Common Fixed Point of Weakly Compatible Maps in Intuitionistic Fuzzy Metric Spaces

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Abstract

In this paper we introduce the concept of semi weakly compatibility of maps and establish some common fixed point theorems by using this concept along with the notion of weakly compatibility of maps in the setting of non-complete intuitionistic fuzzy metric spaces. Our results extend and generalize the result of Pant et al [18]. Some related results and illustrative examples have also been discussed.

Keywords: Common fixed point, intuitionistic fuzzy metric space, semi-weakly compatible maps, weakly compatible maps.

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1. INTRODUCTION

Initially in 1965, Zadeh [24] was introduced the concept of fuzzy sets. Since then, due to the wide applicability of fuzzy sets in various fields such as topology, analysis etc, many authors have expansively developed the theory of fuzzy sets and its applications. In this context Deng [8], Erecg [10], Fang [11], Kaleva and Seikkala [15], Kramosil and Michalek [16] have introduced the concept of fuzzy metric spaces in different ways, which was further modified by by George and Veeramani [12].

Atanassov [5] introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets and later many authors developed the theory intuitionistic fuzzy sets. Further, motivated by the potential applicability of fuzzy topology in quantum particle physics, Park [19] and Alaca et. al. [3] introduced the notion of intuitionistic fuzzy metric spaces, based on the intuitionistic fuzzy sets and fuzzy metric space given by Atanassov [5] and George and...
Veeramani [12] respectively with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space given by Kramosil and Michalek [16]. They have also introduced the notion of Cauchy sequence and gave the intuitionistic fuzzify version of the well-known fixed point theorems of Banach [6] and Edelstein [9] with the help of Grabiec [13] in intuitionistic fuzzy metric spaces. Later on Turkoglu et al [22] proved Jungck’s [14] common fixed point theorem. They have further formulated the notions of weakly commuting and R- weakly commuting mappings and proved the Pant’s [17] theorem in intuitionistic fuzzy metric spaces.

Recently, Sharma et al [21] and Pant et al [18] established some common fixed point theorems for finite number of discontinuous, weakly compatible maps satisfying a new contractive type condition on non complete intuitionistic fuzzy metric spaces.

In this paper we have introduced a new concept namely semi-weakly compatibility of maps and using this concept along with the notion of weakly compatibility of maps establish some common fixed point theorems for such maps on non complete intuitionistic fuzzy metric spaces. Our results extend and generalize the result of Pant et al [18] and others. We also discuss some related results and illustrative examples.

2. PRELIMINARIES
Throughout this paper for the symbols and basic definitions, we refer [1, 3, 18, 19]. Here we describe some relevant definitions and results for further use.

**Definition 2.1 [20]** A binary operation \(* : [0, 1] \times [0, 1] \rightarrow [0, 1]\) is called a continuous t-norm if * satisfies the following conditions:

1. \(* is commutative and associative
2. \(* is continuous
3. \(a * 1 = a \) for all \(a \in [0, 1]\)
4. \(a * b \leq c * d \) whenever \(a \leq c \) and \(b \leq d \) for all \(a, b, c, d \in [0, 1]\).

Examples: (i) \(a * b = \min\{a, b\}\) and (ii) \(a * b = ab\).

**Definition 2.2 [20]** A binary operation \(\triangleright : [0, 1] \times [0, 1] \rightarrow [0, 1]\) is called a continuous t-conorm if \(\triangleright\) satisfies the following conditions:

1. \(\triangleright \) is commutative and associative
2. \(\triangleright \) is continuous
3. \(a \triangleright 0 = a \) for all \(a \in [0, 1]\)
4. \(a \triangleright b \leq c \triangleright d \) whenever \(a \leq c \) and \(b \leq d \) for all \(a, b, c, d \in [0, 1]\).

Examples: (i) \(a \triangleright b = \max\{a, b\}\) and (ii) \(a \triangleright b = \min\{1, a + b\}\).

