

Variable-Basis Fuzzy Interior Operators

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Abstract

For a topological space it is well-known that the associated closure and interior operators provide equivalent descriptions of set-theoretic topology; but it is not generally true in other categories, consequently it makes sense to define and study the notion of interior operators I in the context of fuzzy set theory, where we can find categories in a lattice-theoretical context. Fuzzy interior operators have been studied by U. Höhle, A. Šostak and others, (1999), these works were used to describe L -topologies on a set X . More recently, M. Diker, S. Dost and A. Uğur (2009) present interior and closure operators on texture spaces in the sense of Čech, and F. G. Shi (2009) studies interior operators via L -fuzzy neighborhood systems.

The aim of this paper is to propose a more general theory of variable-basis fuzzy interior operators, employing both categorical tools and the lattice theoretical foundations investigated by S. E. Rodabaugh (1999), where the lattices are usually non-complemented. Furthermore, we construct some topological categories.

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1. Introduction

For a topological space it is well-known that the associated closure and interior operators provide equivalent descriptions of set-theoretic topology; but it is not generally true in other categories, consequently it makes sense to define and study the notion of interior operators I in the context of fuzzy set theory, where we can find categories in a lattice-theoretical context.

Interior operators are very useful tools in several areas of classical mathematics, its applications such as *Geographic information systems* and in general category theory. In

fuzzy set theory, fuzzy interior operators have been studied by U. Höhle, A. Šostak and others, (see e.g. [5]), these works were used to describe L -topologies on a set X .

More recently, M. Diker, S. Dost and A. Uğur present interior and closure operators on texture spaces in the sense of Čech (see [4]), and F. G. Shi ([9]) studies interior operators via L -fuzzy neighborhood systems. On the other hand, W. Shi and K. Liu ([10]) present a development of computational fuzzy topology, which is based on fuzzy interior and closure operators in order to get topological relations between spatial objects and Geographic information systems.

The aim of this paper is to propose a more general theory of variable-basis fuzzy interior operators, employing both categorical tools and the lattice theoretical foundations investigated in [7] and [5], where the lattices are usually non-complemented.

The paper is organized as follows: Following [7] and [5] we introduce, in section 1, the basic lattice theoretical foundations. In section 2, we present the concept of variable-basis fuzzy interior operators and then we construct a topological category $(\text{VBIO-SET}, U)$. In section 3, we study some additional properties of interior operators: idempotent and productive interior operators as well as open fuzzy sets and open morphisms. Finally in section 4, we present some examples of various classes of interior operators.

2. From Lattice Theoretic Foundations

Let (L, \leq) be a complete, infinitely distributive lattice, i.e. (L, \leq) is a partially ordered set such that for every subset $A \subset L$ the join $\bigvee A$ and the meet $\bigwedge A$ are defined, moreover $(\bigvee A) \wedge \alpha = \bigvee \{a \wedge \alpha \mid a \in A\}$ and $(\bigwedge A) \vee \alpha = \bigwedge \{a \vee \alpha \mid a \in A\}$ for every $\alpha \in L$. In particular, $\bigvee L = \top$ and $\bigwedge L = \perp$ are respectively the universal upper and the universal lower bounds in L . We assume that $\perp \neq \top$, i.e. L has at least two elements.

2.1. Complete quasi-monoidal lattices

The definition of complete quasi-monoidal lattices introduced by S. E. Rodabaugh in [7] is the following:

A *cqm*-lattice (short for complete quasi-monoidal lattice) is a triple (L, \leq, \otimes) provided with the following properties

- (1) (L, \leq) is a complete lattice with upper bound \top and lower bound \perp .
- (2) $\otimes : L \times L \rightarrow L$ is a binary operation satisfying the following axioms:
 - (a) \otimes is isotone in both arguments, i.e. $\alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2$ implies $\alpha_1 \otimes \beta_1 \leq \alpha_2 \otimes \beta_2$;
 - (b) \top is idempotent, i.e. $\top \otimes \top = \top$.

