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The Weakly Basicity of System in the Intuitionistic Fuzzy Metric Space

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Abstract

Fuzzy metric space is considered. The concepts of weakly fuzzy completeness, fuzzy minimality, fuzzy biorthogonality, fuzzy basicity and fuzzy space of coefficients are introduced. Weakly completeness of fuzzy space of coefficients with regard to fuzzy metric and weakly basicity of canonical system in this space are proved. Weakly basicity criterion in fuzzy metric space is presented in terms of coefficient operator.

Keywords: weakly fuzzy basicity, fuzzy completeness, fuzzy minimality, fuzzy biorthogonality, fuzzy space of coefficients

Classification 2000: 46A35, 46S40, 26E50

1.Introduction

The concept of the space of coefficients belongs to the theory of bases. As is known, every basis in a Banach space has a Banach space of coefficients which is isomorphic to an initial one (see, e.g., [1;2]). Every nondegenerate system (to be defined later) in a Banach space generates the corresponding Banach space of coefficients with canonical basis (see, e.g., [2;3]). Therefore, space of coefficients plays an important role in the study of approximative properties of systems. It has very important applications in various fields of science, such as solid body physics, molecular physics, multiple production of particles, aviation, medicine, biology, data compression, etc (see, e.g., [4;5] and references within). All these applications are closely related to wavelet analysis, and there arose a great interest in them lately [see, e.g., 5]. It is well known that many topological spaces are nonnormable. Therefore, the study of various properties of the space of coefficients in topological spaces is of special scientific interest. Applications in various branches of mathematics and natural sciences have lately induced a strong interest toward the study of different research problems in terms of fuzzy structures. More details on this topic can be found in [6-9] and references therein. A large number of research works is appearing these days which deal with the concept of fuzzy set-numbers, and the fuzzification of many classical theories has also been made. The concept of Schauder basis in intuitionistic fuzzy normed space and some results related to this concept have recently been studied in [10-12]. These works introduced the concepts of strongly and strongly intuitionistic fuzzy (Schauder) basis in intuitionistic fuzzy Banach spaces (IFBS in short). Some of their properties are revealed. The concepts of strongly and weakly intuitionistic fuzzy approximation properties (sif-AP and wif-AP in short, respectively) are also introduced in these works. It is proved that if the intuitionistic fuzzy space has a sif-basis, then it has a sif-AP. All the results in these works are obtained on condition that IFBS admits equivalent topology using the family of norms generated by t-norm and t-conorm. In our work, we define the basic concepts of classical basis theory in intuitionistic fuzzy metric spaces (IFMS in short). Concept of weakly fuzzy space of coefficients is introduced. Weakly completeness of these spaces with regard to fuzzy metric and weakly basicity of canonical system in them are proved. Weakly basicity criterion in fuzzy metric space is presented in terms of coefficient operator. In Section 2, we recall some notations and concepts. In Section 3, we state our main results. We first define the fuzzy space of coefficients and then introduce the corresponding fuzzy metrics. We prove that for nondegenerate system the corresponding fuzzy space of coefficients is weakly fuzzy complete. Moreover, we show that the canonical system forms a weakly basis for this space. It should be noted that similar results earlier were obtained in the paper [13] in IFBS.

2. Some preliminary notations and concepts

We will use the usual notations: $\mathbb{N}$ will denote the set of all positive integers, $\mathbb{R}$ will be the set of all real numbers, $\mathbb{C}$ will be the set of complex numbers and K will denote a field of scalars ($K \equiv \mathbb{R}$, or $K \equiv \mathbb{C}$), $\mathbb{R}_+ \equiv (0,+\infty)$. ImT is a range of the operator T; $KerT$ is a kernel of the operator T. We state some concepts and facts from IFMS theory to be used later.

Definition 1. Let X be a linear space over a field K. Functions $\mu;\nu : X^2 \times \mathbb{R} \to [0,1]$ are called fuzzy metrics on X if the following conditions hold:

1. $\mu(x;y;t) = 0, \forall t \le 0, \forall x, y \in X$;
2. $\mu(x;y;t) = 1, \forall t > 0 \Rightarrow x = y$;
3. $\mu(x;y;t) = \mu(y;x;t), \forall x, y \in X, \forall t \in \mathbb{R}$;
4. $\mu(x;z;t+s) \ge \min\{\mu(x;y;t);\mu(y;z;s)\}, \forall x, z, y \in X, \forall t, s \in \mathbb{R}$;
5. $\mu(x;y;\cdot) : \mathbb{R} \to [0,1]$ is a non-decreasing function of t for $\forall x, y \in X$ and $\lim\limits_{t\to\infty} \mu(x;y;t) = 1$, $\forall x, y \in X$;

6. $\nu(x;y;t)=1,\forall t\le 0,\forall x,y\in X$;
7. $\nu(x;y,t)=0,\forall t>0\Rightarrow x=y$;
8. $\nu(x;y;t)=\nu(y;x;t),\forall x,y\in X,\forall t\in \mathrm{R}$;
9. $\nu(x;z;t+s)\le\max\{\nu(x;y;t);\nu(y;z;s)\},\forall x,z,y\in X,\forall t,s\in\mathrm{R}$;
10. $\nu(x;y;\cdot):\mathrm{R}\to[0,1]$ is a non-increasing function of t for $\forall x,y\in X$ and $\lim\limits_{t\to\infty}\nu(x;y;t)=0$, $\forall x,y\in X$;
11. $\mu(x;y;t)+\nu(x;y;t)\le 1,\forall x,y\in X,\forall t\in\mathrm{R}$.

Then the triplet $(X;\mu;\nu)$ is called an intuitionistic fuzzy metric space(IFMS in short).

Definition 2. Let $(X;\mu;\nu)$ be a fuzzy metric space and let $\{x_n\}_{n\in\mathrm{N}}\subset X$ be some sequence. Then it is said to be weakly intuitionistic fuzzy convergent to $x\in X$ (denoted by $x_n\xrightarrow{w}x,n\to\infty$, or $w\text{-}\lim\limits_{n\to\infty}x_n=x$ in short) if and only if for $\forall\varepsilon,t>0,\exists n_0=n_0(\varepsilon;t):\mu(x_n;x;t)\ge 1-\varepsilon,\ \nu(x_n;x;t)\le\varepsilon,\forall n\ge n_0$.

Definition 3. Let $(X;\mu;\nu)$ be a fuzzy metric space and let $\{x_n\}_{n\in\mathrm{N}}\subset X$ be some sequence. Then it is said to be weakly Cauchy sequence if $\lim\limits_{n,m\to\infty}\mu(x_n;x_m;t)=1$, $\lim\limits_{n,m\to\infty}\nu(x_n;x_m;t)=0$, $\forall t\in\mathrm{R}$. If every weakly Cauchy sequence converges (weakly) in X , then $(X;\mu;\nu)$ is said to be weakly complete fuzzy metric space.

