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A New Approach for Ranking k+1-Trapezoidal Fuzzy Numbers

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Abstract

We generalize the ranking method introduced in [6] and [7] to rank particular k+1-trapezoidal fuzzy numbers. We build a Java program to implement the approach. We give some numerical examples and suggestions for future work.

Keywords: Particular Fuzzy Numbers, k+1-Trapezoidal Fuzzy Numbers, Fuzzy Comparison Method, Ranking Function.

1. Introduction

Fuzzy set theory is a powerful tool to deal with real life situations. Many articles were published since introducing the concepts of Fuzzy sets and Probability Measure of Fuzzy Events and fuzzy linear programming[1-16]. Ranking fuzzy numbers plays an important role in many fuzzy optimization problems and decision-making procedure ([1-3], [6-7] and [9-12]).

Dadgostar [1] proposed a fuzzy number comparison method called Partial Comparison Method. It is based on the fuzzy number division in comparison, the shapes of the convex numbers do not require special computations during comparison because it relies on the representation of fuzzy numbers as ordered sets of confidence intervals. Dorohonceanu and Marin [2] presented a fuzzy number comparison method based on the fuzzy number representations in fuzzy arithmetic described in [3]. They also described a variant based on the fuzzy number division in comparison used in partial comparison method. Triangular and trapezoidal shapes of membership functions were used to describe the method. In [8], we applied the method presented in [2] to the fuzzy numbers introduced in [8]. In [10] we generalized the particular fuzzy numbers introduced in [8] by defining three particular fuzzy numbers determined by *n* real numbers and constants c_i , $0 \le c_i \le 1$, $i=1, \ldots, k$, and we used the fuzzy comparison methods mentioned in [2] and [9] to compare between them. The

method is based on the fuzzy number representation in fuzzy arithmetic and fuzzy number division in comparison used in [1].
Rao and Shankar [12] proposed a new method which is based on centroid of centroids to rank fuzzy quantities. A trapezoidal fuzzy number was split into three parts, two triangles and a rectangle. Then the centroids of these three parts were calculated followed by the calculation of the centroid of these centroids. Then a ranking procedure was defined depending on area and using mode and spreads in some cases.
In [11], we generalized the method introduced in [12] and used it to rank the fuzzy numbers introduced in [10].
Kumar et al [6-7] proposed a new approach for the ranking of generalized trapezoidal fuzzy numbers. It was shown that the ranking method proposed by Chen and Chen [3] is incorrect. The proposed approach in [6-7] provided the correct ordering of generalized and normal trapezoidal fuzzy numbers. It is simple and easy to apply in the real life problems.In this paper, we generalize the approach in [6-7] for k+1-trapezoidal fuzzy numbers introduced in [10].
In section 2, we introduce the basic concepts, definitions and particular types of fuzzy numbers. In section 3, we mention particular types of fuzzy numbers. In section 4, we introduce the ranking method and proof some useful results. In section 5, we support the generalized method by a computerized solution. Numerical examples are given in section 6. Finally, a conclusion and comments are mentioned in section 7.

2. Fuzzy Sets and Numbers

Definition 1: A fuzzy subset A of some set Ω is defined by its membership function written $A(x)$ which produces values in $[0,1]$ for all x in Ω. That is $A(x)$ is a function mapping Ω into $[0,1]$. We place a bar over a letter to denote a fuzzy set, that is $\bar{A}$. The term crisp means not fuzzy. A crisp set is a regular set.

Definition 2: Let $\Omega = R$. An $\alpha-$cut of $\bar{A}$, written $\bar{A}[\alpha]$, is defined as $\{x : \bar{A}(x) \geq \alpha\}$, for $0 < \alpha \leq 1$. $\bar{A}[0]$, the support of $\bar{A}$ is defined as the closure of the union of all the $\bar{A}[\alpha]$, for $0 < \alpha \leq 1$.

Definition 3: A confidence interval is an interval of real numbers that provides a representation for an imprecise numerical value by means of its sharpest enclosing range.

Definition 4: A presumption level is an estimated truth-value about some knowledge. Presumption levels belong to the [0, 1] interval; the maximum of estimated truth-value is at level 1 and the minimum is at level 0.

Definition 5: A fuzzy number $\bar{N}$ is a fuzzy subset of the real numbers satisfying: (1) $\exists x \in R : \bar{N}(x) = 1$ (2) $\bar{N}[\alpha]$ is a closed and bounded interval for $0 \leq \alpha \leq 1$.

A special type of fuzzy numbers $\bar{M}$ is called a triangular fuzzy number. $\bar{M}$ is defined by three numbers $a_1 < a_2 < a_3$ where (1) $\overline{\overline{M}}(x) = 1$ at $x = a_2$ (2) The graph of $\overline{\overline{M}}(x)$ on $[a_1, a_2]$ is a straight line from $(a_1, 0)$ to $(a_2, 1)$ and also on $[a_2, a_3]$ the graph is a straight line from $(a_2, 1)$ to $(a_3, 0)$ (3) $\overline{\overline{M}}(x) = 0$ for $x \le a_1$ or $x \ge a_3$. We write $\overline{\overline{M}} = (a_1 / a_2 / a_3)$ for triangular fuzzy number $\overline{\overline{M}}$. If at least one of the graphs described above is not a straight line (curve), then $\overline{\overline{M}}$ is called triangular shaped fuzzy number and we write $\overline{\overline{M}} \approx (a_1 / a_2 / a_3)$.

