A New Construction of a Group Acting on Fuzzy Algebraic Structure

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Abstract

The fuzzy group [Rosenfeld (1971), and Q-fuzzy group [Solairaju & Nagarajan (2009)] are studied. In this paper, group action on fuzzy sets, fuzzy groups and their algebraic properties are discussed. Homomorphic image and pre-image of fuzzy groups under algebraic are also given.

INTRODUCTION

The notion of fuzzy sets was first introduced by Zadeh [1965]. Fuzzy set theory has been developed in many directions by many scholars and has evoked great interest among mathematicians working in different fields of mathematics such as topological spaces, functional analysis, loop, and group, ring, near-ring, vector spaces, and automation. There have been wide ranging applications of the theory of fuzzy sets from the design of robots and computer simulation to engineering and water resources planning. Rosenfeld [1971] introduced a fuzzy group.

Section 1: Some previous works: Solairaju & Nagarajan [2009] introduced Q-fuzzy group and normal Q-fuzzy group, Q-fuzzy characteristic, and Q-fuzzy cyclic group. They [2008] also discussed Q-fuzzy left R-subgroup in near ring with respect to T-norm.

Kim and Yun [2000] introduced fuzzy R-subgroup on near-ring. They [2001] discussed normal fuzzy R-subgroup in near rings. The notion of an intuitionistic Q-fuzzy R-subgroups of a near rings is given by Osman kazanci, Sultan Yamark and Serife Yilmaz [2007].

Section 2 – Definitions and Basic Concepts:

**Definition 2.1**: A mapping \( \mu: X \to [0, 1] \), where \( X \) is an arbitrary non-empty set is called a fuzzy set in \( X \).

**Definition 2.2**: Let \( G \) be any group. A mapping \( \mu: G \to [0, 1] \) is a fuzzy group if (FG1). \( \mu(xy) \geq \min\{\mu(x), \mu(y)\} \) and (FG2). \( \mu(x^{-1}) = \mu(x) \), for all \( x, y \in G \).

**Definition 2.3**: Let \( (S, +) \) be a group, and \( G \) be a non-empty set. Then \( G \) acts on \( S \) if there exists a function \( *: G \times S \to S \) (denoted \( * (g, s) = gs \) for all \( g \in G \), and \( s \in S \)) such that \( es = s \) and \( (g + h) \ast s = g \ast (h \ast s) \) for all \( s \in S \), and for all \( g, h \) in \( G \).

**Examples**: For \( n \geq 3 \), each element of the group \( D_n \) acts as a rigid motion of a regular \( n \)-gon in the plane, either a rotation or a reflection.

We can also view \( D_n \) as acting just on the \( n \) vertices of a regular \( n \)-gon. Knowing where the vertices go determines the rest of the rigid motion, so the effect of \( D_n \) on the vertices tells us all we need to know to determine the rigid motion on the \( n \)-gon. Restricting the action of \( D_n \) from an \( n \)-gon to its vertices, and labelling the vertices as 1, 2, 3, ..., \( n \) in some manner, makes \( D_n \) act on \{1, 2, ..., \( n \)\}.

**Definition 2.4**: A group \((G, +)\) with identity 0 acts on a fuzzy group \( A \) on a group \((S, \Delta)\) if (GAFS1) the group \( G \) acts on \( S \) [there exists a function \( *: G \times S \to S \) with the conditions \( g * (h * s) = (g + h) * s \) and \( e * s = s \) or all \( s \in S \), and for all \( g, h \) in \( G \)];

- (GAFG2) \( A(x * (s \Delta t)) \geq \min \{ A(x * s), A(x * t) \} \);
- (GAFG3) \( A((x + y) * s) \geq \min \{ A(x * s), A(y * s) \} \);
- GAFG4 \( A( x * s^{-1} ) \geq A( x * s ) \) for all \( x, y \in G \) and \( s, t \in S \).