The category $CQML$ comprises the following data:

- (a) **Objects:** Complete quasi-monoidal lattices.
- (b) **Morphisms:** All *SET* morphisms, between the above objects, which preserve \otimes and \top and arbitrary \bigvee .
- (c) Composition and identities are taken from *SET*.

The category *LOQML* is the dual of *CQML*, i.e. $LOQML = CQML^{op}$.

2.2. *GL*-monoids

A *GL*-monoid (see [5]) is a complete lattice enriched with a further binary operation \otimes , i.e. a triple $(\mathbb{L}, \leq, \otimes)$ such that:

- (1) \otimes is isotone, commutative and associative;
- (2) $(\mathbb{L}, \leq, \otimes)$ is integral, i.e. \top acts as the unity: $\alpha \otimes \top = \alpha, \forall \alpha \in \mathbb{L}$;
- (3) \perp acts as the zero element in $(\mathbb{L}, \leq, \otimes)$, i.e. $\alpha \otimes \perp = \perp, \forall \alpha \in \mathbb{L}$;
- (4) \otimes is distributive over arbitrary joins, i.e. $\alpha \otimes (\bigvee_{\lambda} \beta_{\lambda}) = \bigvee_{\lambda} (\alpha \otimes \beta_{\lambda}), \forall \alpha \in \mathbb{L}, \forall \{\beta_{\lambda} : \lambda \in I\} \subset \mathbb{L}$;
- (5) $(\mathbb{L}, \leq, \otimes)$ is divisible, i.e. $\alpha \leq \beta$ implies the existence of $\gamma \in \mathbb{L}$ such that $\alpha = \beta \otimes \gamma$.

It is well known that every *GL*-monoid is residuated, i.e. there exists a further binary operation “ \mapsto ” (implication) on \mathbb{L} satisfying the following condition:

$$\alpha \otimes \beta \leq \gamma \iff \alpha \leq (\beta \mapsto \gamma) \quad \forall \alpha, \beta, \gamma \in \mathbb{L}.$$

Explicitly implication is given by

$$\alpha \mapsto \beta = \bigvee \{\lambda \in \mathbb{L} \mid \alpha \otimes \lambda \leq \beta\}.$$

If X is a set and L is a *GL*-monoid (or a complete quasi-monoidal lattice), then the fuzzy powerset L^X in an obvious way can be pointwise endowed with a structure of a *GL*-monoid (or of a complete quasi-monoidal lattice). In particular the L -sets 1_X and 0_X defined by $1_X(x) = \top$ and $0_X(x) = \perp \forall x \in X$ are respectively the universal upper and lower bounds in L^X .

2.3. Powerset operator foundations

We give the powerset operators, developed and justified in detail by S.E. Rodabaugh in [7] and [8]. Let $f \in SET(X, Y)$, $L, M \in |CQML|$, $\phi \in LOQML(L, M)$, and $\wp(X), \wp(Y), L^X, M^Y$ be the classical powerset of X , the classical powerset of Y , the L -powerset of X , and the M -powerset of Y , respectively. Then the following powerset operators are defined:

1. $f^{\rightarrow} : \wp(X) \rightarrow \wp(Y)$ by $f^{\rightarrow}(A) = \{f(x) \mid x \in A\}$
2. $f^{\leftarrow} : \wp(Y) \rightarrow \wp(X)$ by $f^{\leftarrow}(B) = \{x \in X \mid f(x) \in B\}$
3. $f_L^{\rightarrow} : L^X \rightarrow L^Y$ by $f_L^{\rightarrow}(a)(y) = \bigvee_{f(x)=y} a(x)$
4. $f_L^{\leftarrow} : L^Y \rightarrow L^X$ by $f_L^{\leftarrow}(b) = b \circ f$
5. ${}^*\phi : L \rightarrow M$ by ${}^*\phi(\alpha) = \bigwedge \{\beta \in M \mid \alpha \leq \phi^{op}(\beta)\}$
6. $\langle {}^*\phi \rangle : L^X \rightarrow M^X$ by $\langle {}^*\phi \rangle(a) = {}^*\phi \circ a$
7. $\langle \phi^{op} \rangle : M^X \rightarrow L^X$ by $\langle \phi^{op} \rangle(b) = \phi^{op} \circ b$
8. $(f, \Phi)^{\rightarrow} : L^X \rightarrow M^Y$ by $(f, \Phi)^{\rightarrow}(a) = \bigwedge \{b \mid f_L^{\rightarrow}(a) \leq (\Phi^{op})(b)\}$,
9. $(f, \Phi)^{\leftarrow} : M^Y \rightarrow L^X$ by $(f, \Phi)^{\leftarrow}(b) = \Phi^{op} \circ b \circ f$, in other words, that diagram