Another special type of fuzzy numbers $\overline{\overline{M}}$ is called a trapezoidal fuzzy number. $\overline{\overline{M}}$ is defined by four numbers $a_1 < a_2 < a_3 < a_4$ where (1) $\overline{\overline{M}}(x) = 1$ on $[a_2, a_3]$ (2) The graph of $\overline{\overline{M}}(x)$ on $[a_1, a_2]$ is a straight line from $(a_1, 0)$ to $(a_2, 1)$ and also on $[a_3, a_4]$ the graph is a straight line from $(a_3, 1)$ to $(a_4, 0)$ (3) $\overline{\overline{M}}(x) = 0$ for $x \le a_1$ or $x \ge a_4$. We write $\overline{\overline{M}} = (a_1, a_2, a_3, a_4)$ for trapezoidal fuzzy number $\overline{\overline{M}}$. If at least one of the graphs described above is not a straight line (curve), then $\overline{\overline{M}}$ is called trapezoidal shaped fuzzy number and we write $\overline{\overline{M}} \approx (a_1, a_2, a_3, a_4)$.

If $\overline{\overline{M}}(x) = w < 1$ on $[a_2, a_3]$, then it is called a generalized trapezoidal fuzzy number.

A fuzzy number is represented as an ordered set of confidence intervals, each of them provides the related numerical value at a given presumption level $\alpha \in [0,1]$. These confidence intervals should comply with the relation if $\alpha_1 > \alpha_2$ then $A_{\alpha_1} \subset A_{\alpha_2}$, where $\alpha_1, \alpha_2 \in [0,1]$ and $A_{\alpha_1}, A_{\alpha_2}$ are the confidence intervals at presumption levels $\alpha_1, \& \alpha_2$ respectively. More details, properties and operations, can be found in [4-5] and [16].

3. Particular Fuzzy Numbers

3.1 Introduction

In [8], we introduced other types of fuzzy numbers. We considered a fuzzy number that is determined by five real numbers a_1, a_2, a_3, a_4 and c such that $a_1 < a_2 < a_3 < a_4$ and $0 < c < 1$, denoted by $\overline{\overline{N}}_c = (a_1 / a_2 / a_3 / a_4)$ or $(a_1, a_2, a_3, a_4)_c$ whose membership function is given by

$$\overline{\overline{N}}_c(x) = \begin{cases} 0, x \le a_1 \\ \frac{c}{a_2 - a_1}(x - a_1), a_1 \le x \le a_2 \\ 1 - \frac{1-c}{(a_2 - a_3)^2}(2x - a_2 - a_3)^2, a_2 \le x \le a_3 \\ \frac{-c}{a_4 - a_3}(x - a_4), a_3 \le x \le a_4 \\ 0, x \ge a_4 \end{cases}$$

$\overline{\overline{N}}_c(x)$ is a line from $(a_1,0)$ to (a_2,c).

$\overline{\overline{N}}_c(x)$ is a parabola from (a_2,c) to (a_3,c) whose vertex is $(\frac{a_2+a_3}{2},1)$ and focus is $(\frac{a_2+a_3}{2},1-\frac{(a_3-a_2)^2}{16(1-c)})$.

$\overline{\overline{N}}_c(x)$ is a line from (a_3,c) to $(a_4,0)$.

We called such a fuzzy number a Trapezoidal-Parabolic Fuzzy Number and if it is determined by five numbers and it is not of this form, we called it a Trapezoidal-Parabolic Shaped Fuzzy Number.

If we put $c = 0$ in the membership function of the trapezoidal-parabolic fuzzy number, then we get another fuzzy number. We called it parabolic fuzzy number, and is defined by two real numbers a_1 & a_2 with $a_1 < a_2$, denoted by $\overline{\overline{N}} = (a_1 / a_2)$ or $\overline{\overline{N}} = (a_1, a_2)$. The membership function $\overline{\overline{N}}(x)$ is given by

$$\overline{\overline{N}}(x) = \begin{cases} 0, x \le a_1 \\ 1 - \frac{1}{(a_1 - a_2)^2}(2x - a_1 - a_2)^2, a_1 \le x \le a_2 \\ 0, x \ge a_2 \end{cases}$$

$\overline{\overline{N}}(x)$ is parabolic in $[a_1, a_2]$ whose vertex is $(\frac{a_1+a_2}{2},1)$ and focus $(\frac{a_1+a_2}{2},1-\frac{(a_1-a_2)^2}{16})$.

If $\overline{\overline{N}}(x)$ is not parabolic in $[a_1, a_2]$ then it is called a parabolic shaped fuzzy number and is denoted by $\overline{\overline{N}} \approx (a_1 / a_2)$ or $\overline{\overline{N}} \approx (a_1, a_2)$.

$$\overline{\overline{N}}[\alpha] = [\frac{-(a_2 - a_1)}{2}\sqrt{1-\alpha} + \frac{a_2 + a_1}{2},$$
$$\frac{(a_2 - a_1)}{2}\sqrt{1-\alpha} + \frac{a_2 + a_1}{2}]$$

for $0 \le \alpha \le 1$.

$\overline{\overline{N}} \approx (a_1 / \frac{a_1 + a_2}{2} / a_2)$ is a triangular shaped fuzzy number.

In [10] we generalized the fuzzy numbers introduced in [8].