More details on these concepts can be found in [9-12; 14-19].

Let $(X;\mu;\nu)$ be an IFMS, and let $M\subset X$ be some set. By $L[M]$ we denote the linear span of M in X . The weakly intuitionistic fuzzy convergent closure of $L[M]$ will be denoted by $\overline{L_w[M]}$. If X is complete with respect to the weakly intuitionistic fuzzy convergence, then we will call it intuitionistic fuzzy weakly complete metric space (IFM_wS or X_w in short). Let X be an IFM_wS. We denote by X_w^* the linear space of linear and weakly continuous in IFM_wS functionals over the same field K. Now we define the corresponding concepts of basis theory for IFMS. Let $\{x_n\}_{n\in\mathrm{N}}\subset X$ be some system.

Definition 4. System $\{x_n\}_{n\in\mathrm{N}}$ is called w-complete in X_w, if $\overline{L_w[\{x_n\}_{n\in\mathrm{N}}]}\equiv X_w$.

Definition 5. System $\{x_n^*\}_{n\in\mathrm{N}}\subset X_w^*$ is called w-biorthogonal to the system $\{x_n\}_{n\in\mathrm{N}}$, if $x_n^*(x_k)=\delta_{nk},\forall n,k\in\mathrm{N}$, where δ_{nk} is the Kronecker symbol.

Definition 6. System $\{x_n\}_{n\in\mathrm{N}}\subset X_w$ is called w-linearly independent in X , if

$\sum_{n=1}^{\infty}\lambda_n x_n = 0$ in X_w implies $\lambda_n = 0, \forall n \in N$.

Definition 7. System $\{x_n\}_{n\in N} \subset X_w$ is called w-basis for X_w if $\forall x \in X$, $\exists!$ $\{\lambda_n\}_{n\in N} \subset K : \sum_{n=1}^{\infty}\lambda_n x_n = x$ in X_w.

Let $(X;\mu;\nu)$, $(\tilde{X};\tilde{\mu};\tilde{\nu})$ be *IFMS* and $T : X \to \tilde{X}$ be a linear operator. T is called w-continuous, if from $x_n \overset{w}{\to} x, n \to \infty,$ in X it follows that $Tx_n \overset{w}{\to} Tx, n \to \infty,$ in $\tilde{X}$. Operator $T : X \to \tilde{X}$ is called w-isomorphism between X and $\tilde{X}$, if it is w-continuous, $KerT = 0$ and $ImT \equiv \tilde{X}$. We will also need the following concept.

Definition 8. System $\{x_n\}_{n\in N} \subset X$ is called nondegenerate, if $x_n \neq 0, \forall n \in N$. In obtaining of the main results we will use the following conditions on *IFMS*. $\alpha)$ linear operations of addition and multiplication by scalar in IFM_wS is weakly continuous in X, i.e. from $\lambda_n \to \lambda, n \to \infty$, in C and from $x_n \overset{w}{\to} x, y_n \overset{w}{\to} y, n \to \infty$, in X_w it follows that $\lambda_n x_n \overset{w}{\to} \lambda x, x_n + y_n \overset{w}{\to} x + y, n \to \infty$, in X_w. $\beta)$ let $\tau_{\mu,\nu}$ be a topology for X_w, generated by a pair of (μ,ν). We will assume that boundedness of a set in the spaces X_w and IFM_wS $(X;\mu;\nu)$ are equivalent with respect to the topology $\tau_{\mu,\nu}$, i.e. these concepts are same in spaces $(X;\tau_{\mu;\nu})$ and $(X;\mu;\nu)$.

3. Main results

Let $(X;\mu;\nu)$ be some IFM_wS, with conditions $\alpha)$, $\beta)$ and $\{x_n\}_{n\in N} \subset X$ be some nondegenerate system. Assume that

$$K_{\bar{x}}^{w} \equiv \left\{ \{\lambda_n\}_{n\in N} \subset C : \sum_{n=1}^{\infty}\lambda_n x_n \textit{ convergesin } X_w \right\}.$$

It is not difficult to see that $K_{\bar{x}}^{w}$ is linear spaces with regard to component-specific summation and component-specific multiplication by a scalar. Take $\forall \bar{\lambda}, \bar{\mu} \in K_{\bar{x}}^{w}$, $\bar{\lambda} \equiv \{\lambda_n\}_{n\in N}$, $\bar{\mu} \equiv \{\mu_n\}_{n\in N}$ and assume

$$\mu_{K_{\bar{x}}^{w}}(\bar{\lambda};\bar{\mu};t) = \inf_m \mu\left(\sum_{n=1}^{m}\lambda_n x_n ; \sum_{n=1}^{m}\mu_n x_n ; t \right),$$

$$\nu_{K_{\bar{x}}^{w}}(\bar{\lambda};\bar{\mu};t) = \sup_m \nu\left(\sum_{n=1}^{m}\lambda_n x_n ; \sum_{n=1}^{m}\mu_n x_n ; t \right).$$

Let's show that $\mu_{K_{\bar{x}}^{w}}$ and $\nu_{K_{\bar{x}}^{w}}$ satisfy the conditions 1)-11). At first let's consider $\mu_{K_{\bar{x}}^{w}}$. 1) It is clear that $\mu_{K_{\bar{x}}^{w}}(\bar{\lambda};\bar{\mu};t) = 0, \forall t \le 0$. 2) Let $\mu_{K_{\bar{x}}^{w}}(\bar{\lambda};\bar{\mu};t) = 1, \forall t > 0$. Hence,

$\mu_{\mathrm{K}_{\bar{x}}^{w}}\left(\sum_{n=1}^{m}\lambda_n x_n;\sum_{n=1}^{m}\mu_n x_n;t\right)=1,\ \forall m\in\mathrm{N},\forall t>0.$ Suppose that the system $\{x_n\}_{n\in\mathrm{N}}$ is nondegenerate. It follows from the above-stated relations that for $m=1$ we have $\mu(\lambda_1 x_1;\mu_1 x_1;t)=1,\forall t>0.$ Hence, $\lambda_1 x_1=\mu_1 x_1\Rightarrow\lambda_1=\mu_1$. Continuing this process, we get at the end of this process that $\lambda_n=\mu_n,\forall n\in\mathrm{N}$, i.e. $\bar{\lambda}=\bar{\mu}$. 3) It is clear that $\mu_{\mathrm{K}_{\bar{x}}^{w}}\left(\bar{\lambda};\bar{\mu};t\right)=\mu_{\mathrm{K}_{\bar{x}}^{w}}\left(\bar{\mu};\bar{\lambda};t\right),\forall t\in\mathrm{R}$. 4) Let $\bar{\lambda},\bar{\mu},\bar{\nu}\in\mathrm{K}_{\bar{x}}^{w}$ and $s,t\in\mathrm{R}$. We have