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 (f, \Phi)^{\leftarrow}(b) \downarrow & & \downarrow b \\
 L & \xleftarrow{\Phi^{op}} & M
 \end{array}$$

is commutative.

Note that these operators were defined taking into account the Adjoint functor theorem. Consequently, we have that f^{\rightarrow} , f_L^{\rightarrow} , and $(f, \Phi)^{\rightarrow}$ are left adjoints of f^{\leftarrow} , f_L^{\leftarrow} , and $(f, \Phi)^{\leftarrow}$, respectively.

3. Basic properties of variable-basis interior operators

In this section we consider a subcategory \mathcal{D} of CQML in order to construct fuzzy variable-basis interior operators on the category $SET \times \mathcal{D}$ that has as objects all pairs (X, L) , where X is a set and L is an object of \mathcal{D} , as morphisms from (X, L) to (Y, M) all pairs of maps (f, ϕ) with $f \in SET(X, Y)$ and $\phi \in CQML(L, M)$, identities given by $id_{(X, L)} = (id_X, id_L)$, and composition defined by

$$(f, \phi) \circ (g, \psi) = (f \circ g, \phi \circ \psi).$$

Definition 3.1. An interior operator of the category $SET \times \mathcal{D}$ is given by a family

$$\begin{aligned}
 &I = (i_{XL})_{(X, L) \in |SET \times \mathcal{D}|} \text{ of maps} \\
 &i_{XL} : L^X \longrightarrow L^X \text{ that satisfies the requirement:}
 \end{aligned}$$

$$(I_1) \text{ (Contraction) } i_x(u) \leq u \text{ for all } u \in L^X;$$

(I₂) (Monotonicity) if $u \leq v$ in L^X , then $i_X(u) \leq i_X(v)$;

(I₃) (Upper bound) $i_X(1_X) = 1_X$.

Definition 3.2. A fuzzy variable-basis I -space is a triple (X, L, i_{XL}) , where (X, L) is an object of $SET \times \mathcal{D}$ and i_{XL} is an interior map on (X, L) .

Definition 3.3. A morphism $(f, \phi) : (X, L) \longrightarrow (Y, M)$ in $SET \times \mathcal{D}$ is said to be fuzzy I -continuous if

$$(f, \phi)^{\leftarrow}(i_{YM}(v)) \leq i_{XL}((f, \phi)^{\leftarrow}(v)) \text{ for all } v \in M^Y. \quad (1)$$

Proposition 3.4. Consider two fuzzy I -continuous morphisms $(f, \phi) : (X, L) \longrightarrow (Y, M)$ and $(g, \psi) : (Y, M) \longrightarrow (Z, N)$, then the morphism $(g, \psi) \circ (f, \phi)$ is fuzzy I -continuous.

Proof. Since $(g, \psi) : (Y, M) \longrightarrow (Z, N)$ is I -continuous we have

$$(g, \psi)^{\leftarrow}(i_{ZN}(w)) \leq i_{YM}((g, \psi)^{\leftarrow}(w)) \text{ for all } w \in N^Z$$

it follows that

$$(f, \phi)^{\leftarrow}((g, \psi)^{\leftarrow}(i_{ZN}(w))) \leq (f, \phi)^{\leftarrow}(i_{YM}((g, \psi)^{\leftarrow}(w)))$$

now, by the fuzzy I -continuity of (f, ϕ) ,

$$(f, \phi)^{\leftarrow}(i_{YM}(v)) \leq i_{XL}((f, \phi)^{\leftarrow}(v)) \text{ for all } v \in M^Y,$$