3.2 K-Trapezoidal-Triangular Fuzzy Numbers

We considered a fuzzy number that is determined by *n* real numbers $a_1, a_2, ..., a_{2k+3}$ and c_i, $0 \le c_i \le 1$, *i =1, 2, ..., k* where *n* is a multiple of 3 and $a_1 < a_2 < .... < a_{2k+3}$. Denote it by $\overline{A}_{(c_1,....c_k)} = (a_1, a_2, ..., a_{2k+3})$ and whose membership function is given by

$$\mu_{\overline{A}_{(c_1,...,c_k)}}(x) = \begin{cases} 0, x \le a_1 \\ \frac{c_1(x-a_1)}{a_2-a_1}, a_1 \le x \le a_2 \\ . \\ . \\ \frac{-c_1(x-a_{2k+3})}{a_{2k+3}-a_{2k+2}}, a_{2k+2} \le x \le a_{2k+3} \\ 0, x \ge a_{2k+3} \end{cases}$$

1- $\mu_{\overline{A}_{(c_1,...,c_k)}}(x)$ is a line from $(a_1, 0)$ to (a_2, c_1) and from (a_{2k+2}, c_1) to $(a_{2k+3}, 0)$

2- $\mu_{\overline{A}_{(c_1,...,c_k)}}(x)$ is a line from (a_{i+1}, c_i) to (a_{i+2}, c_{i+1}), i=1, 2, …., k-1.

3- $\mu_{\overline{A}_{(c_1,...,c_k)}}(x)$ is a line from (a_{k+3+i}, c_{k-i}) to (a_{k+4+i}, c_{k-i-1}), i=0, 1, 2, …., k-2.

3- $\mu_{\overline{A}_{(c_1,...,c_k)}}(x)$ is a line from (a_{k+1}, c_k) to $(a_{k+2}, 1)$ and from $(a_{k+2}, 1)$ to (a_{k+3}, c_k)

4- $\mu_{\overline{A}_{(c_1,...,c_k)}}(x) = 0$ for $x \le a_1, x \ge a_{2k+3}$.

We called such a fuzzy number a k-trapezoidal-triangular fuzzy number. If k=0, a k-trapezoidal–triangular Fuzzy Number is a triangular Fuzzy Number.

3.3 K+1-Trapezoidal Fuzzy Numbers

We also considered a fuzzy number that is determined by n real numbers $a_1, a_2, ..., a_{2k+3}$ and c_i, $0 \le c_i \le 1$, *i=1, 2, …, k* where n=4, 7, 10, …and $a_1 < a_2 < .... < a_{2k+4}$. Denote it by $\overline{A}_{(c_1,.....,c_k)} = (a_1, ..., a_{2k+4})$ and whose membership function is given by

$$\mu_{\overline{A}_{(c_1,...,c_k)}}(x) = \begin{cases} 0, x \le a_1 \\ \frac{c_1(x-a_1)}{a_2-a_1}, a_1 \le x \le a_2 \\ . \\ . \\ 1, a_{k+2} \le x \le a_{k+3} \\ . \\ . \\ \frac{-c_1(x-a_{2k+4})}{a_{2k+4}-a_{2k+3}}, a_{2k+3} \le x \le a_{2k+4} \\ 0, x \ge a_{2k+4} \end{cases}$$

1- $\mu_{\overline{A}_{(c_1,\dots,c_k)}}(x)$ is a line from $(a_1,0)$ to (a_2,c_1) and from (a_{2k+3},c_1) to $(a_{2k+4},0)$

2- $\mu_{\overline{A}_{(c_1,\dots,c_k)}}(x)$ is a line from (a_{i+1},c_i) to (a_{i+2},c_{i+1}), i=1, 2, …., k-1.

3- $\mu_{\overline{A}_{(c_1,\dots,c_k)}}(x)$ is a line from (a_{k+4+i},c_{k-i}) to (a_{k+5+i},c_{k-i-1}), i=0, 1, 2, …., k-2.

4- $\mu_{\overline{A}_{(c_1,\dots,c_k)}}(x)$ is a line from (a_{k+1},c_k) to $(a_{k+2},1)$ and from $(a_{k+3},1)$ to (a_{k+4},c_k)

5- $\mu_{\overline{A}_{(c_1,\dots,c_k)}}(x) = 1$ if $a_{k+2} \le x \le a_{k+3}$.

6- $\mu_{\overline{A}_{(c_1,\dots,c_k)}}(x) = 0$ for $x \le a_1, x \ge a_{2k+4}$.

We called such a fuzzy number a k+1-trapezoidal fuzzy number. If k=0, a k+1-trapezoidal Fuzzy Number is a trapezoidal Fuzzy Number.

3.4 K-Trapezoidal-Parabolic Fuzzy Numbers

We also Considered a fuzzy number that is determined by n real numbers $a_1, a_2, \dots, a_{2k+2}$ and c_i, $0 \le c_i \le 1$, *i=1, 2, …, k* where n=5, 8, 11, … and $a_1 < a_2 < \dots < a_{2k+2}$. Denote it by $\overline{A}_{(c_1,\dots,c_k)} = (a_1,\dots,a_{2k+2})$ and whose membership function is given by

$$\mu_{\overline{A}_{(c_1,\dots,c_k)}}(x) = \begin{cases} 0, x \le a_1 \\ \dfrac{c_1(x-a_1)}{a_2-a_1}, a_1 \le x \le a_2 \\ . \\ . \\ 1 - \dfrac{(1-c_k)}{(a_{k+1}-a_{k+2})^2}(2x - a_{k+1} - a_{k+2})^2, \\ a_{k+1} \le x \le a_{k+2} \\ . \\ . \\ \dfrac{-c_1(x-a_{2k+2})}{a_{2k+2}-a_{2k+1}}, a_{2k+1} \le x \le a_{2k+2} \\ 0, x \ge a_{2k+2} \end{cases}$$

1- $\mu_{\overline{A}_{(c_1,\dots,c_k)}}(x)$ is a line from $(a_1,0)$ to (a_2,c_1) and from (a_{2k+1},c_1) to $(a_{2k+2},0)$

2- $\mu_{\overline{A}_{(c_1,\dots,c_k)}}(x)$ is a line from (a_{i+1},c_i) to (a_{i+2},c_{i+1}), i=1, 2, …., k-1.