$$\mu_{\mathrm{K}_{\bar{x}}^{w}}\left(\bar{\lambda};\bar{\mu};t+s\right)=\inf_{m}\mu\left(\sum_{n=1}^{m}\lambda_n x_n;\sum_{n=1}^{m}\mu_n x_n;t+s\right)\geq$$

$$\geq\inf_{m}\min\left\{\mu\left(\sum_{n=1}^{m}\lambda_n x_n;\sum_{n=1}^{m}\nu_n x_n;t\right);\mu\left(\sum_{n=1}^{m}\nu_n x_n;\sum_{n=1}^{m}\mu_n x_n;s\right)\right\}=$$

$$=\min\left\{\inf_{m}\mu\left(\sum_{n=1}^{m}\lambda_n x_n;\sum_{n=1}^{m}\nu_n x_n;t\right);\inf_{m}\mu\left(\sum_{n=1}^{m}\nu_n x_n;\sum_{n=1}^{m}\mu_n x_n;s\right)\right\}=$$

$$=\min\left\{\mu_{\mathrm{K}_{\bar{x}}^{w}}\left(\bar{\lambda};\bar{\nu};t\right);\mu_{\mathrm{K}_{\bar{x}}^{w}}\left(\bar{\nu};\bar{\mu};s\right)\right\}.$$

5) Let's show that $\mu_{\mathrm{K}_{\bar{x}}^{w}}\left(\bar{\lambda};\bar{\mu};\cdot\right):R\to[0,1]$ is a non-decreasing function of t for $\forall\bar{\lambda},\bar{\mu}\in\mathrm{K}_{\bar{x}}^{w}$ and $\lim_{t\to\infty}\mu_{\mathrm{K}_{\bar{x}}^{w}}\left(\bar{\lambda};\bar{\mu};t\right)=1$, $\forall\bar{\lambda},\bar{\mu}\in\mathrm{K}_{\bar{x}}^{w}$. As $\mu(x;y;\cdot)$ is a non-decreasing function on R, it is not difficult to see that $\mu_{\mathrm{K}_{\bar{x}}^{w}}\left(\bar{\lambda};\bar{\mu};\cdot\right)$ has the same property. Let us show that $\lim_{t\to\infty}\mu_{\mathrm{K}_{\bar{x}}^{w}}\left(\bar{\lambda};\bar{\mu};t\right)=1$. Take $\forall\varepsilon>0$. Let $S_m^{(1)}=\sum_{n=1}^{m}\lambda_n x_n$, $S_m^{(2)}=\sum_{n=1}^{m}\mu_n x_n$ and $w\text{-}\lim_{m\to\infty}S_m^{(k)}=S^{(k)}\in X_w, k=1,2$. It is clear that $\exists t_0>0$: $\mu\left(S^{(1)};S^{(2)};t_0\right)\geq 1-\varepsilon$. Then it follows from the definition of $w\text{-}\lim$ that $\exists m_0=m_0(\varepsilon;t_0)\in\mathrm{N}$: $\mu\left(S_m^{(k)};S^{(k)};t_0\right)\geq 1-\varepsilon$, $\forall m\geq m_0$, $k=1,2$. Property 4) implies

$$\mu\left(S_m^{(1)};S_m^{(2)};3t_0\right)\geq\min\left\{\mu\left(S_m^{(1)};S^{(1)};t_0\right);\mu\left(S^{(1)};S_m^{(2)};2t_0\right)\right\},$$

$$\mu\left(S^{(1)};S_m^{(2)};2t_0\right)\geq\min\left\{\mu\left(S^{(1)};S^{(2)};t_0\right);\mu\left(S^{(2)};S_m^{(2)};t_0\right)\right\}.$$

Thus

$$\mu\left(S_m^{(1)};S_m^{(2)};3t_0\right)\geq\min\left\{\mu\left(S_m^{(1)};S^{(1)};t_0\right);\mu\left(S^{(1)};S^{(2)};t_0\right);\mu\left(S^{(2)};S_m^{(2)};t_0\right)\right\}.$$

As a result we obtain

$$\mu\left(S_m^{(1)};S_m^{(2)};3t_0\right)\geq 1-\varepsilon,\forall m\geq m_0. \qquad (1)$$

As $\mu(x;y;\cdot)$ is a non-decreasing function of t, it follows from (1) that

$$\mu\left(S_m^{(1)};S_m^{(2)};3t_0\right)\geq 1-\varepsilon,\forall m\geq m_0,\forall t\geq 3t_0. \qquad (2)$$

We have

$$\mu_{\mathrm{K}_{\bar{x}}^{w}}\left(\bar{\lambda};\bar{\mu};t\right)=\min\left\{\mu\left(S_k^{(1)};S_k^{(2)};t\right),k=\overline{1,m_0-1};\inf_{m\geq m_0}\mu\left(S_m^{(1)};S_m^{(2)};t\right)\right\}. \qquad (3)$$

As $\lim\limits_{t\to\infty}\mu\left(S_k^{(1)};S_k^{(2)};t\right)=1$ for $\forall k\in\mathrm{N}$, we have $\exists t_k(\varepsilon);\forall t\geq t_k(\varepsilon):\mu\left(S_k^{(1)};S_k^{(2)};t\right)\geq 1-\varepsilon$, $k=\overline{1,m_0-1}$. Let $t_\varepsilon^0=\max\limits_{1\leq k\leq m_0-1}t_k(\varepsilon)$. Then it is clear that

$$\mu\left(S_k^{(1)};S_k^{(2)};t\right)\geq 1-\varepsilon,\forall t\geq t_\varepsilon^0. \tag{4}$$

It follows from (2) that

$$\inf_{m\geq m_0}\mu\left(S_m^{(1)};S_m^{(2)};t\right)\geq 1-\varepsilon,\forall t\geq 3t_0.$$

Let $t_\varepsilon=\max\left\{3t_0;t_\varepsilon^0\right\}$. Hence we obtain from (3) and (4) that

$$\mu_{\mathrm{K}_{\bar{x}}^w}\left(\bar{\lambda};\bar{\mu};t\right)\geq 1-\varepsilon,\forall t\geq t_\varepsilon.$$