**Remark 2.5**: Note that \((A \circ *)\) is a fuzzy set on \( G \times S \) for each fuzzy set \( A \) on \( S \). Thus the fuzzy set \((A \circ *)\) on \( G \times S \) as \{\((x, s), A \circ * ((x, s))\) : \( x \in G \) and \( s \in S \}\} where \((A \circ *)\) is map from \( G \times S \to [0, 1] \). This fuzzy set \( A \circ * \) is called a fuzzy set in \( G \) acting on \( S \). Define the set \( U(A \circ *, t) = \{ s \in S : \inf_{x \in G} A(x * s) \geq t \} \) where \( t \in [0, 1] \), which is called an upper cut of \((A \circ *)\) acting on \( A \).
Section 3: Contributions for a group acting on fuzzy group

In this chapter, the new classes of algebraic structures introduced by K.H.Kim, [2006] are defuzzified. In this fuzzification, the notion of a group acting on fuzzy group (GAFG) and investigates some of their related properties. Characterization of these groups and normal fuzzy groups (GAFNG) acting on fuzzy groups (GAFCG) are given.

The following results on the properties of fuzzy group acted by a group are obtained.

Theorem 3.1: Let \((G, +)\) be a group acting on a fuzzy group \(A\) on \((S, \Delta)\). Then every subgroup \(H\) of \(G\) acts on \(A\).

Proof: A group \((G, +)\) with identity 0 acts on a fuzzy group \(A\) on \((S, \Delta)\). Then (GAFS1) the group \(G\) acts on \(S\). Also there exists a function \(*: G \times S \to S\) with the conditions
\[
g * (h * s) = (g + h) * s \text{ and } e * s = s \text{ or all } s \text{ in } S, \text{ and for all } g, h \text{ in } G\];
\]
In particular, the restriction map of \(*\) on \(H \times S \to S\) satisfies the conditions
\[
g * (h * s) = (g + h) * s \text{ and } e * s = s \text{ or all } s \text{ in } S, \text{ and for all } g, h \text{ in } H\];
(GAFG2) \(A(x * (s \Delta t)) \geq \min \{A(x * s), A(x * t)\}\);
(GAFG3) \(A((x + y) * s) \geq \min \{A(x * s), A(y * s)\}\); (GAFG5) \(A((-x) * s) \geq A(x * s)\); (GAFG4) \(A(x^{-1} * s) \geq A(x * s)\) for all \(x, y \in H\) and \(s, t \in S\).

Hence every subgroup of \((G, +)\) acts on the given fuzzy group \(A\). Similarly, the group \((G, +)\) acts on each fuzzy subgroup \(H_A\) of \(A\) under \(S\).

Theorem 3.2: Let a group \((G, +)\) act on a fuzzy group \(A\) of a group \((S, \Delta)\). Then (i) \(A(x * s) = A(0 * s)\) for all \(x \in G\) and \(s \in S\). (ii) The subset \(G_A = \{x \in G / A(x * s) = A(0 * s)\}\) is a subgroup of \(G\) acting on the fuzzy group \(A\) on \(S\).

Proof: Let \(x\) be any element of \(G\). Then \(A(x * s) = \min \{A(x * s), A(x * s)\} = \min \{A(x * s), A((-x) * s)\} \leq A((x - x) * s) = A(0 * s)\) implies (i).

To verify (ii), it follows that \(0 \in G_A\), and \(G_A \neq \{\}\).

Let \(x, y \in G_A\) and \(s \in S\). \(A((x - y) * s) = \min \{A(x * s), A((-y) * s)\} = \min \{A(x * s), A(y * s)\} = \min \{A(0 * s), A(0 * s)\} = A(0 * s)\) but from (i) \(A((x - y) * s) \leq A(0 * s)\) for \(x, y \in G\) and \(s \in S\), so \(A((x - y) * s) = A(0 * s)\) which
Corollary 3.3: Let \( G \) be a finite group and \( A \) be a fuzzy group of \( G \) acting on \( S \). Consider the subset \( H \) of \( G \) given by \( H = \{x \in G / A(x * s) = A(0 * s)\} \). Then \( H \) is a crisp subgroup of \( G \) acting on \( S \).

Proof: It is obvious.

Theorem 3.4: Let a group \((G,+)\) act on fuzzy groups \(A\) and \(B\) on \((S, \Delta)\). Then the group \(G\) acts on the fuzzy group \(A \cap B\) on \((S, \Delta)\).