**Definition 2.3 [1,3,19]** A 5-tuple \((X, M, N, *, \diamond)\) is said to be an intuitionistic fuzzy metric space if \(X\) is an arbitrary set, * is a continuous t-norm, \(\diamond \) is a continuous t-conorm and \(M, N\) are fuzzy sets on \(X^2 \times [0, \infty)\) satisfying the following conditions:

(IMF 1) \(M(x, y, t) + N(x, y, t) \leq 1 \) for all \(x, y \in X\) and \(t > 0\),
(IMF 2) \(M(x, y, 0) = 0\) for all \(x, y \in X\),
(IMF 3) \(M(x, y, t) = 1\) for all \(x, y \in X\) and \(t > 0\) if and only if \(x = y\),
(IMF 4) \(M(x, y, t) = M(y, x, t)\) for all \(x, y \in X\) and \(t > 0\),
(IMF 5) \(M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)\) for all \(x, y, z \in X\) and \(s, t > 0\).
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(IMF 6) For all \( x, y \in X \), \( M(x, y, \cdot) : [0, \infty) \to [0, 1] \) is left continuous,

(IMF 7) \( \lim_{t\to\infty} M(x, y, t) = 1 \) for all \( x, y \in X \) and \( t > 0 \),

(IMF 8) \( N(x, y, 0) = 1 \) for all \( x, y \in X \),

(IMF 9) \( N(x, y, t) = 0 \) for all \( x, y \in X \) and \( t > 0 \) if and only if \( x = y \),

(IMF 10) \( N(x, y, t) = N(y, x, t) \) for all \( x, y \in X \) and \( t > 0 \),

(IMF 11) \( N(x, y, t) \to N(y, z, s) \geq N(x, z, t + s) \) for all \( x, y, z \in X \) and \( s, t > 0 \),

(IMF 12) For all \( x, y \in X \), \( N(x, y, \cdot) : [0, \infty) \to [0, 1] \) is right continuous,

(IMF 13) \( \lim_{t\to\infty} N(x, y, t) = 0 \) for all \( x, y \in X \) and \( t > 0 \).

The pair \((M, N)\) is called an intuitionistic fuzzy metric on \( X \). The functions \( M(x, y, \cdot) \) and \( N(x, y, \cdot) \) denote the degree of nearness and degree of non-nearness between \( x \) and \( y \) with respect to \( t \), respectively.

**Remark 2.4** In intuitionistic fuzzy metric space \( M(x, y, \cdot) \) is non-decreasing and \( N(x, y, \cdot) \) is non-increasing for all \( x, y \in X \).

**Remark 2.5** Every fuzzy metric space \((X, M, \ast)\) is an intuitionistic fuzzy metric space of the form \((X, M, I-M, \ast, \triangleright)\) such that \( t\)-norm \( \ast \) and \( t\)-conorm \( \triangleright \) are associated, i.e., \( x \triangleright y = 1 - ((1 - x) \ast (1 - y)) \) for all \( x, y \in X \).

**Example 2.6** [19] Let \((X, d)\) be a metric space. Define \( t\)-norm \( a \ast b = \min\{a, b\} \) and \( t\)-conorm \( a \triangleright b = \max\{a, b\} \) and for all \( x, y \in X \) and \( t > 0 \) \( M \) and \( N \) are defined by

\[
M(x, y, t) = \frac{t}{t + d(x, y)} \quad \text{and} \quad N(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.
\]

Then \((X, M, N, \ast, \triangleright)\) is an intuitionistic fuzzy metric space induced by the metric \( d \). It is obvious that \( N(x, y, t) = 1 - M(x, y, t) \).

**Definition 2.7** [1] Let \((X, M, N, \ast, \triangleright)\) be an intuitionistic fuzzy metric space.

(i) A sequence \( \{x_n\} \) in \( X \) is said to be convergent to a point \( x \in X \) if for all \( t > 0 \),

\[
\lim_{n\to\infty} M(x_n, x, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} N(x_n, x, t) = 0.
\]

Since \( \ast \) and \( \triangleright \) are continuous, the limit is uniquely determined from (IMF5) and (IMF11) in Definition 2.3 respectively.

(ii) A sequence \( \{x_n\} \) is said to be Cauchy sequence if for all \( t > 0 \) and \( p > 0 \),

\[
\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0.
\]

(iii) The intuitionistic fuzzy metric space \( X \) is said to be complete if and only if every Cauchy sequence in \( X \) is convergent.