Thus $\lim\limits_{t\to\infty}\mu_{\mathrm{K}_{\bar{x}}^w}\left(\bar{\lambda};\bar{\mu};t\right)=1$, $\forall\bar{\lambda},\bar{\mu}\in\mathrm{K}_{\bar{x}}^w$. 6) As $\nu(x;y;t)=1,\forall t\leq 0,\forall x,y\in X$, it is clear that $\nu_{\mathrm{K}_{\bar{x}}^w}\left(\bar{\lambda};\bar{\mu};t\right)=1$, $\forall t\leq 0$, $\forall\bar{\lambda},\bar{\mu}\in\mathrm{K}_{\bar{x}}^w$. 7) Assume that $\nu_{\mathrm{K}_{\bar{x}}^w}\left(\bar{\lambda};\bar{\mu};t\right)=0,\forall t>0$. Then $\nu\left(\sum_{n=1}^m\lambda_nx_n;\sum_{n=1}^m\mu_nx_n;t\right)=0$, $\forall t>0,\forall m\in\mathrm{N}$. For $m=1$ we have $\nu(\lambda_1x_1;\mu_1x_1;t)=0,\forall t>0\Rightarrow\ \lambda_1x_1=\mu_1x_1\Rightarrow\lambda_1=\mu_1$, if the system $\{x_n\}_{n\in\mathrm{N}}$ is nondegenerate. Continuing this way, we get $\lambda_n=\mu_n,\forall n\in\mathrm{N}\Rightarrow\ \bar{\lambda}=\bar{\mu}$. 8) Fulfillment of the condition $\nu_{\mathrm{K}_{\bar{x}}^w}\left(\bar{\lambda};\bar{\mu};t\right)=\nu_{\mathrm{K}_{\bar{x}}^w}\left(\bar{\mu};\bar{\lambda};t\right)$ is obvious. 9) Let $\bar{\lambda},\bar{\mu},\bar{\nu}\in\mathrm{K}_{\bar{x}}^w$ ($\bar{\lambda}\equiv\{\lambda_n\}_{n\in\mathrm{N}}$, $\bar{\mu}\equiv\{\mu_n\}_{n\in\mathrm{N}}$, $\bar{\nu}\equiv\{\nu_n\}_{n\in\mathrm{N}}$) and $s,t\in\mathrm{R}$. We have

$$\nu_{\mathrm{K}_{\bar{x}}^w}\left(\bar{\lambda};\bar{\mu};t+s\right)=\sup_m\nu\left(\sum_{n=1}^m\lambda_nx_n;\sum_{n=1}^m\mu_nx_n;s+t\right)\leq$$

$$\leq\sup_m\max\left\{\nu\left(\sum_{n=1}^m\lambda_nx_n;\sum_{n=1}^m\nu_nx_n;s\right);\nu\left(\sum_{n=1}^m\nu_nx_n;\sum_{n=1}^m\mu_nx_n;t\right)\right\}=$$

$$=\max\left\{\sup_m\nu\left(\sum_{n=1}^m\lambda_nx_n;\sum_{n=1}^m\nu_nx_n;s\right);\sup_m\nu\left(\sum_{n=1}^m\nu_nx_n;\sum_{n=1}^m\mu_nx_n;t\right)\right\}=$$

$$=\max\left\{\nu_{\mathrm{K}_{\bar{x}}^w}\left(\bar{\lambda};\bar{\nu};s\right);\nu_{\mathrm{K}_{\bar{x}}^w}\left(\bar{\nu};\bar{\mu};t\right)\right\}.$$

10) It follows from property 10) that $\nu_{\mathrm{K}_{\bar{x}}^w}\left(\bar{\lambda};\bar{\mu};\cdot\right)$ is a non-increasing function on R. Let us show that $\lim\limits_{t\to\infty}\nu_{\mathrm{K}_{\bar{x}}^w}\left(\bar{\lambda};\bar{\mu};t\right)=0$, Take $\forall\varepsilon>0$. Let $S_m^{(1)}=\sum_{n=1}^m\lambda_nx_n$, $S_m^{(2)}=\sum_{n=1}^m\mu_nx_n$ and $w\text{-}\lim\limits_{m\to\infty}S_m^{(k)}=S^{(k)}\in X_w,k=1,2$. It is clear that $\exists t_0>0$: $\nu\left(S^{(1)};S^{(2)};t_0\right)\leq\varepsilon$. Then it follows from the definition of w-lim that $\exists m_0=m_0(\varepsilon;t_0)\in\mathrm{N}$: $\nu\left(S_m^{(k)};S^{(k)};t_0\right)\leq\varepsilon$, $\forall m\geq m_0$, $k=1,2$. Property 9) implies

$$\nu\left(S_m^{(1)};S_m^{(2)};3t_0\right)\leq\max\left\{\nu\left(S_m^{(1)};S^{(1)};t_0\right);\nu\left(S^{(1)};S_m^{(2)};2t_0\right)\right\},$$

$$\nu\left(S^{(1)};S_m^{(2)};2t_0\right)\leq\max\left\{\nu\left(S^{(1)};S^{(2)};t_0\right);\mu\left(S^{(2)};S_m^{(2)};t_0\right)\right\}$$

Thus

$$\nu\left(S_m^{(1)}; S_m^{(2)}; 3t_0\right) \le \max\left\{\nu\left(S_m^{(1)}; S^{(1)}; t_0\right); \nu\left(S^{(1)}; S^{(2)}; t_0\right); \nu\left(S^{(2)}; S_m^{(2)}; t_0\right)\right\}.$$

As a result we obtain

$$\nu\left(S_m^{(1)}; S_m^{(2)}; 3t_0\right) \le \varepsilon, \forall m \ge m_0. \tag{5}$$

As $\nu(x; y; \cdot)$ is a non-increasing function of t, it follows from (5) that

$$\nu\left(S_m^{(1)}; S_m^{(2)}; t\right) \le \varepsilon, \forall m \ge m_0, \forall t \ge 3t_0. \tag{6}$$

We have

$$\nu_{\mathrm{K}_{\bar{x}}^{w}}\left(\bar{\lambda}; \bar{\mu}; t\right) = \max\left\{\nu\left(S_k^{(1)}; S_k^{(2)}; t\right), k = \overline{1, m_0 - 1}; \sup_{m \ge m_0} \nu\left(S_m^{(1)}; S_m^{(2)}; t\right)\right\}. \tag{7}$$

As $\lim\limits_{t \to \infty} \nu\left(S_k^{(1)}; S_k^{(2)}; t\right) = 0$ for $\forall k \in \mathrm{N}$, we have $\exists t_k(\varepsilon); \forall t \ge t_k(\varepsilon): \nu\left(S_k^{(1)}; S_k^{(2)}; t\right) \le \varepsilon$, $k = \overline{1, m_0 - 1}$. Let $t_\varepsilon^0 = \max\limits_{1 \le k \le m_0 - 1} t_k(\varepsilon)$. Then it is clear that

$$\nu\left(S_k^{(1)}; S_k^{(2)}; t\right) \le \varepsilon, \forall t \ge t_\varepsilon^0. \tag{8}$$

It follows from (6) that

$$\sup_{m \ge m_0} \nu\left(S_m^{(1)}; S_m^{(2)}; t\right) \le \varepsilon, \forall t \ge 3t_0.$$

Let $t_\varepsilon = \max\left\{3t_0; t_\varepsilon^0\right\}$. Hence we obtain from (7) and (8) that

$$\nu_{\mathrm{K}_{\bar{x}}^{w}}\left(\bar{\lambda}; \bar{\mu}; t\right) \le \varepsilon, \forall t \ge t_\varepsilon.$$