Proof: Let a group \((G, +)\) act on fuzzy groups \(A\) and \(B\) on \((S, \Delta)\). It is given that \(A\) and \(B\) are fuzzy groups on the group \((S, \Delta)\). It follows that \(A \cap B\) is a fuzzy group on \((S, \Delta)\).

Since the group \((G, +)\) acts on \(S\), then there exists a map \(* : G \times S \rightarrow S\) such that \(g * (h * s) = (g + h) * s\) and \(e * s = s\) for all \(s \in S\), and for all \(g, h \in G\) which gives GAFG1.

Let \(x, y \in G\) and \(s \in S\).

1. \((GAFG2)\ (A \cap B)(x * (s \Delta t)) = \min\{A(x * (s \Delta t)), B(x * (s \Delta t))\}\)
   \[\geq \min\{\min\{A(x * s), A(x + t)\}, \min\{B(x * s), B(x + t)\}\}\]
   \[= \min\{\min\{A(x * s), B(x * s)\}, \min\{A(x + t), B(x + t)\}\}\]
   \[= \min\{(A \cap B)(x * s), (A \cap B)(x * t)\}\]

2. \((GAFG3)\ (A \cap B)((x + y) * s) = \min\{A((x + y) * s), B((x + y) * s)\}\)
   \[\geq \min\{\min\{A(x * s), A(y * s)\}, \min\{B(x * s), B(y * s)\}\}\]
   \[= \min\{\min\{A(x * s), B(x * s)\}, \min\{A(y * s), B(y * s)\}\}\]
   \[= \min\{(A \cap B)(x * s), (A \cap B)(y * s)\}\]

3. \((GAFG4)\ (A \cap B)(x * s^{-1}) = \min\{A(x * s^{-1}), B(x * s^{-1})\}\)
   \[\geq \min\{A(x * s), B(x * s)\}\]
   \[= (A \cap B)(x * s)\].
Thus the group \((G, +)\) acts on the fuzzy group \(A \cap B\) on the group \((S, \Delta)\).

**Theorem 3.5**: If a group \((G, +)\) acts each member in the family \(\{A_i\}_{i \in A}\) of fuzzy groups under \(S\), then a group \((G, +)\) acts on the fuzzy group \(\cap A_i\) under \(S\).

**Proof**: It is obvious.

**Theorem 3.6**: If a group \((G, +)\) acts a fuzzy group \(A\) on \((S, \Delta)\), then the group \(G\) acts on anti-fuzzy group \(A^c\) on \((S, \Delta)\).

**Proof**: It knows that the group \((G, +)\) acts on \(S\). For any \(x, y \in G\), and \(s \in S\), it follows that

\[
(GAFG2) \quad A^c((x * s) \Delta t)) = 1 - A((x * (s \Delta t))
\]

\[
\leq 1 - \min(A(x * s), A(x * t))
\]

\[
= \max(1 - A(x * s), 1 - A(x * t))
\]

\[
= \max(A^c(x * s), A^c(x * t))
\]

\[
(GAFG3) \quad A^c((x + y) * s) = 1 - A((x + y) * s)
\]

\[
\leq 1 - \min(A(x * s), A(y * s))
\]

\[
= \max(1 - A(x * s), 1 - A(y * s))
\]

\[
= \max(A^c(x * s), A^c(y * s))
\]

\[
(GAFG4) \quad A^c(x * s^{-1}) = 1 - A(x * s^{-1})
\]

\[
= 1 - A(x * s) = A^c(x * s).
\]

So the group \(G\) acts on anti-fuzzy group \(A^c\) on \((S, \Delta)\).

**Theorem 3.7**: Let a group \((G, +)\) act on a fuzzy group \(A\) on \((S, \Delta)\). Then \(G\) acts on level cut set \(U(A; t)\) on \(S\) where \(t\) is in \([0, 1]\).

**Proof**: The group \(G\) acts on \(S\), and \(\alpha\) be in \([0, 1]\). Then \(U(A, t) = \{s \in S : inf_{x \in G} A(x * s) \geq \alpha\}\). Let \(s, t\) be in \(U(A, \alpha)\). Then \(inf_{x \in G} A(x * s) \geq \alpha\) and \(inf_{x \in G} A(x * t) \geq \alpha\).