**Lemma 2.8** [1] Let \((X, M, N, \ast, \triangleright)\) be an intuitionistic fuzzy metric space and \( \{y_n\} \) be a sequence in \( X \). If there exists a number \( k \in (0, 1) \) such that:

\[
M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t) \quad \text{and} \quad N(y_{n+2}, y_{n+1}, kt) \leq N(y_{n+1}, y_n, t)
\]

for all \( t > 0 \) and \( n = 1, 2, 3, \ldots \), then \( \{y_n\} \) is a Cauchy sequence in \( X \).

**Definition 2.9** Let \( A \) and \( B \) be maps from an intuitionistic fuzzy metric space \((X, M, N, \ast, \triangleright)\) into itself. Then for all \( t > 0 \), maps \( A \) and \( B \) are said to be

(i) weakly commuting if
\[ M(ABx, BAx, t) \geq M(Ax, Bx, t) \quad \text{and} \quad N(ABx, BAx, t) \leq N(Ax, Bx, t), \quad \forall \ x \in X, \]

(ii) \( M(ABx, BAx, t) \geq M(Ax, Bx, t) \quad \text{and} \quad N(ABx, BAx, t) \leq N(Ax, Bx, t), \quad \forall \ x \in X, \]

\( \lim_{n \to \infty} M(ABx_n, BAx_n, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} N(ABx_n, BAx_n, t) = 0, \]

whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = z \) for some \( z \in X. \) (c.f. 23)

(iii) weakly compatible if for \( x \in X, ABx = BAx \) implies that

\[ M(ABx, BAx, t) = 1 \quad \text{and} \quad N(ABx, BAx, t) = 0, \]

(iv) semi weakly compatible if

\[ M(ABz, BAz, t) = 1 \quad \text{and} \quad N(ABz, BAz, t) = 0, \quad \text{where} \ z \ \text{is the fixed point of either} \ A \ \text{or} \ B. \]

Proposition 2.10 For two self maps \( A \) and \( B \) on an intuitionistic fuzzy metric space \( (X, M, N, *, \odot) \), the notion of commutativity \( \Rightarrow \) weakly commutativity \( \Rightarrow \) compatibility \( \Rightarrow \) weakly compatibility \( \Rightarrow \) commutativity at common fixed points, but the converse is not true always.

**Proof.** If \( A \) and \( B \) are commuting maps, then \( ABx = BAx \) for all \( x \) in \( X \), then

\[ 1 = M(ABx, BAx, t) \geq M(Ax, Bx, t) \quad \text{and} \quad 0 = N(ABx, BAx, t) \leq N(Ax, Bx, t), \quad \forall \ x \in X \]

implies that \( A \) and \( B \) are weakly commuting maps.

If \( A \) and \( B \) are weakly commuting maps and there exists a sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = y \in X \), then for all \( t > 0 \), we have

\[ M(ABx_n, BAx_n, t) \geq M(Ax_n, Bx_n, t) \to 1 \quad \text{and} \quad N(ABx_n, BAx_n, t) \leq M(Ax_n, Bx_n, t) \to 0 \]

implies that \( A \) and \( B \) are compatible maps.

If \( A \) and \( B \) are compatible maps and take \( x_n = x \) for all \( n \), then \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = Ax( = Bx) \in X \). Therefore for all \( t > 0 \), we have

\[ M(ABx, BAx, t) = M(ABx_n, BAx_n, t) \to 1 \quad \text{and} \quad N(ABx, BAx, t) = N(ABx_n, BAx_n, t) \to 0 \]

as \( n \to \infty \) yields that \( A \) and \( B \) are weakly compatible maps.

If suppose that \( A \) and \( B \) are weakly compatible maps and \( x \) is the common fixed point of \( A \) and \( B \) then \( x = Ax = Sx \) implies that \( M(ABx, BAx, t) = 1 \) and \( N(ABx, BAx, t) = 0 \) implies that \( ABx = BAx \).

Proposition 2.11. Let \( A \) and \( B \) be compatible and continuous self-maps on an intuitionistic fuzzy metric space \( (X, M, N, *, \odot) \). If there exists a sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = y \in X \), where \( y \) is fixed point of either \( A \) or \( B \). Then \( A \) and \( B \) are semi weakly compatible maps.