Thus $\lim\limits_{t \to \infty} \nu_{\mathrm{K}_{\bar{x}}^{w}}\left(\bar{\lambda}; \bar{\mu}; t\right) = 0$, $\forall \bar{\lambda}, \bar{\mu} \in \mathrm{K}_{\bar{x}}^{w}$. 11) We have

$$\mu_{\mathrm{K}_{\bar{x}}^{w}}\left(\bar{\lambda}; \bar{\mu}; t\right) + \nu_{\mathrm{K}_{\bar{x}}^{w}}\left(\bar{\lambda}; \bar{\mu}; t\right) = \inf_m \mu\left(\sum_{n=1}^{m} \lambda_n x_n; \sum_{n=1}^{m} \mu_n x_n; t\right) + \sup_m \nu\left(\sum_{n=1}^{m} \lambda_n x_n; \sum_{n=1}^{m} \mu_n x_n; t\right)$$

$$\le \sup_m \left[\mu\left(\sum_{n=1}^{m} \lambda_n x_n; \sum_{n=1}^{m} \mu_n x_n; t\right) + \nu\left(\sum_{n=1}^{m} \lambda_n x_n; \sum_{n=1}^{m} \mu_n x_n; t\right)\right] \le 1, \forall \bar{\lambda}, \bar{\mu} \in \mathrm{K}_{\bar{x}}^{w}, \forall t \in \mathrm{R}.$$

Thus, we have proved the validity of the following

Theorem 1. Let $(X; \mu; \nu)$ be a weakly fuzzy metric space and let $\{x_n\}_{n \in \mathrm{N}} \subset X$ be a nondegenerate system. Then the space of coefficients $\left(\mathrm{K}_{\bar{x}}^{w}; \mu_{\mathrm{K}_{\bar{x}}^{w}}; \nu_{\mathrm{K}_{\bar{x}}^{w}}\right)$ is also weakly fuzzy metric space.

3.2. Completeness of the space of coefficients. Subsequently, we assume that $(X; \mu; \nu)$ is weakly complete IFMS. Let us show that $\left(\mathrm{K}_{\bar{x}}^{w}; \mu_{\mathrm{K}_{\bar{x}}^{w}}; \nu_{\mathrm{K}_{\bar{x}}^{w}}\right)$ is also a weakly fuzzy complete metric space. First we prove the following

Lemma 1. Let $x_0 \ne 0, x_0 \in X$, and let $\{\lambda_n\}_{n \in \mathrm{N}} \subset \mathrm{R}$ be some sequence. If w-

$\lim\limits_{n\to\infty}(\lambda_n x_0)=0$, i.e. $\forall\varepsilon;t>0$, $\exists n_0=n_0(\varepsilon;t)$: $\mu(\lambda_n x_0;0;t)>1-\varepsilon$, $\nu(\lambda_n x_0;0;t)<\varepsilon$, $\forall n\ge n_0$; then $\lambda_n\to 0, n\to\infty$.

Indeed, assume that the relation $\lim\limits_{n\to\infty}\lambda_n=0$ is not true. Suppose that $\{\lambda_n\}_{n\in\mathbb{N}}$ has a bounded subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$. Then $\exists\lambda_0\in C:\lambda_{n_k}\to\lambda_0, k\to\infty$. We have $\lambda_{n_k}x_0\overset{w}{\to}\lambda_0x_0, k\to\infty$, and hence $\lambda_0=0$, since w-convergent sequence has unique limit. Assume that the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ has an unbounded subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ $:\lambda_{n_k}\to\infty, k\to\infty$. Consequently, $\lambda_{n_k}^{-1}\to 0, k\to\infty$. We have $\lambda_{n_k}^{-1}\cdot\lambda_{n_k}x=x\neq 0, \forall k\in\mathbb{N}$. On the other hand $\lim\limits_{k\to\infty}\left(\lambda_{n_k}^{-1}\lambda_{n_k}x\right)=\lim\limits_{k\to\infty}\lambda_{n_k}^{-1}\lim\limits_{k\to\infty}\left(\lambda_{n_k}x\right)=0$. So, we came upon a contradiction which proves the Lemma. Take w-fundamental sequence $\{\overline{\lambda}_n\}_{n\in\mathbb{N}}\subset K_{\bar{x}}^{w}$, $\lambda_n\equiv\{\lambda_k^{(n)}\}_{k\in\mathbb{N}}$. Then

$\lim\limits_{n,m\to\infty}\mu_{K_{\bar{x}}^{w}}\left(\overline{\lambda}_n;\overline{\lambda}_m;t\right)=1$ and $\lim\limits_{n,m\to\infty}\nu_{K_{\bar{x}}^{w}}\left(\overline{\lambda}_n;\overline{\lambda}_m;t\right)=0$, $\forall t\in\mathbb{R}$, i.e.

$$\left.\begin{aligned}&\lim_{n,m\to\infty}\inf_r\mu\left(\sum_{k=1}^{r}\lambda_k^{(n)}x_k;\sum_{k=1}^{r}\lambda_k^{(m)}x_k;t\right)=1,\\&\lim_{n,m\to\infty}\sup_r\nu\left(\sum_{k=1}^{r}\lambda_k^{(n)}x_k;\sum_{k=1}^{r}\lambda_k^{(m)}x_k;t\right)=0,\end{aligned}\right\}\tag{9}$$

$\forall t\in\mathbb{R}$. In the further we will assume that the functions μ and ν are invariant with respect to the shift, i.e. the following condition holds: 12) $\mu(x;y;t)=\mu(x-z;y-z;t), \nu(x;y;t)=\nu(x-z;y-z;t), \forall x,y,z\in X, \forall t\in\mathbb{R}$. Take into account the conditions 3) and 8), hence we directly obtain that

$$\mu(x;0;t)=\mu(-x;0;t), \nu(x;0;t)=\nu(-x;0;t), \forall x\in X, \forall t\in\mathbb{R}.$$

It is absolutely clear that the functions $\mu_{K_{\bar{x}}^{w}}$ and $\nu_{K_{\bar{x}}^{w}}$ also possess these conditions. Thus

$$\mu(x;y;t)=\mu(-x;-y;t), \nu(x;y;t)=\nu(-x;-y;t), \forall x,y\in X, \forall t\in\mathbb{R}.\tag{10}$$

We have $\mu\left(\lambda_1^{(n)}x_1;\lambda_1^{(m)}x_1;t\right)\to 1, \nu\left(\lambda_1^{(n)}x_1;\lambda_1^{(m)}x_1;t\right)\to 0, n,m\to\infty$, $\forall t\in\mathbb{R}$. It directly follows from the relation (9). Consider

$$\begin{aligned}&\mu\left(\lambda_2^{(n)}x_2;\lambda_2^{(m)}x_2;t\right)\ge\min\left\{\mu\left(\lambda_2^{(n)}x_2;\lambda_2^{(m)}x_2+\lambda_1^{(m)}x_1-\lambda_1^{(n)}x_1;t\right);\right.\\&\left.\mu\left(\lambda_2^{(m)}x_2+\lambda_1^{(m)}x_1-\lambda_1^{(n)}x_1;\lambda_2^{(m)}x_2;t\right)\right\}=\min\left\{\mu\left(\lambda_1^{(n)}x_1+\lambda_2^{(n)}x_{2_0};\lambda_1^{(m)}x_1+\lambda_2^{(m)}x_2;t\right);\right.\\&\left.\mu\left(-\lambda_1^{(n)}x_1;-\lambda_1^{(m)}x_1;t\right)\right\}.\end{aligned}$$