in particular for $v = (g, \psi)^{\leftarrow}(w)$,

$$(f, \phi)^{\leftarrow}(i_{YM}((g, \psi)^{\leftarrow}(w))) \leq i_{XL}((f, \phi)^{\leftarrow}((g, \psi)^{\leftarrow}(w))),$$

therefore

$$((g, \psi) \circ (f, \phi))^{\leftarrow}(i_{ZN}(w)) \leq i_{XL}(((g, \psi) \circ (f, \phi))^{\leftarrow}(w)).$$

■

As a consequence we obtain

Definition 3.5. The category VBIO-SET that has as objects all triples (X, L, i_{XL}) where (X, L) is an object of $SET \times \mathcal{D}$ and $i_{XL} : L^X \longrightarrow L^X$ is a fuzzy interior map, as morphisms from (X, L, i_{XL}) to (Y, M, i_{YM}) all pairs of fuzzy I -continuous functions $(f, \phi) : (X, L, i_{XL}) \longrightarrow (Y, M, i_{YM})$, identities and composition as in $SET \times \mathcal{D}$.

3.1. The lattice structure of all interior operators

We consider the collection

$$I(SET, L) \text{ for all } L \in \mathcal{D}$$

of all interiors operators on $SET \times \{L\}$. It is ordered by

$$I \leq J \leftrightarrow i_X(u) \leq j_X(u), \text{ for all set } X, \text{ and for all } u \in L^X.$$

This way $I(SET, L)$ inherits a lattice structure from L :

Proposition 3.6. Every family $(I_\lambda)_{\lambda \in \Lambda}$ in $I(SET, L)$ has a join $\bigvee_{\lambda \in \Lambda} I_\lambda$ and a meet $\bigwedge_{\lambda \in \Lambda} I_\lambda$ in $I(SET, L)$. The discrete interior operator

$$I_D = (i_{D_X})_{X \in |SET|} \text{ with } i_{D_X}(u) = u \text{ for all } u \in L^X$$

is the largest element in $I(SET, L)$, and the trivial interior operator

$$I_T = (i_{T_X})_{X \in |SET|} \quad \text{with} \quad (i_{T_X})(u) = \begin{cases} 0_X & \text{for all } u \neq 1_X \\ 1_X & \text{if } u = 1_X \end{cases}$$

is the least one.

Proof. For $\Lambda \neq \emptyset$, let $\tilde{I} = \bigvee_{\lambda \in \Lambda} I_\lambda$, then

$$\tilde{i}_X = \bigvee_{\lambda \in \Lambda} i_{\lambda_X},$$

where X is an arbitrary set, satisfies

- $\tilde{i}_X(u) \leq u$, because $i_{\lambda_X}(u) \leq u$ for all $u \in L^X$ and for all $\lambda \in \Lambda$.
- If $u_1 \leq u_2$ in L^X then $i_{\lambda_X}(u_1) \leq i_{\lambda_X}(u_2)$ for all $\lambda \in \Lambda$, therefore $\tilde{i}_X(u_1) \leq \tilde{i}_X(u_2)$.
- Since $i_{\lambda_X}(1_X) = 1_X$ for all $\lambda \in \Lambda$, we have that $\tilde{i}_X(1_X) = 1_X$.

Similarily $\bigvee_{\lambda \in \Lambda} I_{\lambda_X}$, I_{T_X} and I_{D_X} are interior operators. ■

Consequently,

Corollary 3.7. For every set X

$$I(X) = \{i_X \mid i_X \text{ is an interior map on } X\}$$

is a complete lattice.

3.2. Initial variable-basis interior operator

Let (Y, M, i_{YM}) be an object of the category VBIO-SET and let (X, L) be an object of the category $SET \times \mathcal{D}$.