3- $\mu_{\overline{A}_{(c_1,\dots,c_k)}}(x)$ is a line from (a_{k+2+i},c_{k-i}) to (a_{k+3+i},c_{k-i-1}), i=0, 1, 2, …., k-2.

4- $\mu_{\overline{A}_{(c_1,\dots,c_k)}}(x)$ is a parabola from (a_{k+1},c_k) to (a_{k+2},c_k) whose vertex $(\dfrac{a_{k+1}+a_{k+2}}{2},1)$

and focus is $(\frac{a_{k+1}+a_{k+2}}{2}, 1-\frac{(a_{k+2}-a_{k+1})^2}{16(1-c_k)})$

6- $\mu_{\bar{A}_{(c_1,\dots,c_k)}}(x)=0$ for $x \le a_1, x \ge a_{2k+2}$.

We called such a fuzzy number a k-trapezoidal –parabolic fuzzy number. If *k=0*, a k-trapezoidal – parabolic Fuzzy Number is a parabolic Fuzzy Number. If k=1, a k-trapezoidal– parabolic Fuzzy Number is a trapezoidal–parabolic Fuzzy Number.

Definition 6 Let $T_1=(a_1,a_2,a_{2k+2},a_{2k+3};c_1),\dots,T_i=(a_i,a_{i+1},a_{2k+3-i},a_{2k+3-(i-1)};c_i-c_{i-1}), i=2,\dots,k$ be generalized trapezoidal fuzzy number and $T_{k+1}=(a_{k+1},a_{k+2},a_{k+3};1-c_k)$ be a generalized triangular fuzzy number. If

$\bar{A}=(T_1,T_2,\dots,T_k,T_{k+1})=\bar{A}_{(c_1,\dots,c_k)}=(a_1,\dots,a_{2k+3})$ & $\bar{B}=(T_{1^-},T_{2^-},\dots,T_{k^-},T_{k+1^-})=\bar{B}_{(s_1,\dots,s_k)}=(b_1,\dots,b_{2k+3})$ are two k-trapezoidal-triangular fuzzy numbers. Then:

1) $\bar{A}\oplus\bar{B}=(T_1\oplus T_{1^-},T_2\oplus T_{2^-},\dots,T_k\oplus T_{k^-},T_{k+1}\oplus T_{k+1^-})$

where $T_i\oplus T_{i^-}=(a_i+b_i,a_{i+1}+b_{i+1},a_{2k+3-i}+b_{2k+3-i},a_{2k+3-(i-1)}+b_{2k+3-(i-1)};\min(c_i-c_{i-1},s_i-s_{i-1}))$

and $T_{k+1}\oplus T_{k+1^-}=(a_{k+1}+a_{k+1^-},a_{k+2}+a_{k+2^-},a_{k+3}+a_{k+3^-};\min(1-c_k,1-s_k))$

2) $\bar{A}\Theta\bar{B}=(T_1\Theta T_{k+1^-},T_2\Theta T_{k^-},\dots,T_k\Theta T_{2^-},T_{k+1}\Theta T_{1^-})$

where $T_i\Theta T_{i^-}=(a_i-b_{2k+3-(i-1)},a_{i+1}-b_{2k+3-i},a_{2k+3-i}-b_{i+1},a_{2k+3-(i-1)}-b_i;\min(c_i-c_{i-1},s_i-s_{i-1}))$

and $T_{k+1}\Theta T_{k+1^-}=(a_{k+1}-b_{k+3},a_{k+2}-b_{k+2},a_{k+3}-b_{k+1};\min(1-c_k,1-s_k))$

3) If *M>0*, then

$M\bar{A}=(MT_1,MT_2,\dots,MT_k,MT_{k+1})$ where

$MT_i=(Ma_i,Ma_{i+1},Ma_{2k+3-i},Ma_{2k+3-(i-1)};c_i-c_{i-1})$ and

$MT_{k+1}=(Ma_{k+1},Ma_{k+2},Ma_{k+3};1-c_k)$

and if *M<0*, then

$M\bar{A}=(MT_{k+1},MT_k,\dots,MT_2,MT_1)$ where

$MT_i=(Ma_{2k+3-(i-1)},Ma_{2k+3-i},Ma_{i+1},Ma_i;c_i-c_{i-1})$

and

$MT_{k+1}=(Ma_{k+1},Ma_{k+2},Ma_{k+3};1-c_k)$.

Notice that the same operations can be defined similarly for k+1-trapezoidal fuzzy numbers and all fuzzy numbers mentioned above.