The group \(G\) acts the fuzzy group \(A\).

Then \(A(x * (s \Delta t)) \geq \min\{inf_{x \in G} A(x * s), inf_{x \in G} A(x * t)\}\) for all \(x, y\) in \(G\).
So \[ A \{ \inf_{x \in G} (x \ast (s \Delta t)) \} \]
\[ \geq \min \{ \inf_{x \in G} A(x \ast s), \inf_{y \in G} A(x \ast t) \} \]
\[ \geq \min \{ \alpha, \alpha \} = \alpha \]
implies \((s \Delta t) \in U(A; t)\).

Let \( s \in U(A; t) \). Then \( A \{ (x \ast s^{-1}) = A \{ (x \ast s) \} \). It gives that \((\inf_{x \in G} A(x \ast s^{-1})) = \inf_{x \in G} A(x \ast s) \geq \alpha \) implies that \( s^{-1} \in U(A; t) \). Thus \( U(A, t) \) is a subgroup of \((S, \Delta)\), and \( A_{U(A, t)} \) (the restriction map of \( A \) on \( U(A, t) \)) is a fuzzy subgroup of \( S \). Then the group \((G, +)\) acts the fuzzy group \( A_{U(A, t)} \) under \((S, \Delta)\).

Section 4: Homomorphic properties for a group acting on fuzzy group

**Definition 4.1:** Let \( \theta \) be a mapping from \( X \) to \( Y \). (i) Let \((G', +')\) a group acting on fuzzy group \( B \) under \( S \). Then the inverse image of \( B \) under \( \theta \) denoted by \( \theta^{-1}(B) \) is fuzzy set in \((G, +)\) defined by \( \theta^{-1}(B) = \mu_{(B)}(x) = \mu_{B}(\theta(x)) \); (ii) Let \((G, +)\) be a group acting on fuzzy group \( A \) under \( S \). Then the image of \( A \) under \( \theta \) denoted by \( \theta(A) \), where \( \mu_{\theta(A)}(y) = \{ \sup \mu_{A}(x)s : (x \ast s) \in (\theta^{-1}(y) \ast s) \} \) for all \( s \) in \( S \) Also \( \mu_{\theta(A)}(y) \ast s = \{ \sup \mu_{A}(x)s : (x \ast s) \in (\theta^{-1}(y) \ast s) \} \) for all \( s \) in \( S \) if \( \theta^{-1}(y) \neq 0 \); 0, otherwise.

**Theorem 4.2:** Let \((G', +')\) a group acting on fuzzy group \( B \) under \( S \), and \( \theta: G \to G' \) be an onto homomorphism on groups. Then the group \((G, +)\) acts on the fuzzy set \( \theta^{-1}(B) \) under \( S \).

**Proof:** There exists a map \( *' : G' \times S \to S \) such that \((g +' h) *' s = g *' (h *' s) \) and \( e' *' s = s \) for all \( g, h \) in \( G' \) and \( s \) in \( S \). For \( a, b \) in \( G \), define \( \mu_{\theta(A)}(a \ast s) = B((\theta(a) \ast s) \). Using \( *' \) and \( \theta \), there is a map \( * : G \times S \to S \) defined by \( *(a, s) = \theta(a) *' s \) with (i) \((a + b) * s = a * (b * s) \) and \( e \ast s = s \) for all \( a, b \) in \( G \), and \( s \) in \( S \). Thus \((G, +)\) acts on the fuzzy subset \( \theta^{-1}(B) \) under \( S \).

**Theorem 4.3:** Let \((G, +)\) a group acting on fuzzy group \( A \) under \( S \), and \( \theta: G \to G' \) be an onto homomorphism on groups. Then the group \((G', +')\) acts on the fuzzy set \( \theta(A) \) under \( S \).