For each morphism $(f, \phi) : (X, L) \longrightarrow (Y, M)$ in $SET \times \mathcal{D}$ we define on (X, L) the map $\hat{i}_{XL} : L^X \longrightarrow L^X$ by

$$\hat{i}_{XL} = (f, \phi)^\leftarrow \circ i_{YM} \circ (f, \phi)_*, \quad (2)$$

where $(f, \phi)_*$ is the right adjoint of $(f, \phi)^\leftarrow$. In other words, the following diagram is commutative

$$\begin{array}{ccc} L^X & \xrightarrow{(f, \phi)_*} & M^Y \\ \hat{i}_{XL} \downarrow \dots & & \downarrow i_{YM} \\ L^X & \xleftarrow{(f, \phi)^\leftarrow} & M^Y \end{array}$$

Proposition 3.8. \hat{i}_{LX} an interior map.

Proof.

(1) (Contraction)

$$\begin{aligned} \hat{i}_{LX}(u) &= ((f, \phi)^\leftarrow \circ i_{MY} \circ (f, \phi)_*)(u) \\ &\leq ((f, \phi)^\leftarrow \circ (f, \phi)_*)(u) \leq u; \end{aligned}$$

(2) (Monotonicity) If $u_1 \leq u_2$ then $(f, \phi)_*(u_1) \leq (f, \phi)_*(u_2)$, therefore

$$i_{MY}((f, \phi)_*(u_1)) \leq i_{MY}((f, \phi)_*(u_2)),$$

consequently

$$(f, \phi)^\leftarrow(i_{MY}((f, \phi)_*(u_1))) \leq (f, \phi)^\leftarrow(i_{MY}((f, \phi)_*(u_2)))$$

that is, $\hat{i}_{LX}(u_1) \leq \hat{i}_{LX}(u_2)$;

(3) (Upper bound)

$$\begin{aligned} \hat{i}_{LX}(1_X) &= (f, \phi)^\leftarrow \circ i_{MY} \circ (f, \phi)_*(1_X) = (f, \phi)^\leftarrow \circ i_{MY}(1_Y) \\ &= (f, \phi)^\leftarrow(1_Y) = 1_X. \end{aligned}$$

■

3.2.1 Structural source

Proposition 3.9. Let (X, L) be an object of $SET \times \mathcal{D}$, let $(Y_\lambda, M_\lambda, i_{Y_\lambda, M_\lambda})$ be a family of fuzzy variable I -spaces, where $\lambda \in \Lambda$ for some indexed set Λ , and let $(f_\lambda, \phi_\lambda) : (X, L) \rightarrow (Y_\lambda, M_\lambda)$ be a family of morphisms in $SET \times \mathcal{D}$. Then the structured source $[(X, L), ((f_\lambda, \phi_\lambda), (Y_\lambda, M_\lambda))]_{\lambda \in \Lambda}$ w.r.t the forgetful functor U from VBIO-SET to $SET \times \mathcal{D}$ has a unique initial lift $((X, L, \tilde{i}_{XL}) \rightarrow (Y_\lambda, M_\lambda, i_{Y_\lambda, M_\lambda}))$, where \tilde{i}_{XL} is the meet $\bigwedge_{\lambda \in \Lambda} \hat{i}_{XL}^\lambda$

of all initial interior maps \hat{i}_{XL}^λ w.r.t. $(f_\lambda, \phi_\lambda)$, where $\lambda \in \Lambda$.

Proof. We must show that for every object (Z, N, i_{ZN}) of VBIO-SET, each morphism $(g, \psi) : (Z, N) \rightarrow (X, L)$ is I-continuous iff each $(f_\lambda, \phi_\lambda) \circ (g, \psi)$ is I-continuous, for all $\lambda \in \Lambda$ and for all $u \in L^X$. In fact,

