

$$td_{\mu(G_1)}(u_1) + td_{\mu(G_2)}(u_2) - c_1 = td_{\mu(G_1)}(v_1) + td_{\mu(G_2)}(v_2) - c_1$$

$$\Rightarrow td_{\mu(G_1)}(u_1) + td_{\mu(G_2)}(u_2) = td_{\mu(G_1)}(v_1) + td_{\mu(G_2)}(v_2) \quad \text{(3.5)}$$

$$td_{\nu(G_1 \times G_2)}(u_1, u_2) = td_{\nu(G_1 \times G_2)}(v_1, v_2)$$

$$td_{\nu(G_1)}(u_1) + td_{\nu(G_2)}(u_2) - \nu_1(u_1) \land \nu'_1(u_2) = td_{\nu(G_1)}(v_1) + td_{\nu(G_2)}(v_2) - \nu_1(v_1) \land \nu'_1(v_2)$$

$$td_{\nu(G_1)}(u_1) + td_{\nu(G_2)}(u_2) - c_2 = td_{\nu(G_1)}(v_1) + td_{\nu(G_2)}(v_2) - c_2$$

$$\Rightarrow td_{\nu(G_1)}(u_1) + td_{\nu(G_2)}(u_2) = td_{\nu(G_1)}(v_1) + td_{\nu(G_2)}(v_2) \quad \text{(3.6)}$$

Fix $u \in V_1$ and consider $(u, u_2)$ and $(u, v_2)$ in $V_1 \times V_2$, where $u_2, v_2 \in V_2$ are arbitrary.

From (3.5) $$td_{\mu(G_1)}(u) + td_{\mu(G_2)}(u_2) = td_{\mu(G_1)}(v_1) + td_{\mu(G_2)}(v_2) \Rightarrow td_{\mu(G_2)}(u_2) = td_{\mu(G_2)}(v_2)$$

From (3.6) $$td_{\nu(G_1)}(u) + td_{\nu(G_2)}(u_2) = td_{\nu(G_1)}(v_1) + td_{\nu(G_2)}(v_2) \Rightarrow td_{\nu(G_2)}(u_2) = td_{\nu(G_2)}(v_2)$$

This is true for all $u_2, v_2 \in V_2$. Thus $G_2$ is a totally regular intuitionistic fuzzy graph.

Fix $v \in V_2$ and consider $(u_1, v)$ and $(v_1, v)$ in $V_1 \times V_2$, where $u_1, v_1 \in V_1$ are arbitrary.

From (3.5) $$td_{\mu(G_1)}(u_1) + td_{\mu(G_2)}(v) = td_{\mu(G_1)}(v_1) + td_{\mu(G_2)}(v) \Rightarrow td_{\mu(G_1)}(u_1) = td_{\mu(G_1)}(v_1)$$

From (3.6) $$td_{\nu(G_1)}(u_1) + td_{\nu(G_2)}(v) = td_{\nu(G_1)}(v_1) + td_{\nu(G_2)}(v) \Rightarrow td_{\nu(G_1)}(u_1) = td_{\nu(G_1)}(v_1)$$

This is true for all $u_1, v_1 \in V_1$. Thus $G_1$ is a totally regular intuitionistic fuzzy graph.

**Notation:** The relation $\mu_1 \leq \mu'_2$ means $\mu_1(u) \leq \mu'_2(e) \forall u \in V_1$ and $\forall e \in E_2$. Where $\mu_1$ is an intuitionistic fuzzy subset of $V_1$ and $\mu'_2$ is an intuitionistic fuzzy subset of $E_2$. Similarly for $\nu_1 \leq \nu'_2$. 
Theorem 3.8: Let $G_1: (\mu_1, \mu_2)$ and $G_2: (\mu_1', \mu_2')$ be two intuitionistic fuzzy graphs such that $\mu_1 \leq \mu'_2$ and $\mu_1$ is a constant function, $\nu_1 \leq \nu'_2$ and $\nu_1$ is a constant function then $G_1 \times G_2$ is totally regular intuitionistic fuzzy graph if and only if $G_1$ is a regular intuitionistic fuzzy graph and $G_2$ is a partially regular intuitionistic fuzzy graph.