**Proof:** There exists a map \( * : G \times S \to S \) such that \((a + b) * s = a \ast (a \ast s) \) and \( e \ast s = s \) for all \( a, b \) in \( G \) and \( s \) in \( S \). Define a fuzzy set \( \theta(A) \) on \( G' \) by \( \mu_{\theta(A)}(y) = \{ \sup \mu_{A}(x)s : x \in \theta^{-1}(y) \} \) if \( \theta^{-1}(y) \neq 0 \); 0, otherwise. There exists a map \( *' : G' \times S \to S \) such that \((g +' h) *' s = g *' (h *' s) \) and \( e' *' s = s \) for all \( g, h \) in \( G' \) and \( s \) in \( S \). The group \((G, +)\) acts on the fuzzy subset \( \theta(A) \) under \( S \).
**Theorem 4.4:** Let \( (G', +') \) a group act on a fuzzy group \( B \) under \( S \), and \( \theta: G \to G' \) be an onto homomorphism on groups. Then the group \( (G, +') \) acts on the fuzzy group \( \theta^{-1}(B) \) under \( S \).

**Proof:** Let \( G^1 \) be a group acting on a fuzzy group \( B \) under \( S \). Then the group \( G \) acts on the fuzzy set \( \theta^{-1}(B) \) under \( S \) by (4.2).

Let \( x, y \in G \), and \( s, t \in S \).

\[
\begin{align*}
(GAFG2) \quad \mu_{0^{-1}(B)}(x \ast (s \triangle t)) &= \mu_B((\theta(x) \ast^1 (s \Delta t)) \\
&= \mu_B((\theta(x) \ast^1 (s \Delta t)) \\
&\geq \min\{\mu_B(\theta(x) \Delta s), \mu_B(\theta(x) \Delta t)\} \\
&= \min\{\mu_{0^{-1}(B)}(x \ast s), \mu_{0^{-1}(B)}(x \ast t)\}.
\end{align*}
\]

\[
\begin{align*}
(GAFG3) \quad \mu_{0^{-1}(B)}((x + y) \ast s) &= \mu_B((\theta(x + y)) \ast^1 s) \\
&= \mu_B((\theta(x) +^1 \theta(y) \ast^1 s) \\
&\geq \min\{\mu_B(\theta(x) \Delta s), \mu_B(\theta(y) \Delta s)\} \\
&= \min\{\mu_{0^{-1}(B)}(x \ast s), \mu_{0^{-1}(B)}(y \ast s)\}.
\end{align*}
\]

\[
\begin{align*}
(GAFG4) \quad \mu_{0^{-1}(B)}(x \ast^1 s) &= \mu_B(\theta(x) \Delta s^{-1}) = \mu_B(\theta(x) \Delta s) = \mu_{0^{-1}(B)}(x \ast s)
\end{align*}
\]

Therefore the group \( (G, +') \) acts the fuzzy group \( \theta^{-1}(B) \) under \( S \).

**Theorem 4.5:** Let \( \theta: G \to G^1 \) be an epimorphism and a group \( G \) act on a fuzzy group \( A \) under \( S \). Then the group \( G^1 \) acts on the fuzzy group \( \theta(A) \) under \( S \).

**Proof:** Let \( G \) be a group acting on a fuzzy group \( A \) under \( S \). Then the group \( G^1 \) acts on the fuzzy set \( \theta(A) \) under \( S \) by (4.1). Also \( \theta(A) \) is a fuzzy group of \( (G', +') \).

Let \( g, h \) be in \( G^1 \) and \( s, t \) in \( S \).

It follows that

\[
\begin{align*}
(GAFG2) \quad \theta(A) (g \ast' (s \triangle t)) \\
&= \sup_{x \in X} \{ A(x \ast (s \triangle t)) : x \ast s \in \theta^{-1}(g) \ast' s \text{ if } \theta^{-1}(x) \ast' s \neq 0 \} \\
&\geq \sup_{x \in X} \min A((x \ast s) \Delta (x \ast t)) \\
&\geq \min \{ \theta(A)(g \ast' s), \theta(A)(g \ast t) \}
\end{align*}
\]

\[
\begin{align*}
(GAFG3) \quad \theta(A)((g + h) \ast' s) \\
&= \sup_{y \in X} (\text{where } x \in X \text{ is fixed}) \{ A((x + y) \ast s) : x \ast s \in \theta^{-1}(g + h) \ast' s \text{ if } \theta^{-1}(y) \ast' s \neq 0 \}
\end{align*}
\]
\( \geq \operatorname{Sup}_{y \in X} \text{where } x \in X \text{ is fixed} \min A( (x * s) \Delta (y * s) ) \)
\( \geq \min \{ \theta(A) ( g *' s), \theta(A) (h *' s) \} \)

\[(\text{GAFG4}) \theta(A) ( g *' s^{-1}) = (\operatorname{Sup}_{x \in X} \ A (x * s) ) = (\operatorname{Sup}_{x \in X} \ A (x * s)) = \theta(A) ( g *' s) \]