$$\begin{aligned} (g, \psi)^\leftarrow(\tilde{i}_{XL}(u)) &= (g, \psi)^\leftarrow\left(\bigwedge_{\lambda \in \Lambda} \hat{i}_{XL}^\lambda(u)\right) \\ &= \bigwedge_{\lambda \in \Lambda} (g, \psi)^\leftarrow(\hat{i}_{XL}^\lambda(u)) \\ &= \bigwedge_{\lambda \in \Lambda} (g, \psi)^\leftarrow((f_\lambda, \phi_\lambda)^\leftarrow \circ i_{Y_\lambda, M_\lambda} \circ (f_\lambda, \phi_\lambda)_*(u)) \\ &= \bigwedge_{\lambda \in \Lambda} [(g, \psi)^\leftarrow \circ (f_\lambda, \phi_\lambda)^\leftarrow] \circ i_{Y_\lambda, M_\lambda} \circ (f_\lambda, \phi_\lambda)_*(u), \end{aligned}$$

but as the composition $(f_\lambda, \phi_\lambda) \circ (g, \psi)$ for all $\lambda \in \Lambda$ is a continuous then

$$\begin{aligned} (g, \psi)^\leftarrow(\tilde{i}_{XL}(u)) &\leq \bigwedge_{\lambda \in \Lambda} i_{ZN} [(g, \psi)^\leftarrow \circ (f_\lambda, \phi_\lambda)^\leftarrow] \circ (f_\lambda, \phi_\lambda)_*(u) \\ &= \bigwedge_{\lambda \in \Lambda} i_{ZN}(g, \psi)^\leftarrow \circ [(f_\lambda, \phi_\lambda)^\leftarrow \circ (f_\lambda, \phi_\lambda)_*](u) \\ &\leq \bigwedge_{\lambda \in \Lambda} i_{ZN}(g, \psi)^\leftarrow(u) \\ &= i_{ZN}(g, \psi)^\leftarrow(u) \end{aligned}$$

then

$$(g, \psi)^\leftarrow(\tilde{i}_{XL}(u)) \leq i_{ZN}(g, \psi)^\leftarrow(u).$$

■

As a consequence of corollary (3.7), proposition (3.8) and proposition (3.9), we obtain

Theorem 3.10. The concrete category $(\text{VBIO-SET}, U)$ over $SET \times \mathcal{D}$ is a topological category.

4. Some additional properties of interior operators

Definition 4.1. The interior operator $I = (i_{XL})_{(X,L) \in |SET \times \mathcal{D}|}$ of definition (3.2) is called idempotent if the condition

$$i_{XL}(i_{XL}(u)) = i_{XL}(u) \quad \text{for all } u \in L^X$$

holds for every pair $(X, L) \in |SET \times \mathcal{D}|$.

Proposition 4.2. Let $I = (i_{XL})_{(X,L) \in |SET \times \mathcal{D}|}$ be an idempotent interior operator. Then the initial interior operator $\hat{I} = (\hat{i}_{XL})_{(X,L) \in |SET \times \mathcal{D}|}$ defined by

$$\hat{i}_{XL} = (f, \phi)^\leftarrow \circ i_{YM} \circ (f, \phi)_*$$

for each morphism $(f, \phi) : (X, L) \longrightarrow (Y, M)$ in $SET \times \mathcal{D}$ is also idempotent.

Proof. Suppose that $I = (i_{XL})_{(X,L) \in |SET \times \mathcal{D}|}$ is an idempotent interior operator and let $(f, \phi) : (X, L) \longrightarrow (Y, M)$ be a morphism. Then

$$\begin{aligned} \hat{i}_{XL} \circ \hat{i}_{XL} &= ((f, \phi)^\leftarrow \circ i_{YM} \circ (f, \phi)_*) \circ ((f, \phi)^\leftarrow \circ i_{YM} \circ (f, \phi)_*) \\ &\geq (f, \phi)^\leftarrow \circ (i_{YM} \circ i_{YM}) \circ (f, \phi)_* \\ &= (f, \phi)^\leftarrow \circ i_{YM} \circ (f, \phi)_* \\ &= \hat{i}_{XL}. \end{aligned}$$

On the other hand, the monotonicity condition of interior operators implies that

$$\hat{i}_{XL} \leq \hat{i}_{XL} \circ \hat{i}_{XL}. \quad \blacksquare$$

Definition 4.3. The interior operator $I = (i_{XL})_{(X,L) \in |SET \times \mathcal{D}|}$ of definition (3.1) is called

1. productive if the condition

$$i_{XL}(u \wedge v) = i_{XL}(u) \wedge i_{XL}(v) \quad \text{for all } u, v \in L^X$$

holds for every set X .