**Proof.** Suppose that \( y \) is a fixed point of \( A \) then \( Ay = y \). By the continuity of \( A \), \( ABx_n \to Ay \) as \( n \to \infty \). Now for \( s, t > 0 \)

\[ M(BAx_n, Ay, s + t) \geq M(BAx_n, ABx_n, s) \odot M(ABx_n, Ay, t) \to 1 \quad \text{and} \quad N(BAx_n, Ay, s + t) \leq N(BAx_n, ABx_n, s) \odot N(ABx_n, Ay, t) \to 1 \]
Letting $n \to \infty$ and using the compatibility of $A$ and $B$, we have $BAx_n \to Ay$. But by the continuity of $B$, $BSAx_n \to By$. Now by the uniqueness of the limit $Ay = By = y$ implies that $ABy = Ay = y = By = BAy$. Hence $A$ and $B$ are semi weakly compatible maps.

**Remark 2.12.** From the propositions (2.10) and (2.11), it is clear that for two self maps $A$ and $B$ on an intuitionistic fuzzy metric space $(X, M, N, *, \triangleleft)$, (i) Commutativity $\Rightarrow$ semi weakly compatibility of maps. (ii) Compatibility $\Rightarrow$ semi weakly compatibility of maps, if both the maps are continuous and $y = \limn_{\to \infty}Ax_n = \limn_{\to \infty}Bx_n \in X$ is a fixed point of either $A$ or $B$. But the converse of (i) and (ii) are not true always as we can see in the following examples.

**Example 2.13.** Let $(X, M, N, *, \triangleleft)$ be an intuitionistic fuzzy metric space, where $X = \mathbb{R}$, $M(x, y, t) = \frac{t}{t+|d(x,y)|}$ and $N(x, y, t) = \frac{d(x,y)}{t+|d(x,y)|}$, $\forall x, y \in X, t > 0$. Define self maps $A$ and $B$ on $X$ by $Ax = x^2$, $Bx = x^2 + 1$. Then we have $M(ABx, BAx, t) = \frac{t}{t+|x^2+(x^2+1)|^2-(x^2+1)^4} \not= 1$ and $N(ABx, BAx, t) = \frac{|(x^2+1)^2-(x^2+1)|}{t+|x^2+(x^2+1)^2-(x^2+1)|} \not= 0 \forall x \in X, t > 0$, implies that $A$ and $B$ are non-commuting maps. On the other hand at the fixed point $0$ of $A$, we have $M(AB0, BA0, t) = \frac{t}{t+|1-1|} = 1$ and $N(AB0, BA0, t) = \frac{|1-1|}{t+|1-1|} = 0$, implies that the maps $A$ and $B$ are semi weakly compatible.

**Example 2.14.** Let $(X, M, N, *, \triangleleft)$ be an intuitionistic fuzzy metric space, where $X = [0, 2]$, $M(x, y, t) = \frac{t}{t+|d(x,y)|}$ and $N(x, y, t) = \frac{d(x,y)}{t+|d(x,y)|}$, $\forall x, y \in X, t > 0$. Define self maps $A$ and $B$ on $X$ by

$$Ax = \begin{cases} 
0, & \text{if } x = 0, \\
\frac{x+1}{2}, & \text{otherwise}
\end{cases}, \quad Bx = \begin{cases} 
0, & \text{if } x = 0, \\
\frac{2x+2}{3}, & \text{otherwise}
\end{cases}.$$ 

Then it is easy to verify that $A$ and $B$ are non-commuting, non-continuous and non-compatible maps. On the other hand at the fixed point $1$ of $A$ and $2$ of $B$, we have $M(AB1, BA1, t) = \frac{t}{t+|0-0|} = 1$, $N(AB1, BA1, t) = \frac{|0-0|}{t+|0-0|} = 0$ and $M(AB2, BA2, t) = \frac{t}{t+|0-0|} = 1$, $N(AB2, BA2, t) = \frac{|0-0|}{t+|0-0|} = 0$, implies that the maps $A$ and $B$ are semi weakly compatible.

### 3. RESULTS

Pant et al [18] proved the following theorem on common fixed points of weakly compatible maps satisfying a new contractive type condition on non complete intuitionistic fuzzy metric spaces.