Then the group \( G' \) acts on the fuzzy group \( \theta(A) \) under \( S \).

**Theorem 4.6:** If a group \( (G,+) \) acts all non-empty level subset \( U(A, t) \) under \( (S, \Delta) \), then the group \( (G,+) \) acts \( A \) under \( S \).

**Proof:** Then \( U(A, t) = \{ s \in S: \inf_{x \in G} A(x * s) \geq \alpha \} \).

Let \( s, t \in U(A, t_0) \). Then \( \inf_{x \in G} A(x * t) \geq \alpha \) and \( \inf_{x \in G} A(x * s) \geq \alpha \).

\[ (A(\inf_{x \in G} x * (s \Delta t)) \geq \min \{ A(\inf_{x \in G} x * s), A(\inf_{x \in G} x * t) \} \geq \min \{ \alpha, \alpha \} = \alpha \]

Thus \( s \Delta t \in U(A, t_0) \).

Further \( s \in U(A; t_0) \) and if and only if \( A(\inf_{x \in G} (x * s^{-1}) = A(\inf_{x \in G} (x * s) \geq \alpha \) which implies \( s^{-1} \in U(A, t_0) \). Therefore \( U(A, t_0) \) is a subgroup of \( (S, \Delta) \). Then the group \( (G, +) \) acts on fuzzy group \( A_{U(A, t_0)} \) \( [ \text{the restriction map of } A \text{ on subgroup } U(A: t) ] \) under by \( (3.7) \).

But \( A = \bigcup_{t \in [0,1]} A_{U(A:t)} \) and every level fuzzy subgroup \( A_{U(A:t_1)} \) is contained in other level fuzzy subgroup \( A_{U(A:t_2)} \) for all \( t_1, t_2 \in [0, 1] \). So \( A = \bigcup_{t \in [0,1]} A_{U(A:t)} \) is a fuzzy group on \( (S, \Delta) \). Since the group \( (G, +) \) acts on each level set \( U(A: t) \) on \( S \), then the group \( (G, +) \) acts on fuzzy group \( A \) on \( S \) by \( (3.7) \).

**Theorem 4.7:** A set of necessary and sufficient conditions for a group \( (G, +) \) acting on fuzzy group \( A \) under \( (S, \Delta) \) is that \( A( x * (s \Delta t)^{-1}) \geq \min \{ A(x * s), A(x * t) \} \) for all \( x, y \) in \( G \) and \( s, t \) in \( S \), where every element of \( G \) has its own inverse under addition.

**Proof:** Let a group \( (G, +) \) act on a fuzzy group \( A \) under \( (S, \Delta) \).

Then \( A( x * (s \Delta t)^{-1}) \geq \min \{ A(x * s), (x * t)^{-1} \} \)

\[ = \min \{ A(x * s), (x * t) \} \text{ for all } x, y \in G \text{ and } s, t \in S. \]

For the converse, suppose that the condition holds.

Thus \( A( x * (s \Delta t)^{-1}) \geq \min \{ A(x * s), A(x * t) \} \) for all \( x, y \) in \( G \) and \( s, t \) in \( S \).