Take into consideration the relations (9) and (10), from here we have $\mu\left(\lambda_2^{(n)}x_2;\lambda_2^{(m)}x_2;t\right)\to 1, n,m\to\infty$, $\forall t\in\mathbb{R}$. Similarly we obtain that $\nu\left(\lambda_2^{(n)}x_2;\lambda_2^{(m)}x_2;t\right)\to 0, n,m\to\infty$, $\forall t\in\mathbb{R}$. Continuing this reasoning, we get $\mu\left(\lambda_k^{(n)}x_k;\lambda_k^{(m)}x_k;t\right)\to 1$, $\nu\left(\lambda_k^{(n)}x_k;\lambda_k^{(m)}x_{k_0};t\right)\to 0, n,m\to\infty$, $\forall t\in\mathbb{R}$, for each fixed

$k \in \mathrm{N}$, i.e. $w\text{-}\lim\limits_{n,m\to\infty}\left(\lambda_k^{(n)}-\lambda_k^{(m)}\right)x_k = 0, \forall k \in \mathrm{N}$. By Lemma 1, from here it follows that the sequence $\left\{\lambda_k^{(n)}\right\}_{n\in \mathrm{N}}$ is fundamental for $\forall k\in\mathrm{N}$, and let $\lambda_k^{(n)} \to \lambda_k, n\to\infty$. Assume that $\overline{\lambda} \equiv \langle\lambda_k\rangle_{k\in\mathrm{N}}$ and let us show that $\lim\limits_{n\to\infty}\mu_{\mathrm{K}_{\bar{x}}^w}\left(\overline{\lambda}_n;\overline{\lambda};t\right)=1$ and $\lim\limits_{n\to\infty}\nu_{\mathrm{K}_{\bar{x}}^w}\left(\overline{\lambda}_n;\overline{\lambda};t\right)=0$, $\forall t\in\mathrm{R}$. Let us establish it with respect to $\mu_{\mathrm{K}_{\bar{x}}^w}$. Take $\forall\varepsilon;t>0$. It is clear that $\exists n_0$, $\forall n\ge n_0, \forall p\in\mathrm{N}$:

$$\mu_{\mathrm{K}_{\bar{x}}^w}\left(\overline{\lambda}_n;\overline{\lambda}_{n+p};t\right)>1-\varepsilon.$$

Consequently

$$\inf_r \mu\left(\sum_{k=1}^{r}\lambda_k^{(n)}x_k;\sum_{k=1}^{r}\lambda_k^{(n+p)}x_k;t\right)>1-\varepsilon, \forall n\ge n_0, \forall p\in\mathrm{N}. \tag{11}$$

For the further reasoning we will need the following condition:

13) from $\lambda_n\to\lambda, n\to\infty$, it follows that $w\text{-}\lim\limits_{n\to\infty}(\lambda_n x)=\lambda x$, i.e. $\lim\limits_{n\to\infty}\mu(\lambda_n x;\lambda x;t)=1$, $\lim\limits_{n\to\infty}\nu(\lambda_n x;\lambda x;t)=0$, $\forall t\in\mathrm{R}$, $\forall x\in X$.

From here it directly follows that, if $\lambda_n^{(k)}\to\lambda^{(k)}, n\to\infty, \forall k=\overline{1,r}$, then

$$\lim_{n\to\infty}\mu\left(\sum_{k=1}^{r}\lambda_n^{(k)}x_k;y;t\right)=\mu\left(\sum_{k=1}^{r}\lambda^{(k)}x_k;y;t\right),$$

$$\lim_{n\to\infty}\nu\left(\sum_{k=1}^{r}\lambda_n^{(k)}x_k;y;t\right)=\nu\left(\sum_{k=1}^{r}\lambda^{(k)}x_k;y;t\right), \forall\{x_1;...;x_r;y\}\subset X, \forall t\in\mathrm{R}.$$

Indeed, without loss of generality we will consider the case $r=2$. It is sufficient to lead the proof with respect to ν. Since, this scheme is applied to $\tilde{\mu}=1-\mu$. Let $\lambda_n\to\lambda,\ \mu_n\to\mu, n\to\infty.$ By definition

$$\nu(x;y;t)\le\max\{\nu(x;z;t);\nu(y;z;t)\}\le\nu(x;z;t)+$$
$$\nu(y;z;t), \forall x,y,z\in X, \forall t\in\mathrm{R}.$$

Hence

$$\nu(x;y;t)-\nu(x;z;t)\le\nu(y;z;t).$$

Similarly we obtain

$$\nu(x;z;t)-\nu(x;y;t)\le\nu(y;z;t).$$

Thus

$$|\nu(x;y;t)-\nu(x;z;t)|\le\nu(y;z;t). \tag{12}$$

Taking here $y=\lambda_n a, z=\lambda a$, we obtain $\lim\limits_{n\to\infty}\nu(x;\lambda_n a;t)=\nu(x;\lambda a;t)$, $\forall t\in\mathrm{R}$, and for $\forall x,a\in X$. On the other hand we have

$$\nu(\lambda_n x+\mu_n y;\lambda x+\mu y;t)\le\nu(\lambda_n x+\mu_n y;\mu_n y+\lambda x;t)+\nu(\mu_n y+\lambda x;\lambda x+\mu y;t).$$

Take into account the property 12) we have

$$\nu(\lambda_n x+\mu_n y;\lambda x+\mu y;t)\le\nu(\lambda_n x;\lambda x;t)+\nu(\mu_n y;\mu y;t).$$

Consequently, $w\text{-}\lim\limits_{n\to\infty}(\lambda_n x+\mu_n y)=\lambda x+\mu y$, $\forall x,y\in X$. From here it directly follows that, if $\lambda_n^{(k)} \to \lambda^{(k)}, n \to \infty, \forall k = \overline{1,r}$, then $w\text{-}\lim_{n\to\infty}\left(\sum_{k=1}^{r}\lambda_n^{(k)}x_k\right) = \sum_{k=1}^{r}\lambda^{(k)}x_k$, $\forall \{x_k\}_1^r \subset X$. If in (12) we assume $y = \sum_{k=1}^{r}\lambda_n^{(k)}x_k$ and $z = \sum_{k=1}^{r}\lambda^{(k)}x_k$, then we have $\lim_{n\to\infty}\nu\left(x;\sum_{k=1}^{r}\lambda_n^{(k)}x_k;t\right) = \nu\left(x;\sum_{k=1}^{r}\lambda^{(k)}x_k;t\right)$, $\forall t \in \mathbb{R}$, and $\forall\{x;x_1;...;x_r\} \subset X$. The similar results are also true with respect to μ. Then, passing to the limit in the inequality (11) as $p \to \infty$ we get

$$\inf_r \mu\left(\sum_{k=1}^{r}\lambda_k^{(n)}x_k;\sum_{k=1}^{r}\lambda_k x_k;t\right) \ge 1-\varepsilon, \forall n \ge n_0. \tag{13}$$