Definition7 Let $\bar{A}_{(c_1,\dots,c_k)}=(a_1,\dots,a_{2k+4})$ and $\bar{B}_{(s_1,\dots,s_k)}=(b_1,\dots,b_{2k+4})$ be k+1 trapezoidal fuzzy numbers and $w_i = \min(c_i, s_i)$ $i=1, ..k$, $w_0 = 0$ then:

(i) $\Re(\overline{A}) = \sum_{i=0}^{k-1} \frac{w_i}{4}(a_i - a_{i+2} - a_{2k+3-i} + a_{2k+5-i}) + \frac{w_k}{4}(a_k - a_{k+2} - a_{k+3} + a_{k+5}) + \frac{1}{4}(a_{k+1} + a_{k+2} + a_{k+3} + a_{k+4})$

(ii) $\text{mode}(\overline{A}) = \sum_{i=0}^{k-1} \frac{w_i}{2}(a_{i+1} - a_{i+2} - a_{2k+3-i} + a_{2k+4-i}) + \frac{w_k}{2}(a_{k+1} - a_{k+2} - a_{k+3} + a_{k+4}) + \frac{a_{k+2} + a_{k+3}}{2}$

(iii) spread $(\overline{A}) = \sum_{i=0}^{k-1} w_i(-a_i + a_{i+1} - a_{2k+4-i} + a_{2k+5-i})$

(iv) $+ w_k(-a_k + a_{k+1} - a_{k+4} + a_{k+5}) + a_{k+4} - a_{k+1}$

left spread $(\overline{A}) = \sum_{i=0}^{k-1} w_i(-a_i + 2a_{i+1} - a_{i+2})$

(v) $+ w_k(-a_k + 2a_{k+1} - a_{k+2}) + a_{k+2} - a_{k+1}$ right spread $(\overline{A}) = \sum_{i=0}^{k-1} w_i(a_{2k+3-i} - 2a_{2k+4-i} + a_{2k+5-i})$ $+ w_k(a_{k+3} - 2a_{k+4} + a_{k+5}) + a_{k+4} - a_{k+3}$

In [9-10] we used the method in [1-2] to compare between such numbers.

4. Ranking Method

Let $\overline{A}_{(c_1,\dots,c_k)} = (a_1,..,a_{2k+4})$ and $\overline{B}_{(s_1,\dots,s_k)} = (b_1,...,b_{2k+4})$ be two k+1-trapezoidal fuzzy numbers and $w_i = \min(c_i, s_i)$, $w_0 = 0$. We use the following steps to compare between $\overline{A}$ and $\overline{B}$:

Step 1: We calculate $\Re(\overline{A})$ and $\Re(\overline{B})$

Case (i) $\Re(\overline{A}) > \Re(\overline{B}) \Rightarrow \overline{A} \succ \overline{B}$

Case (ii) $\Re(\overline{A}) < \Re(\overline{B}) \Rightarrow \overline{A} \prec \overline{B}$

Case (iii) If $\Re(\overline{A}) = \Re(\overline{B})$ then go to step 2.

Step 2: We calculate mode $(\overline{A})$ and mode $(\overline{B})$

Case (i) mode $(\overline{A})$ > mode $(\overline{B}) \Rightarrow \overline{A} \succ \overline{B}$

Case (ii) mode $(\overline{A})$ < mode $(\overline{B}) \Rightarrow \overline{A} \prec \overline{B}$

Case (iii) If mode $(\overline{A})$ = mode $(\overline{B})$ then go to Step 3.

Step3: We calculate spread $(\overline{A})$ and spread $(\overline{B})$

case(i) spread $(\overline{A})$ > spread $(\overline{B}) \Rightarrow \overline{A} \succ \overline{B}$

case(ii) spread $(\overline{A})$ < spread $(\overline{B}) \Rightarrow \overline{A} \prec \overline{B}$

case(iii) If spread $(\overline{A})$ = spread $(\overline{B})$ then go to step 4.

Step 4: We calculate left spread $(\overline{A})$ and left spread $(\overline{B})$

case(i) left spread $(\overline{A})$ > left spread $(\overline{B}) \Rightarrow \overline{A} \succ \overline{B}$

case(ii) left spread$(\overline{A})$ < *left* spread$(\overline{B}) \Rightarrow \overline{A} \prec \overline{B}$

case(iii) left spread$(\overline{A})$ = left spread$(\overline{B})$ then go to step 5.

Step5: Finally we find $c = \sum_{i=1}^{k} c_i$ and $s = \sum_{i=1}^{k} s_i$

Case(i) if c>s then $\overline{A} \succ \overline{B}$

Case(ii) if c<s then $\overline{A} \prec \overline{B}$

Case(iii) if c=s then $\overline{A} \approx \overline{B}$

Now we state and prove some useful results for the ranking method.

Proposition 4.1 let $\overline{A}_{(c_1,..,c_k)} = (a_1,..,a_{2k+4})$ and $\overline{B}_{(s_1,...,s_k)} = (b_1,...,b_{2k+4})$ be two k+1-trapezoidal fuzzy numbers and $w_i = \min(c_i, s_i)$, $w_0 = 0$ such that

(i) $\Re(\overline{A}) = \Re(\overline{B})$ (ii) mode$(\overline{A})$ = mode$(\overline{B})$

(iii) spread$(\overline{A})$ = spread$(\overline{B})$ then

a) left spread$(\overline{A})$ > left spread$(\overline{B})$ if and only if

$$Z(\overline{A}) = \sum_{i=0}^{k-1} w_i (a_{2k+3-i} - a_{2k+5-i}) + w_k (a_{k+3} - a_{k+5}) + (-a_{k+4} - a_{k+3}) > Z(\overline{B})$$

b) left spread$(\overline{A})$ < left spread$(\overline{B})$ if and only if

$Z(\overline{A}) < Z(\overline{B})$

c) left spread$(\overline{A})$ = left spread$(\overline{B})$ if and only if

$Z(\overline{A}) = Z(\overline{B})$

Proof:

a)left spread$(\overline{A})$ > left spread$(\overline{B}) \Rightarrow$ left spread$(\overline{A})$ -spread$(\overline{A})$ -mode$(\overline{A})$ > left spread $(\overline{B})$ -spread$(\overline{B})$ -mode$(\overline{B})$ (since spread$(\overline{A})$ = spread$(\overline{B})$ and mode$(\overline{A})$ = mode$(\overline{B})$)

Now, $Z(\overline{A})$ = left spread$(\overline{A})$ -spread$(\overline{A})$ -mode$(\overline{A})$ and $Z(\overline{B})$ = left spread$(\overline{B})$ -spread $(\overline{B})$ -mode$(\overline{B})$

Thus we get $Z(\overline{A}) > Z(\overline{B})$.