It follows that \( A(0 * 1) = A ((x + (-x)) * (s \Delta s^{-1})) \)
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\[ \geq \min \{ A(x \ast s), A(x \ast s) \} = A(x \ast s) \text{ for all } x, y \in G \text{ and } s, t \in S. \]

Since \( A \) is a fuzzy group on \((S, \Delta)\), then it gives that

\[ A(x \ast 1) = A((x \ast (s \Delta s^{-1}))) \geq \min \{ A(x \ast s), A(0 \ast s) \} \]

\[ A(0 \ast s) = A((x - x) \ast (s \Delta 1)) \]

\[ \geq \min \{ A(x \ast s), A(x \ast 1) \} = A(x \ast s) \text{ for all } x, y \in G \text{ and } s, t \in S \]

\[ \geq \min \{ A(y \ast s), A(y \ast t) \} \]

\[ A(y - y, s \Delta t^{-1}) \]

\[ \geq \min \{ A(y \ast s), A((-y) \ast t) \} \]

\[ \geq \min \{ A(y \ast s), A(y \ast t) \} \]

Thus \( A(0 \ast s) \geq A(y \ast s) \) for all \( s \) in \( S \).

Further \( A((-y) \ast (s \Delta t)) \geq \min \{ A(0 \ast s), A((-y) \ast t) \} \geq A((-y) \ast t) \) for all \( s, t \) in \( S \). It implies that \( A((-y) \ast s) \geq A(y \ast s) \) for all \( y \) in \( G \). Therefore \( A((-y) \ast s) = A(y \ast s) \) for all \( y \) in \( G \).

Also \( A((x + y) \ast (s \Delta t)) \geq \min \{ A(x \ast s), A((-y) \ast s) \} = \min \{ A(x \ast s), A((-y) \ast s) \} \) for all \( x, y \) in \( G \) and \( s, t \) in \( S \). Hence the group \((G, +)\) acts on the fuzzy group \( A \) under \((S, \Delta)\).

Section 5: Fuzzy normal group

**Definition 5.1:** Let a group \((G, +)\) act on a fuzzy group \( A \) under \((S, \Delta)\), and \( \theta \) is a homomorphism from \( S \) into \( S \). Define \( A^\theta = \{ x \ast s \mid s \in S, \inf_{x \in G} A(\theta(x) \ast s) \} \) under \((S, \Delta)\).

**Definition 5.2:** A fuzzy group \( A \) on \((S, \Delta)\) is normal if

\[ \inf_{x \in G} A(x \ast s) = \inf_{x \in G} A((x \ast (t^{-1}s \Delta t))) \text{ for all } s, t \in S. \]

**Definition 5.3:** A fuzzy group \( A \) of \( G \) acting on \( S \) is a fuzzy characteristic of \( G \) acting on \( S \) if \( A^\theta = A \) for some homomorphism \( \theta \) on \( S \).

**Definition 5.4:** Let a group \((G, +)\) acts on a normal fuzzy group \( A \) under \((S, \Delta)\) if

\[ \inf_{x \in G} A(x \ast s) = \inf_{x \in G} A((x \ast (t \Delta s \Delta t^{-1}))) \text{ for all } s, t \in S. \]

**Definition 5.5:** In a group \((S, \Delta)\), define \([s, t] = (t^{-1}s \Delta s \Delta t)\)
The following propositions are proved:

**Theorem 5.6:** Let a group \((G, +)\) act on a fuzzy group \(A\) under \((S, \Delta)\), and \(\theta\) is a homomorphism from \(S\) into \(S\). The group \(G\) acts on the fuzzy subgroup \(A^0 = \{ < s \in S, \inf_{x \in G} A(\theta(x) \ast s) > \}\) under \((S, \Delta)\).

**Proof:** Let \(s, t \in A^0\).

Then it follows that \(A((x + y) \ast (s \Delta t^{-1})) \geq \min \{ A(x \ast s), A(y \ast t) \}\) for \(x, y \in G\).

\[
A(\theta((x + y)) \ast (s \Delta t^{-1})) \geq \min \{ A(\theta(x) \ast s), A(\theta(y) \ast t) \}\) for \(x, y \in G\).

So \((\inf_{x \in G} A(\theta((x + y)) \ast (s \Delta t^{-1}))) \geq \min \{ \inf_{x \in G} A(\theta(x) \ast s), \inf_{x \in G} A(\theta(y) \ast t) \}\) for \(x \in G\).

Thus \(s \Delta t^{-1} \in A^0\). This proves that \(A^0\) is a fuzzy subgroup of \((S, \Delta)\). Therefore the group \((G, +)\) acts on the fuzzy subset \(A^0\) under \((S, \Delta)\).