**Theorem 3.1 [18]** Let $(X, M, N, *, \triangleleft)$ be an intuitionistic fuzzy metric space (IFM-space) with continuous $t$-norm* and continuous $t$-conorm$\triangleleft$ defined by $t^*t \geq t$ and $(1-t)\triangleleft(1-t) \leq (1-t)$ for all $t \in [0, 1]$. Let $A, B, S, T, P$ and $Q$ be mappings from $X$ into itself such that:

1. $(P(X) \subseteq ST(X))$, $(Q(X) \subseteq AB(X))$, ...
There exists a constant $k \in (0, 1)$ such that
\[
M^2(Px, Qy, kt) [M(ABx, Px, kt) \cdot M(STy, Qy, kt)] \\
\geq [\alpha M(ABx, Px, t) + \beta M(ABx, STy, t)] M(ABx, Qy, 2kt)
\]
and
\[
N^2(Px, Qy, kt) [N(ABx, Px, kt) \cdot N(STy, Qy, kt)] \\
\leq [\alpha N(ABx, Px, t) + \beta N(ABx, STy, t)] N(ABx, Qy, 2kt)
\]
for all $x, y \in X$ and $t > 0$ where $0 < \alpha, \beta < 1$ such that $\alpha + \beta = 1$,

If one of $P(X), ST(X), AB(X), Q(X)$ is a complete subspace of $X$ then:
(i) $P$ and $AB$ have a coincidence point and (ii) $Q$ and $ST$ have a coincidence point.
Moreover, if

(iii) $AB = BA, ST = TS, PB = BP, QT = TQ$,
(iv) pairs $(P, AB)$ and $(Q, ST)$ are weakly compatible.
Then $A, B, S, T, P$ and $Q$ have a unique common fixed point in $X$.

Using the concept of weakly compatibility and semi-weakly compatibility of maps we extend and generalize the result of Pant et al [21] by proving the following results of common fixed point for such maps on non complete intuitionistic fuzzy metric spaces.

**Theorem 3.2** Let $(X, M, N, *, \circ)$ be an intuitionistic fuzzy metric space (IFM-space) with continuous $t$-norm* and continuous $t$-conorm* defined by $t * t \geq t$ and $(1 - t) \circ (1 - t) \leq (1 - t)$ for all $t \in [0, 1]$. Let $A, B, S, T, P$ and $Q$ be mappings from $X$ into itself such that:

(i) $P(X) \subseteq ST(X)$, $Q(X) \subseteq AB(X)$

(ii) There exists a constant $k \in (0, 1)$ such that
\[
M^2(Px, Qy, kt) [M(ABx, Px, kt) \cdot M(STy, Qy, kt)] \\
\geq [\alpha M(ABx, Px, t) + \beta M(ABx, STy, t)] M(ABx, Qy, 2kt)
\]
and
\[
N^2(Px, Qy, kt) [N(ABx, Px, kt) \cdot N(STy, Qy, kt)] \\
\leq [\alpha N(ABx, Px, t) + \beta N(ABx, STy, t)] N(ABx, Qy, 2kt)
\]
for all $x, y \in X$ and $t > 0$ where $0 < \alpha, \beta < 1$ such that $\alpha + \beta = 1$,

(iii) If one of $P(X), ST(X), AB(X), Q(X)$ is a complete subspace of $X$ then:

(a) $P$ and $AB$ have a coincidence point and (b) $Q$ and $ST$ have a coincidence point.
Moreover, if

(iv) pairs $(P, AB)$ and $(Q, ST)$ are weakly compatible.
(v) pairs $(A, B)$ and $(S, T)$ are commuting maps,
(vi) pairs $(P, A)$, $(P, B)$, $(S, Q)$ and $(T, Q)$ are semi-weakly compatible maps.
Then $A, B, S, T, P$ and $Q$ have a unique common fixed point in $X$.

**Proof.** Let $x_0$ be an arbitrary point of $X$. By (3.2.1), there exist $x_1, x_2 \in X$ such that $Px_0 = STx_1 = y_0$ and $Qx_1 = ABx_2 = y_1$. Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that $Px_{2n} = STx_{2n+1} = y_{2n}$ and $Qx_{2n+1} = ABx_{2n+2} = y_{2n+1}$ for $n = 0, 1, 2, \ldots$. By taking $x = x_{2n}$ and $y = x_{2n+1}$ in (3.2.2), we have
\[
M^2(Px_{2n}, Qx_{2n+1}, kt) [M(ABx_{2n}, Px_{2n}, kt) \cdot M(STx_{2n+1}, Qx_{2n+1}, kt)] \\
\geq [\alpha M(ABx_{2n}, Px_{2n}, t) + \beta M(ABx_{2n}, STx_{2n+1}, t)] M(ABx_{2n}, Qx_{2n+1}, 2kt)
\]
and...
By putting \( x = w, y = x_{2n+1} \) in (3.2.2) and taking limit as \( n \to \infty \), we have