Conversely, If $Z(\overline{A}) > Z(\overline{B})$, then $Z(\overline{A})$ + spread$(\overline{A})$ + mode$(\overline{A})$ > $Z(\overline{B})$ + spread $(\overline{B})$ + mode$(\overline{B})$ (since spread$(\overline{A})$ = spread$(\overline{B})$ and mode$(\overline{A})$ = mode$(\overline{B})$)

Now, left spread$(\overline{A})$ = $Z(\overline{A})$ + spread$(\overline{A})$ + mode$(\overline{A})$ and left spread$(\overline{B})$ = $Z(\overline{B})$ + spread$(\overline{B})$ + mode$(\overline{B})$

Thus we get left spread$(\overline{A})$ > left spread$(\overline{B})$ b) left spread$(\overline{A})$ < left spread$(B) \Rightarrow$ left spread$(\overline{A})$ -spread$(\overline{A})$ -mode$(\overline{A})$ < left spread$(\overline{B})$ -spread$(\overline{B})$ -mode$(\overline{B})$ since spread $(\overline{A})$ = spread$(\overline{B})$ and mode$(\overline{A})$ = mode$(\overline{B})$

Now, $Z(\overline{A})$ = left spread$(\overline{A})$ -spread$(\overline{A})$ -mode$(\overline{A})$ and $Z(\overline{B})$ = left spread$(\overline{B})$ -spread $(\overline{B})$ -mode$(\overline{B})$

Thus we get $Z(\overline{A}) < Z(\overline{B})$.

Conversely, If $Z(\overline{A}) < Z(\overline{B})$, then $Z(\overline{A})$ + spread$(\overline{A})$+ mode$(\overline{A})$ < $Z(\overline{B})$+ spread $(\overline{B})$+ mode$(\overline{B})$ since spread$(\overline{A})$= spread$(\overline{B})$ and mode$(\overline{A})$= mode$(\overline{B})$

Now, left spread$(\overline{A})$= $Z(\overline{A})$ + spread$(\overline{A})$+ mode$(\overline{A})$ and left spread$(\overline{B})$= $Z(\overline{B})$+ spread$(\overline{B})$+ mode$(\overline{B})$

Thus we get left spread$(\overline{A})$ < left spread$(\overline{B})$ c) left spread$(\overline{A})$ = left spread$(\overline{B})$ $\Rightarrow$ left spread$(\overline{A})$-spread$(\overline{A})$-mode$(\overline{A})$= left spread$(\overline{B})$-spread$(\overline{B})$-mode$(\overline{B})$ (since spread$(\overline{A})$= spread$(\overline{B})$ and mode$(\overline{A})$= mode$(\overline{B})$).

Now, $Z(\overline{A})$= left spread$(\overline{A})$-spread$(\overline{A})$-mode$(\overline{A})$ and $Z(\overline{B})$= left spread$(\overline{B})$-spread $(\overline{B})$-mode$(\overline{B})$

Thus we get $Z(\overline{A}) = Z(\overline{B})$.

Conversely, If $Z(\overline{A}) = Z(\overline{B})$, then $Z(\overline{A})$ + spread$(\overline{A})$+ mode$(\overline{A})$ = $Z(\overline{B})$+ spread $(\overline{B})$+ mode$(\overline{B})$ (since spread$(\overline{A})$= spread$(\overline{B})$ and mode$(\overline{A})$= mode$(\overline{B})$).

Now, left spread$(\overline{A})$= $Z(\overline{A})$ + spread$(\overline{A})$+ mode$(\overline{A})$ and left spread$(\overline{B})$= $Z(\overline{B})$+ spread$(\overline{B})$+ mode$(\overline{B})$

Thus we get left spread$(\overline{A})$ = left spread$(\overline{B})$.

Corollary 4.1 All the results of proposition 4.1 also hold for right spread.

Proof: a)right spread$(\overline{A})$ > right spread$(\overline{B})$ $\Rightarrow$right spread$(\overline{A})$-$\Re(\overline{A})$-spread$(\overline{A})$ + mode$(\overline{A})$ > right spread$(\overline{B})$-$\Re(\overline{B})$-spread$(\overline{B})$+ mode$(\overline{B})$ (since $\Re(\overline{A})$= $\Re(\overline{B})$, spread$(\overline{A})$= spread$(\overline{B})$ and mode$(\overline{A})$= mode$(\overline{B})$).

Now, $Z(\overline{A})$= right spread$(\overline{A})$-$\Re(\overline{A})$-spread$(\overline{A})$ + mode$(\overline{A})$ and $Z(\overline{B})$= right spread $(\overline{B})$-$\Re(\overline{B})$-spread$(\overline{B})$+ mode$(\overline{B})$

Thus we get $Z(\overline{A}) > Z(\overline{B})$.