In the same way we obtain that $\exists m_0 \in \mathbb{N}$:

$$\sup_r \nu\left(\sum_{k=1}^{r}\lambda_k^{(n)}x_k;\sum_{k=1}^{r}\lambda_k x_k;t\right) \le \varepsilon, \forall n \ge m_0. \tag{14}$$

We have

$$\mu\left(\sum_{k=r}^{r+p}\lambda_k^{(n)}x_k;\sum_{k=r}^{r+p}\lambda_k x_k;t\right) = \mu\left(\sum_{k=r}^{r+p}\left(\lambda_k^{(n)}-\lambda_k\right)x_k;0;t\right) \ge$$

$$\ge \min\left\{\mu\left(\sum_{k=r}^{r+p}\left(\lambda_k^{(n)}-\lambda_k\right)x_k;-\sum_{k=1}^{r-1}\left(\lambda_k^{(n)}-\lambda_k\right)x_k;\frac{t}{2}\right);\mu\left(-\sum_{k=1}^{r-1}\left(\lambda_k^{(n)}-\lambda_k\right)x_k;0;\frac{t}{2}\right)\right\}$$

$$= \min\left\{\mu\left(\sum_{k=1}^{r+p}\left(\lambda_k^{(n)}-\lambda_k\right)x_k;0;\frac{t}{2}\right);\mu\left(\sum_{k=1}^{r-1}\left(\lambda_k^{(n)}-\lambda_k\right)x_k;0;\frac{t}{2}\right)\right\}.$$

Take into account the inequality (13) we obtain

$$\mu\left(\sum_{k=r}^{r+p}\lambda_k^{(n)}x_k;\sum_{k=r}^{r+p}\lambda_k x_k;t\right) \ge 1-\varepsilon, \forall n \ge n_0, \forall r,p \in \mathbb{N}. \tag{15}$$

As $\overline{\lambda}_n \in K_{\bar{x}}^{w}$ it is clear that $\exists m_0^{(n)}$:

$$\mu\left(\sum_{k=m}^{m+p}\lambda_k^{(n)}x_k;0;t\right) > 1-\varepsilon, \forall m \ge m_0^{(n)}, \forall p \in \mathbb{N}. \tag{16}$$

We have

$$\mu\left(\sum_{k=m}^{m+p}\lambda_k x_k;0;t\right) \ge \min\left\{\mu\left(\sum_{k=m}^{m+p}\lambda_k x_k;\sum_{k=m}^{m+p}\lambda_k^{(n)}x_k;t\right);\mu\left(\sum_{k=m}^{m+p}\lambda_k^{(n)}x_k;0;t\right)\right\}.$$

Take into account the relations (15) and (16) from here we obtain

$$\mu\left(\sum_{k=m}^{m+p}\lambda_k x_k;0;t\right) \ge 1-\varepsilon, \forall m \ge m_0^{(n)}, \forall p \in \mathbb{N}.$$

In the same way we establish that $\exists m_1 \in \mathbb{N}: \nu\left(\sum_{k=m}^{m+p}\lambda_k x_k;0;t\right) \le \varepsilon, \forall m \ge m_1$, $\forall p \in \mathbb{N}$. It follows that the series $\sum_{k=1}^{\infty}\lambda_k x_k$ is weakly fuzzy convergent in X, i.e. if X is weakly complete, then $\exists w\text{-}\lim_{m\to\infty}\sum_{k=1}^{m}\lambda_k x_k$. Consequently, $\overline{\lambda} \in K_{\bar{x}}^{w}$, and the relations (13),(14) implie that $\lim\limits_{n\to\infty}\mu_{\mathrm{K}_{\bar{x}}^{w}}\left(\bar{\lambda}_n;\bar{\lambda};t\right)=1$, $\lim\limits_{n\to\infty}\nu_{\mathrm{K}_{\bar{x}}^{w}}\left(\bar{\lambda}_n;\bar{\lambda};t\right)=0$, $\forall t\in\mathrm{R}$. As a result we obtain that the space $\left(\mathrm{K}_{\bar{x}}^{w};\mu_{\mathrm{K}_{\bar{x}}^{w}};\nu_{\mathrm{K}_{\bar{x}}^{w}}\right)$ is weakly fuzzy complete. Thus, we have proved the following

Theorem 2. Let $(X;\mu;\nu)$ be a fuzzy weakly complete metric space with conditions $\alpha),\beta)$, 12) and 13). If $\{x_n\}_{n\in\mathrm{N}}\subset X$ is a nondegenerate system, then the space of coefficients $\left(\mathrm{K}_{\bar{x}}^{w};\mu_{\mathrm{K}_{\bar{x}}^{w}};\nu_{\mathrm{K}_{\bar{x}}^{w}}\right)$ is a weakly fuzzy complete metric space.

Consider operator $T:\mathrm{K}_{\bar{x}}^{w}\to X$ defined by

$$T\bar{\lambda}=\sum_{n=1}^{\infty}\lambda_n x_n,\ \bar{\lambda}\equiv\{\lambda_n\}_{n\in\mathrm{N}}\in\mathrm{K}_{\bar{x}}^{w}.$$

Let $w\text{-}\lim\limits_{n\to\infty}\bar{\lambda}_n=\bar{\lambda}$ in $\mathrm{K}_{\bar{x}}^{w}$, where $\bar{\lambda}_n\equiv\left\{\lambda_k^{(n)}\right\}_{k\in\mathrm{N}}\in\mathrm{K}_{\bar{x}}^{w}$. We have

$$\mu\left(T\bar{\lambda}_n;T\bar{\lambda};t\right)=\mu\left(\sum_{k=1}^{\infty}\left(\lambda_k^{(n)}-\lambda_k\right)x_k;0;t\right)\geq$$

$$\inf_m\mu\left(\sum_{k=1}^{m}\left(\lambda_k^{(n)}-\lambda_k\right)x_k;0;t\right)=\mu_{\mathrm{K}_{\bar{x}}^{w}}\left(\bar{\lambda}_n;\bar{\lambda};t\right).$$