**Theorem 5.7:** Let a group \((G, +)\) act on a fuzzy group \(A\) under \((S, \Delta)\). Let \(A^+\) be a fuzzy set \(\{ < s \in S, A^+(s) >: A^+(s) = \inf_{x \in G} A(x \ast s) + 1 - A(0 \ast s) \}\). Then the group \(G\) acts on the fuzzy group \(A^+\) under \((S, \Delta)\).

**Proof:** It follows that \(A^+(s) = \inf_{x \in G} A(x \ast s) + 1 - A(0 \ast s)\) for all \(s \in S\).

So \(A^+(s \Delta t^{-1}) = \inf_{x \in G} A(x \ast (s \Delta t^{-1})) + 1 - A(0 \ast (s \Delta t^{-1}))\)

\[
= [\inf_{x \in G} A(x \ast s) + 1 - A(0 \ast s)], \inf_{x \in G} A(x \ast t) + 1 - A(0 \ast t)\) for \(s, t \in S\).

Thus \(s, t \in A^+\) implies that \(s \Delta t^{-1} \in A^+. \) Then \(A^+\) is a fuzzy subgroup of \((S, \Delta)\). Hence the group \(G\) acts on the fuzzy group \(A^+\) under \((S, \Delta)\).

So \(A^+(x \ast (t \Delta s \Delta t^{-1})) = \inf_{x \in G} A(x \ast (t \Delta s \Delta t^{-1})) + 1 - A(0 \ast ((t \Delta s \Delta t^{-1}))\)

\[
= A(x \ast s) + 1 - A(0 \ast s)
\]

\(= A^+(x \ast s)\) for \(s, t \in S\).

Hence \(A^+\) is normal fuzzy group of \(G\) acting on \(S\).

**Theorem 5.8:** Let a group \((G, +)\) act on a normal fuzzy group \(A\) under \((S, \Delta)\). Then \(\inf_{x \in G} A(x \ast [s, t]) = A(0 \ast s)\) for all \(s, t \in S\).

**Proof:** Since a group \((G, +)\) acts on a normal fuzzy group \(A\) under \((S, \Delta)\), it gives that

\[
\inf_{x \in G} A(x \ast s) = \inf_{x \in G} A(x \ast (t \Delta s \Delta t^{-1})\) for all \(s, t \in S\). Replacing \(s\) by \((s^{-1})\) and \(t\) by \((t^{-1})\), \(\inf_{x \in G} A(x \ast s^{-1}) = \inf_{x \in G} A(x \ast (t^{-1} \Delta s \Delta t)\) for all \(s, t \in S\). Then it becomes that

\[
\inf_{x \in G} A(x \ast [s, t]) = A(0 \ast s)\) for all \(s, t \in S\).
Theorem 5.9: Let a group \((G, +)\) act on a fuzzy characteristic group \(A\) with respect to \(\theta\) under \((S, \Delta)\). Then \(A\) is fuzzy normal subgroup of \((S, \Delta)\).

Proof: Let \(x, y \in G\). Consider the map \(\theta_t: S \to S\) by \(\theta_t(s) = t \Delta s \Delta t^{-1}\), where \(t\) fixed in \(S\). Clearly \(\theta_t\) is an automorphism of \(S\).

Now \(A (x * s) = A^{\theta_t} (x * s) = A (\theta_t (x * s))\) for all \(s\) in \(S\)
\[
= A ((x * (t^{-1} \Delta s \Delta t)) \text{ for all } s, t \text{ in } S.
\]
It follows that \(\inf_{x \in G} A (x * s) = \inf_{x \in G} A ((x * (t \Delta s \Delta t^{-1})) \text{ for all } s, t \text{ in } S.
\]
So \(A\) is fuzzy normal subgroup of \((S, \Delta)\), and hence the group \((G, +)\) acts on normal fuzzy group \(A\) under \((S, \Delta)\).

CONCLUSION
Fuzzy group theory has vast and potential applications in many core areas like physics, chemistry, communications, coding theory, computer science etc. The concept of group acting on fuzzy group and their properties are studied.

REFERENCE