Conversely, If $Z(\overline{A}) > Z(\overline{B})$, then $Z(\overline{A})$ +$\Re(\overline{A})$+ spread$(\overline{A})$-mode$(\overline{A})$ > $Z(\overline{B})$+ $\Re(\overline{B})$+ spread$(\overline{B})$-mode$(\overline{B})$

Now, right spread$(\overline{A})$ = $Z(\overline{A})$ +$\Re(\overline{A})$+ spread$(\overline{A})$-mode$(\overline{A})$ and right spread$(\overline{B})$= $Z(\overline{B})$+$\Re(\overline{B})$+ spread$(\overline{B})$-mode$(\overline{B})$

Thus we get right spread$(\overline{A})$ > right spread$(\overline{B})$ b) right spread$(\overline{A})$ < right spread$(\overline{B})$ $\Rightarrow$right spread$(\overline{A})$-$\Re(\overline{A})$-spread$(\overline{A})$ + mode$(\overline{A})$ < right spread$(\overline{B})$-$\Re(\overline{B})$-spread $(\overline{B})$+ mode$(\overline{B})$ (since $\Re(\overline{A})$= $\Re(\overline{B})$, spread$(\overline{A})$= spread$(\overline{B})$ and mode$(\overline{A})$= mode $(\overline{B})$).

Now, $Z(\overline{A})$= right spread$(\overline{A})$-$\Re(\overline{A})$-spread$(\overline{A})$ + mode$(\overline{A})$ and $Z(\overline{B})$= right spread

$(\overline{B})$ - $\Re(\overline{B})$ -spread $(\overline{B})$ + mode $(\overline{B})$

Thus we get $Z(\overline{A}) < Z(\overline{B})$.

Conversely, If $Z(\overline{A}) < Z(\overline{B})$, then $Z(\overline{A})$ $+\Re(\overline{A})+$ spread $(\overline{A})$ -mode $(\overline{A})$ < $Z(\overline{B})+$ $\Re(\overline{B})+$ spread $(\overline{B})$ -mode $(\overline{B})$.

Now, right spread $(\overline{A})$ = $Z(\overline{A})$ $+\Re(\overline{A})+$ spread $(\overline{A})$ -mode $(\overline{A})$ and right spread $(\overline{B})$ = $Z(\overline{B})+\Re(\overline{B})+$ spread $(\overline{B})$ -mode $(\overline{B})$

Thus we get right spread $(\overline{A})$ < right spread $(\overline{B})$ .c) right spread $(\overline{A})$ = right spread $(\overline{B})$ $\Rightarrow$right spread $(\overline{A})$ - $\Re(\overline{A})$ -spread $(\overline{A})$ + mode $(\overline{A})$ = right spread $(\overline{B})$ - $\Re(\overline{B})$ -spread $(\overline{B})$ + mode $(\overline{B})$ (since $\Re(\overline{A}) = \Re(\overline{B})$, spread $(\overline{A})$ = spread $(\overline{B})$ and mode $(\overline{A})$ = mode $(\overline{B})$).

Now, $Z(\overline{A}) =$ right spread $(\overline{A})$ - $\Re(\overline{A})$ -spread $(\overline{A})$ + mode $(\overline{A})$ and $Z(\overline{B})=$ right spread $(\overline{B})$ - $\Re(\overline{B})$ -spread $(\overline{B})$ + mode $(\overline{B})$

Thus we get $Z(\overline{A}) = Z(\overline{B})$.

Conversely, If $Z(\overline{A}) = Z(\overline{B})$ then $Z(\overline{A})$ $+\Re(\overline{A})+$ spread $(\overline{A})$ -mode $(\overline{A})$ = $Z(\overline{B})+$ $\Re(\overline{B})+$ spread $(\overline{B})$ -mode $(\overline{B})$

Now, right spread $(\overline{A})$ = $Z(\overline{A})$ $+\Re(\overline{A})+$ spread $(\overline{A})$ -mode $(\overline{A})$ and right spread $(\overline{B})$ = $Z(\overline{B})+\Re(\overline{B})+$ spread $(\overline{B})$ -mode $(\overline{B})$

Thus we get right spread $(\overline{A})$ = right spread $(\overline{B})$.

Proposition 4.2 let $\overline{A}_{(c_1,..,c_k)} = (a_1,..,a_{2k+4})$ and $\overline{B}_{(s_1,..,s_k)} = (b_1,...,b_{2k+4})$ be two k+1-trapezoidal fuzzy numbers and $w_i = \min(c_i, s_i)$, $w_0 = 0$ such that

(i) $\Re(\overline{A}) = \Re(\overline{B})$

(ii)mode $(\overline{A})$ = mode $(\overline{B})$

(iii)spread $(\overline{A})$ = spread $(\overline{B})$ then

a) left spread $(\overline{A})$ > left spread $(\overline{B})$ if and only if right spread $(\overline{A})$ > right spread $(\overline{B})$ b) left spread $(\overline{A})$ < left spread $(\overline{B})$ if and only if right spread $(\overline{A})$ < right spread $(\overline{B})$ c) left spread $(\overline{A})$ = left spread $(\overline{B})$ if and only if right spread $(\overline{A})$ = right spread $(\overline{B})$