2. Fully productive if the condition

$$i_{XL}\left(\bigwedge_{\lambda \in \Lambda} u_\lambda\right) = \bigwedge_{\lambda \in \Lambda} i_{XL}(u_\lambda) \quad \text{for all } \{u_\lambda \mid \lambda \in \Lambda\} \subseteq L^X \text{ holds for every set } X.$$

Proposition 4.4. Let $I = (i_{XL})_{(X,L) \in |SET \times \mathcal{D}|}$ be a fully productive interior operator. Then the initial interior operator $\hat{I} = (\hat{i}_{XL})_{(X,L) \in |SET \times \mathcal{D}|}$ defined by

$$\hat{i}_{XL} = (f, \phi)^\leftarrow \circ i_{YM} \circ (f, \phi)_*$$

for each morphism $(f, \phi) : (X, L) \longrightarrow (Y, M)$ is also fully productive.

Proof. Suppose that $I = (i_{XL})_{(X,L) \in |SET \times \mathcal{D}|}$ be a fully productive interior operator and let $f : X \rightarrow Y$ be a function. Since $(f, \phi)_*$ is a right adjoint, it preserves all existing meets, and since $(f, \phi)^\leftarrow$ is both left and right adjoint, it preserves all existing joins and meets, so for all $\{u_\lambda \mid \lambda \in \Lambda\} \subseteq L^X$

$$\begin{aligned} \hat{i}_{XL}(\bigwedge_{\lambda \in \Lambda} u_\lambda) &= (f, \phi)^\leftarrow(i_{YM}((f, \phi)_*(\bigwedge_{\lambda \in \Lambda} u_\lambda))) \\ &= (f, \phi)^\leftarrow(i_{YM}(\bigwedge_{\lambda \in \Lambda} (f, \phi)_*(u_\lambda))) \\ &= (f, \phi)^\leftarrow(\bigwedge_{\lambda \in \Lambda} i_{YM}((f, \phi)_*(u_\lambda))) \\ &= \bigwedge_{\lambda \in \Lambda} (f, \phi)^\leftarrow(i_{YM}((f, \phi)_*(u_\lambda))) \\ &= \bigwedge_{\lambda \in \Lambda} \hat{i}_{XL} \end{aligned}$$

■

4.1. Open fuzzy sets

Definition 4.5. An L -fuzzy subset u of X is called I -open in (X, L) if it is equal to its interior, i.e., $i_{XL}(u) = u$. The fuzzy I -continuity condition (1) implies that I -openness is preserved by inverse images.

Proposition 4.6. Let $(f, \phi) : (X, L, c_{XL}) \rightarrow (Y, M, c_{YM})$ be a morphism in VBIO-SET. If $v \in M^Y$ is I -open then $(f, \phi)^\leftarrow(v)$ is i -open in (X, L) .

Proof. If $v = i_{YM}(v)$, for $v \in M^Y$, then $(f, \phi)^\leftarrow(v) = (f, \phi)^\leftarrow(i_{YM}(v)) \leq i_{XL}((f, \phi)^\leftarrow(v))$, so $i_{XL}((f, \phi)^\leftarrow(v)) = (f, \phi)^\leftarrow(v)$. ■

4.2. I -open morphisms

Definition 4.7. A morphism $(f, \phi) : (X, L, i_{XL}) \rightarrow (Y, M, i_{YM})$ between variable-basis fuzzy I -spaces is I -open if

$$i_{XL}((f, \phi)^\leftarrow(v)) \leq (f, \phi)^\leftarrow(i_{YM}(v)) \text{ for all } v \in L^Y. \quad (3)$$

Proposition 4.8. Let $(f, \phi) : (X, L, i_{XL}) \rightarrow (Y, M, i_{YM})$ and $(g, \psi) : (Y, M, i_{YM}) \rightarrow (Z, N, i_{ZN})$ be two I -open morphisms, then the morphism $(f, \phi) \circ (g, \psi)$ is I -open.