It follows directly that $w\text{-}\lim\limits_{n\to\infty}T\bar{\lambda}_n=T\bar{\lambda}$, i.e. the operator T is weakly fuzzy continuous. Let $\bar{\lambda}\in KerT$, i.e. $T\bar{\lambda}=0\Rightarrow\sum_{n=1}^{\infty}\lambda_n x_n=0$, where $\bar{\lambda}\equiv\{\lambda_n\}_{n\in\mathrm{N}}\in\mathrm{K}_{\bar{x}}^{w}$. It is clear that if the system $\{x_n\}_{n\in\mathrm{N}}$ is w-linearly independent, then $\lambda_n=0$, $\forall n\in\mathrm{N}$, and, as a result, $KerT=\{0\}$. In this case $\exists T^{-1}:X\supset ImT\to\mathrm{K}_{\bar{x}}^{w}$. Denote by $\{\bar{e}_n\}_{n\in\mathrm{N}}\subset\mathrm{K}_{\bar{x}}^{w}$ a canonical system in $\mathrm{K}_{\bar{x}}^{w}$, where $\bar{e}_n=\{\delta_{nk}\}_{k\in\mathrm{N}}\in\mathrm{K}_{\bar{x}}^{w}$. Obviously, $T\bar{e}_n=x_n$, $\forall n\in\mathrm{N}$. Let us prove that $\{\bar{e}_n\}_{n\in\mathrm{N}}$ forms an w-basis for $\mathrm{K}_{\bar{x}}^{w}$. Take $\forall\bar{\lambda}\equiv\{\lambda_n\}_{n\in\mathrm{N}}\in\mathrm{K}_{\bar{x}}^{w}$ and show that the series $\sum_{n=1}^{\infty}\lambda_n\bar{e}_n$ is weakly fuzzy convergent in $\mathrm{K}_{\bar{x}}^{w}$. In fact, the existence of w - $\lim\limits_{m\to\infty}\sum_{n=1}^{m}\lambda_n x_n$ in X_w implies that $\forall\varepsilon;t>0$, $\exists m_0\in\mathrm{N}$:

$$\mu\left(\sum_{n=m}^{m+p}\lambda_n x_n;0;t\right)>1-\varepsilon,\forall m\geq m_0,\forall p\in\mathrm{N}.$$

We have

$$\mu_{\mathrm{K}_{\bar{x}}^{w}}\left(\sum_{n=m}^{m+p}\lambda_n\bar{e}_n;0;t\right)=\inf_r\left(\sum_{n=m}^{r}\lambda_n x_n;0;t\right)\geq1-\varepsilon,\forall m\geq m_0,\forall p\in\mathrm{N}.$$

It follows that the series $\sum_{n=1}^{\infty}\lambda_n\bar{e}_n$ is weakly fuzzy convergent in $\mathrm{K}_{\bar{x}}^{w}$. Moreover

$$\mu_{\mathrm{K}_{\bar{x}}^{w}}\left(\bar{\lambda}-\sum_{n=1}^{m}\lambda_n\bar{e}_n;0;t\right)=\mu_{\mathrm{K}_{\bar{x}}^{w}}\left(\{\ldots;0;\lambda_{m+1};\ldots\};0;t\right)=$$

$$= \inf_{r} \mu\left(\sum_{n=m+1}^{r} \lambda_n x_n;0;t \right) \geq 1-\varepsilon, \forall m \geq m_0.$$

Consequently, $w\text{-}\lim_{m\to\infty} \sum_{n=1}^{m} \lambda_n \bar{e}_n = \bar{\lambda}$, i.e. $\bar{\lambda} \overset{w}{=} \sum_{n=1}^{m} \lambda_n \bar{e}_n$. Consider the functionals $e_n^*(\bar{\lambda}) = \lambda_n$, $\forall n \in \mathrm{N}$. Let us show that they are w-continuous. Let $w\text{-}\lim_{n\to\infty} \bar{\lambda}_n = \bar{\lambda}$, where $\bar{\lambda}_n \equiv \{\lambda_k^{(n)}\}_{k\in \mathrm{N}} \in \mathrm{K}_{\bar{x}}^{w}$. As established in the proof of Theorem 2, we have $\lambda_k^{(n)} \to \lambda_k$ as $n \to \infty$, i.e. $e_k^*(\bar{\lambda}_n) \to e_k^*(\bar{\lambda})$, $n \to \infty$, for $\forall k \in \mathrm{N}$. Thus, e_k^* is w-continuous in $\mathrm{K}_{\bar{x}}^{w}$ for $\forall k \in \mathrm{N}$. On the other hand, it is easy to see that $e_n^*(\bar{e}_k) = \delta_{nk}$, $\forall n,k \in \mathrm{N}$, i.e. $\{e_n^*\}_{n\in \mathrm{N}}$ is w-biorthogonal to $\{\bar{e}_n\}_{n\in \mathrm{N}}$. As a result we obtain that the system $\{\bar{e}_n\}_{n\in \mathrm{N}}$ forms an w-basis for $\mathrm{K}_{\bar{x}}^{w}$. So we get the validity of the following

Theorem 3. Let $(X;\mu;\nu)$ be a fuzzy weakly complete metric space with conditions $\alpha),\beta)$, 12) and 13). Let $\{x_n\}_{n\in \mathrm{N}} \subset X$ be a nondegenerate system. Then the corresponding space of coefficients $\left(\mathrm{K}_{\bar{x}}^{w};\mu_{\mathrm{K}_{\bar{x}}^{w}};\nu_{\mathrm{K}_{\bar{x}}^{w}}\right)$ is weakly fuzzy complete with canonical w-basis $\{\bar{e}_n\}_{n\in \mathrm{N}}$.

Suppose that the system $\{x_n\}_{n\in \mathrm{N}}$ is w-linearly independent and ImT is closed. Then it is easily seen that $\{x_n\}_{n\in \mathrm{N}}$ forms an w-basis for ImT and, in case of its w-completeness in X_w, it forms an w-basis for X_w. In this case, $\mathrm{K}_{\bar{x}}^{w}$ and X_w are w-isomorphic, and T is an w-isomorphism between them. The opposite of it is also true, i.e. if the above-defined operator T is an w-isomorphism between $\mathrm{K}_{\bar{x}}^{w}$ and X_w, then the system $\{x_n\}_{n\in \mathrm{N}}$ forms an w-basis for X_w. We will call T a coefficient operator. Thus, the following theorem holds.

Theorem 4. Let $(X;\mu;\nu)$ be a fuzzy weakly complete metric space with conditions $\alpha),\beta)$, 12) and 13). Let $\{x_n\}_{n\in \mathrm{N}} \subset X$ be a nondegenerate system, $\left(\mathrm{K}_{\bar{x}}^{w};\mu_{\mathrm{K}_{\bar{x}}^{w}};\nu_{\mathrm{K}_{\bar{x}}^{w}}\right)$ be a corresponding weakly fuzzy complete normed space and $T:\mathrm{K}_{\bar{x}}^{w} \to X_w$ be a correspondence coefficient operator. System $\{x_n\}_{n\in \mathrm{N}}$ forms an w-basis for X_w if and only if the operator T is an isomorphism between $\mathrm{K}_{\bar{x}}^{w}$ and X_w.

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