Proof: We have $\mu_1 \leq \mu'_2$. Hence $\mu_1' \geq \mu_2$ and $\nu_1 \leq \nu'_2$. Hence $\nu_1' \geq \nu_2$. Let $\mu_1(u_1) = c_1$, $\nu_1(u_1) = c_2$, for all $u_1 \in V_1$, where $c_1, c_2$ are constants.

Suppose that $G_1$ is a regular intuitionistic fuzzy graph of degree $k$ and $G_2^*$ is a regular graph of degree $r_2$.

By definition, for any $(u_1, u_2) \in V_1 \times V_2$,
\[
\text{td}_{\mu(G_1 \times G_2)}(u_1, u_2) = \sum_{u_1 = v_1, u_2 v_2 \in E_2} \mu_1(u_1) \land \mu'_2(u_2 v_2) + \sum_{u_2 = v_2, u_1 v_1 \in E_1} \mu'_1(u_2) \land \mu_2(u_1 v_1) + \mu_1(u_1) \land \mu'_1(u_2) \leq \mu'_1 \]
\[
= \sum_{u_2 v_2 \in E_2} c_1 + \sum_{u_1 v_1 \in E_1} \mu_2(u_1 v_1) + \mu_1(u_1) \quad [\text{since } \mu_1 \leq \mu'_2, \mu'_1 \geq \mu_2 \text{ and } \mu_1] \]
\[
\text{td}_{\mu(G_1 \times G_2)}(u_1, u_2) = c_1 r_2 + k + c_1 = c_1 (r_2 + 1) + k \quad \text{------------------ (3.7)}
\]
\[
\text{td}_{\nu(G_1 \times G_2)}(u_1, u_2) = \sum_{u_1 = v_1, u_2 v_2 \in E_2} \nu_1(u_1) \lor \nu'_2(u_2 v_2) + \sum_{u_2 = v_2, u_1 v_1 \in E_1} \nu'_1(u_2) \lor \nu_2(u_1 v_1) + \nu_1(u_1) \lor \nu'_1(u_2) \leq \nu'_1 \]
\[
= \sum_{u_2 v_2 \in E_2} c_2 + \sum_{u_1 v_1 \in E_1} \nu_2(u_1 v_1) + \nu_1(u_1) \quad [\text{since } \nu_1 \leq \nu'_2, \nu'_1 \geq \nu_2 \text{ and } \nu_1] \]
\[
\text{td}_{\nu(G_1 \times G_2)}(u_1, u_2) = c_2 d_{G_2}^*(u_2) + d_{G_1}(u_1) + c_2 \quad \text{------------------ (3.8)}
\]
So, $G_1 \times G_2$ is a totally regular intuitionistic fuzzy graph, conversely, assume that
$G_1 \times G_2$ is a totally regular intuitionistic fuzzy graph then for any two points $(u_1, u_2)$
and $(v_1, v_2)$ in $V_1 \times V_2$.

\[ td_{\mu(G_1 \times G_2)}(u_1, u_2) = td_{\nu(G_1 \times G_2)}(v_1, v_2) \]

\[ c_1 d_{G_2}^*(u_2) + d_{\mu(G_1)}(u_1) + c_1 = c_1 d_{G_2}^*(v_2) + d_{\mu(G_1)}(v_1) + c_1 \]

\[ c_1 d_{G_2}^*(u_2) + d_{\mu(G_1)}(u_1) = c_1 d_{G_2}^*(v_2) + d_{\mu(G_1)}(v_1) \] \hspace{1cm} (3.9)

\[ td_{\nu(G_1 \times G_2)}(u_1, u_2) = td_{\nu(G_1 \times G_2)}(v_1, v_2) \]

\[ c_2 d_{G_2}^*(u_2) + d_{\nu(G_1)}(u_1) + c_2 = c_2 d_{G_2}^*(v_2) + d_{\nu(G_1)}(v_1) + c_2 \]

\[ c_2 d_{G_2}^*(u_2) + d_{\nu(G_1)}(u_1) = c_2 d_{G_2}^*(v_2) + d_{\nu(G_1)}(v_1) \] \hspace{1cm} (3.10)

Fix $u \in V_1$ and consider $(u, u_2)$ and $(u, v_2)$ in $V_1 \times V_2$, where $u_2, v_2 \in V_2$ are arbitrary.