Proof:

a) left spread $(\overline{A})$ > left spread $(\overline{B})$ $\Leftrightarrow Z(\overline{A}) > Z(\overline{B}) \Leftrightarrow$ right spread $(\overline{A})$ > right spread $(\overline{B})$ (using Proposition 4.1 and Corollary 4.1)

b) left spread $(\overline{A})$ < left spread $(\overline{B})$ $\Leftrightarrow Z(\overline{A}) < Z(\overline{B}) \Leftrightarrow$ right spread $(\overline{A})$ < right spread $(\overline{B})$ (using Proposition 4.1 and Corollary 4.1)

c) left spread $(\overline{A})$ = left spread $(\overline{B})$ $\Leftrightarrow Z(\overline{A}) = Z(\overline{B})$ $\Leftrightarrow$right spread $(\overline{A})$ = right spread $(\overline{B})$ (using Proposition 4.1 and Corollary 4.1)

5. Java Implementation

For the purpose of supporting our method, we designed a computerized solution. This solution was implemented as a program using the JAVA language. Our program aims at comparing between such kind of fuzzy numbers based on the described ranking method. Currently, it applies to small k's but with the ability to include large k's in the future work. The class diagram that represents our program is shown below:

Table 1 The Class Diagram of The Program

TrapezoidalFuzzyNumbers
-k: **int**
+ findTrapFuzzyNoByStep1(XAs[]: **double**, XBs[]: **double**, cAs[]: **double**, cBs[]: **double**): **double** + findTrapFuzzyNoByStep2(XAs[]: **double**, XBs[]: **double**, cAs[]: **double**, cBs[]: **double**): **double** + findTrapFuzzyNoByStep3(XAs[]: **double**, XBs[]: **double**, cAs[]: **double**, cBs[]: **double**): **double** + findTrapFuzzyNoByStep4(XAs[]: **double**, XBs[]: **double**, cAs[]: **double**, cBs[]: **double**): **double** + findTrapFuzzyNoByStep5(XAs[]: **double**, XBs[]: **double**, cAs[]: **double**, cBs[]: **double**): **double** + findMinCs(cAs[]:**double**, cBs[]:**double**):**double**

6. Numerical Examples

We apply the method shown above to our own examples

Set 6.1 Let $\overline{A}_{(0.35)} = (0.2,0.3,0.4,0.6,0.7,0.8)$ and $\overline{B}_{(0.7)} = (0.1,0.16,0.2,0.3,0.35,0.4)$

Step 1 $\Re(\overline{A}) = \sum_{i=0}^{0} \frac{w_i}{4}(a_i - a_{i+2} - a_{5-i} + a_{7-i}) + \frac{w_1}{4}(a_1 - a_3 - a_4 + a_6) + \frac{1}{4}(a_2 + a_3 + a_4 + a_5)$

$\Re(\overline{A}) = 0 + \frac{0.35}{4}(0.2 - 0.4 - 0.6 + 0.8) + \frac{1}{4}(0.3 + 0.4 + 0.6 + 0.7) = 0.5$

and $\Re(\overline{B}) = 0 + \frac{0.35}{4}(0.1 - 0.2 - 0.3 + 0.4) + \frac{1}{4}(0.16 + 0.2 + 0.3 + 0.35) = 0.2525$

$\Re(\overline{A}) > \Re(\overline{B}) \Rightarrow \overline{A} \succ \overline{B}$

Set 6.2 Let $\overline{A}_{(0.5)} = (0.1,0.15,0.2,0.3,0.4,0.5)$ and $\overline{B}_{(0.5)} = (0.1,0.15,0.25,0.3,0.375,0.5)$

Step 1 $\Re(\overline{A}) = 0 + \frac{0.5}{4}(0.1 - 0.2 - 0.3 + 0.5) + \frac{1}{4}(0.15 + 0.2 + 0.3 + 0.4) = 0.275$

and $\Re(\overline{B}) = 0 + \frac{0.5}{4}(0.1 - 0.25 - 0.3 + 0.5) + \frac{1}{4}(0.15 + 0.25 + 0.3 + 0.375) = 0.275$

Since $\Re(\overline{A}) = \Re(\overline{B})$, we go to step 2 .

Step 2 $\text{mode}(\overline{A}) = \sum_{i=0}^{0} \frac{w_i}{2}(a_{i+1} - a_{i+2} - a_{5-i} + a_{6-i}) + \frac{w_1}{2}(a_2 - a_3 - a_4 + a_5) + \frac{a_3 + a_4}{2}$

$\text{mode}(\overline{A}) = 0 + \frac{0.5}{2}(0.15 - 0.2 - 0.3 + 0.4) + \frac{0.2 + 0.3}{2} = 0.2625$ and mode

$(\overline{B}) = 0 + \frac{0.5}{2}(0.15 - 0.25 - 0.3 + 0.375) + \frac{0.25 + 0.3}{2}$ = 0.26875 mode $(\overline{A}) <$ mode $(\overline{B}) \Rightarrow \overline{A} \prec \overline{B}$

According to our java program, the results of the previous sets are matching,
Set 6. 1:
R (A's):0.49999999999999994 and
R (B's):0.25249999999999995
Thus R (A's) > R(B's)
Hence $A \succ B$
Set. 6.2:
0.27499999999999997
0.275
After rounding R(A's) = R(B's)
Thus we go to step 2 (Mode), we find:
Mode (A's):0.25 and
Mode (B's):0.275
Thus Mode(B's) > Mode (A's)
Hence $B \succ A$

7. Conclusion

The fuzzy comparison method mentioned in [6-7] can be applied to many classes of fuzzy numbers like the particular fuzzy numbers (i.e k+1-Trapoziodal fuzzy numbers) mentioned in this work. The constructed JAVA program is currently limited to k=3, but it can be modified for larger k. We will work to modify it for other ranking methods and other kinds of fuzzy numbers.

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