\[
M^2(Pw, z, kt)*[M(z, Pw, kt)-M(z, z, kt)] \geq [\alpha M(z, Pw, t) + \beta M(z, z, t)]M(z, z, 2kt)
\]

and

\[
N^2(Pw, z, kt)*[N(z, Pw, kt)-N(z, z, kt)] \geq [\alpha N(z, Pw, t) + \beta N(z, z, t)]N(z, z, 2kt)
\]

Thus it follows that

\[
M(z, Pw, kt) \geq \alpha M(z, Pw, t) + \beta \geq \alpha M(z, Pw, t) + \beta + \beta = M(z, Pw, kt) \geq [\beta/(1-\alpha)] = 1
\]

and

\[
N(z, Pw, kt) \leq 0.
\]

Hence \( z = Pw \). Since \( ABw = z \) thus we have \( Pw = z = ABw \) that is \( w \) is a coincidence point of \( P \) and \( AB \). Since \( P(X) \subseteq ST(X) \), \( Pw = z \) implies that \( z \in ST(X) \). Let \( v = ST^{-1}z \). Then \( STv = z \).

By putting \( x = x_{2n} \) and \( y = v \) in (3.2.2) and taking limit as \( n \to \infty \), we have

\[
M^2(z, Qv, kt)*[M(z, z, kt)-M(z, Qv, kt)] \geq [\alpha M(z, z, t) + \beta M(z, z, t)]M(z, Qv, 2kt)
\]
and

\[ N^2(z, Qv, kt) \circ [N(z, z, kt) - N(z, Qv, kt)] \leq [\alpha N(z, z, t) + \beta N(z, z, t)]N(z, Qv, 2kt). \]

Thus we have \( M(z, Qv, kt) \geq 1 \) and \( N(z, Qv, kt) \leq 0 \). Thus, \( z = Qv \). Since \( STv = z \), we have \( Qv = z = STv \) that is \( v \) is coincidence point of \( Q \) and \( ST \). This proves (b). The remaining two cases pertain essentially to the previous cases. Indeed if \( P(X) \) or \( Q(X) \) is complete then by (3.2.1) \( z \in P(X) \subset ST(X) \) or \( z \in Q(X) \subset AB(X) \). Thus (a) and (b) are completely established.

Since the pair \( (P, AB) \) is weakly compatible therefore \( P \) and \( AB \) commute at their coincidence point that is \( z = Qv = z = ABz \). Similarly \( (Q, ST) \) is weakly compatible therefore \( Q \) and \( ST \) commute at their coincidence point that is \( z = STz = z \).

By putting \( x = z, y = x_{2n+1} \) in (3.2.2) and taking limit as \( n \to \infty \), we have

\[ M^2(Pz, Qz, kt) \circ [M(ABz, Pz, kt) - M(z, z, kt)] \geq [\alpha M(ABz, Pz, t) + \beta M(ABz, z, t)]M(ABz, z, 2kt). \]

and

\[ N^2(Pz, Qz, kt) \circ [N(ABz, Pz, kt) - N(z, z, kt)] \leq [\alpha N(ABz, Pz, t) + \beta N(ABz, z, t)]N(ABz, z, 2kt). \]

Then we have \( M(z, Qz, kt) \geq 1 \) and \( N(z, Qz, kt) \leq 0 \), therefore \( z = Qz \). Hence \( Qz = STz = z \), i.e., \( z \) is the unique common fixed point of \( P \), \( Q \), \( AB \) and \( ST \).