From (3.9) \hspace{1cm} $c_1 d_{G_2}^{^*}(u_2) + d_{\mu(G_1)}(u) = c_1 d_{G_2}^{^*}(v_2) + d_{\mu(G_1)}(v) \Rightarrow c_1 d_{G_2}^{^*}(u_2) = c_1 d_{G_2}^{^*}(v_2) \Rightarrow d_{G_2}^{^*}(u_2) = d_{G_2}^{^*}(v_2)$

This is true for all $u_2, v_2 \in V_2$. Thus $G_2^{^*}$ is a regular graph. Hence $G_2$ is a partially
regular intuitionistic fuzzy graph.

Fix $v \in V_2$ and consider $(u_1, v)$ and $(v_1, v)$ in $V_1 \times V_2$, where $u_1, v_1 \in V_1$ are arbitrary.

From (3.9) \hspace{1cm} $c_1 d_{G_2}^{^*}(v) + d_{\mu(G_1)}(u_1) = c_1 d_{G_2}^{^*}(v) + d_{\mu(G_1)}(v_1) \Rightarrow d_{\mu(G_1)}(u_1) = d_{\mu(G_1)}(v_1)$

From (3.10) \hspace{1cm} $c_2 d_{G_2}^{^*}(v) + d_{\nu(G_1)}(u_1) = c_2 d_{G_2}^{^*}(v) + d_{\nu(G_1)}(v_1) \Rightarrow d_{\nu(G_1)}(u_1) = d_{\nu(G_1)}(v_1)$

This is true for all $u_1, v_1 \in V_1$. Thus $G_1$ is a regular intuitionistic fuzzy graph.

**Theorem 3.9:** Let $G_1:(\mu_1, \mu_2)$ and $G_2:(\mu'_1, \mu'_2)$ be two intuitionistic fuzzy graphs
such that $\mu_1 \leq \mu'_2, \mu_1$ is a constant function and $\nu_1 \leq \nu'_2, \nu_1$ is a constant function then
$G_1 \times G_2$ is totally regular intuitionistic fuzzy graph if and only if $G_1$ is a totally regular
intuitionistic fuzzy graph and $G_2$ is a partially regular intuitionistic fuzzy graph.

**Proof:** We have $\mu_1 \leq \mu'_2$. Hence $\mu'_1 \geq \mu_2$, $\mu_1 \leq \mu'_1$ and $\nu_1 \leq \nu'_2$. Hence $\nu'_1 \geq \nu_2$, $\nu_1 \leq \nu'_1$.

Let $\mu_1(u_1) = c_1$, $\nu_1(u_1) = c_2$, for all $u_1 \in V_1$, where $c_1, c_2$ are constants.
Totally Regular Property of Cartesian product of Intuitionistic Fuzzy Graphs

Suppose that $G_1$ is a totally regular intuitionistic fuzzy graph of degree $k$ and $G_2^*$ is a regular graph of degree $r_2$.

By definition, for any $(u_1, u_2) \in V_1 \times V_2$,

$$td_{\mu(G_1 \times G_2)}(u_1, u_2) = \sum_{u_1=\nu_1, u_2=\nu_2 \in E_2} \mu_1(u_1) \land \mu'_2(u_2 v_2)$$

$$+ \sum_{u_2=\nu_2, u_1=\nu_1 \in E_1} \mu'_1(u_2) \land \mu_2(u_1 v_1) + \mu_1(u_1) \land \mu'_1(u_2)$$

$$= \sum_{u_2=\nu_2, u_1=\nu_1 \in E_2} \mu_2(u_1 v_1) + \mu_1(u_1) \big[\text{since } \mu_1 \leq \mu'_2, \mu'_1 \big]$$