Proof. Since $(g, \psi) : (Y, M, i_{YM}) \rightarrow (Z, N, i_{ZN})$ is I -open, we have

$$i_{YM}((g, \psi)^\leftarrow(w)) \leq (g, \psi)^\leftarrow(i_{ZN}(w)) \text{ for all } w \in N^Z,$$

it follows that

$$(f, \phi)^\leftarrow(i_{YM}((g, \psi)^\leftarrow(w))) \leq (f, \phi)^\leftarrow((g, \psi)^\leftarrow(i_{ZN}(w)))$$

now, by the I -openness of (f, ϕ) ,

$$i_{XL}((f, \phi)^{\leftarrow}(v)) \leq (f, \phi)^{\leftarrow}(i_{YM}(v)) \quad \text{for all } v \in M^Y,$$

in particular for $v = (g, \psi)^{\leftarrow}(u)$,

$$i_{XL}((f, \phi)^{\leftarrow}((g, \psi)^{\leftarrow}(w))) \leq (f, \phi)^{\leftarrow}(i_{YM}((g, \psi)^{\leftarrow}(w)))$$

therefore

$$i_{XL}(((g, \psi) \circ (f, \phi))^{\leftarrow}(w)) \leq (((g, \psi) \circ (f, \phi))^{\leftarrow}(i_{ZN}(w))).$$

■

If we replace in the category VBIO-SET fuzzy I -continuous morphisms by I -open morphisms, we obtain another topological category. The morphisms $(f, \phi) : (X, L, i_{XL}) \longrightarrow (Y, M, i_{YM})$ between variable-basis fuzzy I -spaces which are bijective, fuzzy I -continuous and I -open, forms a group. We can say that a way of seeing variable-basis fuzzy topology is studying invariants of the action of these groups over the category $SET \times \mathcal{D}$.

5. Some examples of fuzzy interior operators

Example 5.1. Let $L = [0, 1]$ be the unit interval considered as an ordered subset of the real numbers \mathbb{R} , as well as a complete (not complemented) lattice.

- (i) For each topological space X , we define $i_X : I^X \rightarrow L^X$ by

$$i_X(u) = \bigvee \{v \in L^X \mid v \text{ is lower semi-continuous and } v \leq u\}.$$

Clearly, the family $I = (i_X)_{X \in |TOP|}$ is a fuzzy interior operator of the category TOP . Since the fixed points of the restriction of i_X to 2^X produces the open sets of X , this operator is an extension of the usual interior in TOP .

- (ii) For each compact topological space Y , we define $d_Y : L^Y \rightarrow L^Y$ by $d_Y(v) = m_v$, where m_v is the constant function on L^Y whose value is $m_v = \min\{v(y) \mid y \in Y\}$.

Undoubtedly, the family $D = (d_Y)_{Y \in |COMP|}$ is a fuzzy interior operator of the category $COMP$ (of compact topological spaces).

- (iii) Every map $f : X \rightarrow Y$ from a topological space X to a compact space Y is fuzzy ID-continuous since each constant map is lower semi-continuous.
- (iv) On the other hand, the only fuzzy DI-continuous maps between compact spaces on topological spaces are the constant.

Example 5.2. Let $X = \{x\}$ be a single point set and $L = [0, 1]$ be the usual unit interval. The maps $i_n : L^X \rightarrow L^X$ defined by $i_n(t) = t^n$, for $n = 1, 2, \dots$ are interior maps, from which just i_1 and $i_\infty = \lim_{n \rightarrow \infty} t^n$ are idempotent.

Example 5.3. For a GL -monoid \mathbb{L} and for an \mathbb{L} -topology $\tau \subseteq \mathbb{L}^X$, we define

$$i_x(u) = \bigvee \{v \in \tau \mid v \leq u\}.$$

These maps produce an interior operator of the category $L\text{-TOP}$ whose associated closure operator is

$$c_x(u) = \bigwedge_{v \in \tau} \{v \mid u \leq v\}.$$

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