From (3.2.5 & 3.2.6), we have \( Az = A(ABz) = A(ABz) = (AB)Az \); \( Az = APz = PAz \) and \( Bz = B(ABz) = (BA)Bz = (AB)Bz \); \( Bz = BPz = PBz \), implies that \( Az \) and \( Bz \) are common fixed points of \( (AB, P) \) therefore \( z = Az = Bz = Pz = ABz \). Similarly, \( Sz \) and \( Tz \) are common fixed points of \( (ST, Q) \) therefore \( z = Sz = Tz = Qz = STz \). Hence \( z \) is the unique common fixed point of \( A, B, S, T, P \) and \( Q \). Further, since \( z \) is the unique common fixed point of \( P, Q, AB \) and \( ST \), consequently it is the unique common fixed point of \( A, B, S, T, P \) and \( Q \). This completes the proof.
Common Fixed Point of Weakly Compatible Maps in Intuitionistic Fuzzy Metric Spaces

If we put $B = T = I_X$ (The identity map on $X$) in Theorem 3.2, we have the following:

**Corollary 3.3** Let $(X, M, N, *, \tilde{\circ})$ be an intuitionistic fuzzy metric space (IFM-space) with continuous $t$-norm $*$ and continuous $t$-conorm $\tilde{\circ}$ defined by $t * t \geq t$ and $(1 - t)\tilde{\circ}(1 - t) \leq (1 - t)$ for all $t \in [0, 1]$. Let $A, S, P$ and $Q$ be mappings from $X$ into itself such that:

(3.3.1) $P(X) \subseteq S(X), Q(X) \subseteq A(X),$  
(3.3.2) There exists a constant $k \in (0, 1)$ such that  
\[
M^2(Px, Qy, kt) \ast [M(Ax, Px, kt) \ast M(Sy, Qy, kt)]  
\geq [\alpha M(Ax, Px, t) + \beta M(Ax, Sy, t)] M(Ax, Qy, 2kt)  
\]
and  
\[
N^2(Px, Qy, kt) \ast [N(Ax, Px, kt) \ast N(Sy, Qy, kt)]  
\leq [\alpha N(Ax, Px, t) + \beta N(Ax, Sy, t)] N(Ax, Qy, 2kt)  
\]
for all $x, y \in X$ and $t > 0$ where $0 < \alpha, \beta < 1$ such that $\alpha + \beta = 1$.

(3.3.3) If one of $P(X), S(X), A(X), Q(X)$ is a complete subspace of $X$ then,  
(a) $P$ and $A$ have a coincidence point and  
(b) $Q$ and $S$ have a coincidence point.

Moreover, if  
(3.3.4) pairs $(P, A)$ and $(Q, S)$ are weakly compatible.

Then $A, S, P$ and $Q$ have a unique common fixed point in $X$.

If we take $A = S$ and $P = Q$ in Corollary 3.3, we have the following:

**Corollary 3.4** Let $(X, M, N, *, \tilde{\circ})$ be an intuitionistic fuzzy metric space (IFM-space) with continuous $t$-norm $*$ and continuous $t$-conorm $\tilde{\circ}$ defined by $t * t \geq t$ and $(1 - t)\tilde{\circ}(1 - t) \leq (1 - t)$ for all $t \in [0, 1]$. Let $A$ and $P$ be mappings from $X$ into itself such that:

(3.4.1) $P(X) \subseteq A(X),$  
(3.4.2) There exists a constant $k \in (0, 1)$ such that  
\[
M^2(Px, Py, kt) \ast [M(Ax, Px, kt) \ast M(Ay, Py, kt)]  
\geq [\alpha M(Ax, Px, t) + \beta M(Ax, Ay, t)] M(Ax, Py, 2kt)  
\]
and  
\[
N^2(Px, Py, kt) \ast [N(Ax, Px, kt) \ast N(Ay, Py, kt)]  
\leq [\alpha N(Ax, Px, t) + \beta N(Ax, Ay, t)] N(Ax, Py, 2kt)  
\]
for all $x, y \in X$ and $t > 0$ where $0 < \alpha, \beta < 1$ such that $\alpha + \beta = 1$.

(3.4.3) If one of $A(X), P(X)$ is a complete subspace of $X$ then $P$ and $A$ have a coincidence point.

Moreover, if  
(3.4.4) pair $(P, A)$ is weakly compatible.

Then $A$ and $P$ have a unique common fixed point in $X$. 
4. CONCLUSION

Our result is an improved extension of the results of Alaca, Altun and Turkoglu [2], Alaca, Turkoglu and Yildiz [4] and Pant et al [18] in the sense that we have taken discontinuous maps with a new contractive condition on non-complete intuitionistic fuzzy metric space. Also, our result has been proved for weakly compatible and semi weakly compatible maps as these concepts are most general among all the commutativity concepts.

REFERENCES


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