$$\geq \mu_2 \text{ and } \mu_1 \leq \mu'_1$$

$$td_{\mu(G_1 \times G_2)}(u_1, u_2) = c_1 d_{G_2}^*(u_2) + d_{\mu(G_1)}(u_1) + \mu_1(u_1) \quad \text{------------------ (3.11)}$$

$$td_{\mu(G_1 \times G_2)}(u_1, u_2) = c_1 d_{G_2}^*(u_2) + td_{\mu(G_1)}(u_1) \quad \text{------------------ (3.12)}$$

$$\Rightarrow td_{\mu(G_1 \times G_2)}(u_1, u_2) = c_1 r_2 + k$$

$$td_{\nu(G_1 \times G_2)}(u_1, u_2) = \sum_{u_1=\nu_1, u_2=\nu_2 \in E_2} \nu_1(u_1) \lor \nu'_2(u_2 v_2) + \sum_{u_2=\nu_2, u_1=\nu_1 \in E_1} \nu'_1(u_2) \lor \nu_2(u_1 v_1)$$

$$+ \nu_1(u_1) \lor \nu'_1(u_2)$$

$$= \sum_{u_2=\nu_2, u_1=\nu_1 \in E_2} \nu_1(u_1) \lor \nu'_2(u_2 v_2) + \sum_{u_2=\nu_2, u_1=\nu_1 \in E_1} \nu'_1(u_2) \lor \nu_2(u_1 v_1)$$

$$\big[\text{since } \nu_1 \leq \nu'_2, \nu'_1 \geq \nu_2 \text{ and } \nu_1 \leq \nu'_1 \big]$$

$$td_{\nu(G_1 \times G_2)}(u_1, u_2) = c_2 d_{G_2}^*(u_2) + d_{\nu(G_1)}(u_1) + \nu_1(u_1) \quad \text{------------------ (3.13)}$$

$$td_{\nu(G_1 \times G_2)}(u_1, u_2) = c_2 d_{G_2}^*(u_2) + td_{\nu(G_1)}(u_1) \quad \text{------------------ (3.14)}$$

$$\Rightarrow td_{\nu(G_1 \times G_2)}(u_1, u_2) = c_2 r_2 + k$$

So, $G_1 \times G_2$ is a totally regular intuitionistic fuzzy graph, conversely, assume that $G_1 \times G_2$ is a totally regular intuitionistic fuzzy graph then for any two points $(u_1, u_2)$ and $(v_1, v_2)$ in $V_1 \times V_2$.

$$td_{\mu(G_1 \times G_2)}(u_1, u_2) = td_{\mu(G_1 \times G_2)}(v_1, v_2)$$

From (3.12) $c_1 d_{G_2}^*(u_2) + td_{\mu(G_1)}(u_1) = c_1 d_{G_2}^*(v_2) + td_{\mu(G_1)}(v_1) \quad \text{------------------ (3.15)}$

$$td_{\nu(G_1 \times G_2)}(u_1, u_2) = td_{\nu(G_1 \times G_2)}(v_1, v_2)$$

From (3.14) $c_2 d_{G_2}^*(u_2) + td_{\nu(G_1)}(u_1) = c_2 d_{G_2}^*(v_2) + td_{\nu(G_1)}(v_1) \quad \text{------------------ (3.16)}$
Fix $u \in V_1$ and consider $(u, u_2)$ and $(u, v_2)$ in $V_1 \times V_2$, where $u_2, v_2 \in V_2$ are arbitrary.

From (3.15) $c_1d_{G_2}^*(u_2) + td_{\mu(G_1)}(u) = c_1d_{G_2}^*(v_2) + td_{\mu(G_1)}(u)$
\[\Rightarrow c_1d_{G_2}^*(u_2) = c_1d_{G_2}^*(v_2) \Rightarrow d_{G_2}^*(u_2) = d_{G_2}^*(v_2)\]

This is true for all $u_2, v_2 \in V_2$. Thus $G_2^*$ is a regular graph. Hence $G_2$ is a partially regular intuitionistic fuzzy graph.

Fix $v \in V_2$ and consider $(u_1, v)$ and $(v_1, v)$ in $V_1 \times V_2$, where $u_1, v_1 \in V_1$ are arbitrary.

From (3.15) $c_1d_{G_2}^*(v) + td_{\mu(G_1)}(u_1) = c_1d_{G_2}^*(v) + td_{\mu(G_1)}(v_1) \Rightarrow td_{\mu(G_1)}(u_1) = td_{\mu(G_1)}(v_1)$

From (3.16) $c_2d_{G_2}^*(v) + td_{\nu(G_1)}(u_1) = c_2d_{G_2}^*(v) + td_{\nu(G_1)}(v_1) \Rightarrow td_{\nu(G_1)}(u_1) = td_{\nu(G_1)}(v_1)$

This is true for all $u_1, v_1 \in V_1$. Thus $G_1$ is a totally regular intuitionistic fuzzy graph.

**Theorem 3.10:** Let $G_1: (\mu_1, \mu_2)$ and $G_2: (\mu_1', \mu_2')$ be two intuitionistic fuzzy graphs such that $\mu_1 \leq \mu_2'$ and $v_1 \leq v_2'$. If $G_1 \times G_2$ and $G_1$ are totally regular intuitionistic fuzzy graphs and if $G_2$ is a partially regular intuitionistic fuzzy graph then $\mu_1$ and $v_1$ are constant functions.

**Proof:** Suppose that $G_1$ is a $k$-totally regular intuitionistic fuzzy graph and $G_2$ is a partially regular intuitionistic fuzzy graph $\Rightarrow G_2^*$ is a $r_2$-regular graph. Since $\mu_1 \leq \mu_2'$, $\mu_1 \leq \mu_1'$, $v_1 \leq v_2'$, $v_1 \leq v_1'$.

\[td_{\mu(G_1 \times G_2)}(u_1, u_2) = \mu_1(u_1)d_{G_2}^*(u_2) + td_{\mu(G_1)}(u_1)\]
\[td_{\nu(G_1 \times G_2)}(u_1, u_2) = \nu_1(u_1)d_{G_2}^*(u_2) + td_{\nu(G_1)}(u_1)\]

Since $G_1 \times G_2$ is a totally regular intuitionistic fuzzy graph, for any two points $(u_1, u_2)$ and $(v_1, v_2)$ in $V_1 \times V_2$.

\[td_{\mu(G_1 \times G_2)}(u_1, u_2) = td_{\mu(G_1 \times G_2)}(v_1, v_2)\]
\[\Rightarrow \mu_1(u_1)d_{G_2}^*(u_2) + td_{\mu(G_1)}(u_1) = \mu_1(v_1)d_{G_2}^*(v_2) + td_{\mu(G_1)}(v_1)\]
\[\Rightarrow \mu_1(u_1) \leq \mu_1(v_1) \text{ for all } u_1, v_1 \in V_1 \Rightarrow \mu_1 \text{ is a constant function.}\]

\[td_{\nu(G_1 \times G_2)}(u_1, u_2) = td_{\nu(G_1 \times G_2)}(v_1, v_2)\]
\[\Rightarrow v_1(u_1)d_{G_2}^*(u_2) + td_{\nu(G_1)}(u_1) = v_1(v_1)d_{G_2}^*(v_2) + td_{\nu(G_1)}(v_1)\]
\[\Rightarrow v_1(u_1) \leq v_1(v_1) \text{ for all } u_1, v_1 \in V_1 \Rightarrow \nu_1 \text{ is a constant function.}\]
\( \Rightarrow v_1(u_1) = v_1(v_1) \) for all \( u_1, v_1 \in V_2 \) \( \Rightarrow v_1 \) is a constant function.

Hence \( \mu_1, \nu_1 \) are constant functions.

**Remark:** Converse of the theorem 3.10 need not be true. For example, consider \( G_1 \) is both regular and totally regular IFG, \( G_2^* \) is regular but \( G_1 \times G_2 \) is not a totally regular IFG. Here \( \mu_1, \nu_1 \) are constant functions.

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**REFERENCES**

