

Ranking Particular Fuzzy Numbers Using Area, Mode, Spreads and Weights

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Abstract

Using area, mode, spreads and weights, we generalize the ranking method introduced in [8]. We apply the resulting generalization to rank particular fuzzy numbers introduced in some of our previous work. In addition, we give some numerical examples from which we give some conclusions and suggestions for some future work.

Keywords k-Trapezoidal-Triangular Fuzzy Numbers, k+1-Trapezoidal Fuzzy Numbers, Fuzzy Comparison Method, Ranking Function.

1. Introduction

Fuzzy set theory and its applications have become a subject that generates much interest among areas of research in the last few decades for many authors. Many articles were published since introducing the concepts of Fuzzy sets and Probability Measure of Fuzzy Events and fuzzy linear programming [9-11]. Ranking fuzzy numbers plays an important role in many fuzzy optimization problems and decision-making procedure [1-2], [6-8].

Dadgostar [1] proposed a fuzzy number comparison method called Partial Comparison Method. It is based on the fuzzy number division in comparison, the shapes of the convex numbers do not require special computations during comparison because it relies on the representation of fuzzy numbers as ordered sets of confidence intervals. Dorohonceanu and Marin [2] presented a fuzzy number comparison method based on the fuzzy number representations in fuzzy arithmetic described in [3]. They also described a variant based on the fuzzy number division in comparison used in partial comparison method. Triangular and trapezoidal shapes of membership functions were used to describe the method. In [6], we applied the method presented

in [2] to the fuzzy numbers introduced in [5]. In [7] we generalized the particular fuzzy numbers introduced in [5] by defining three particular fuzzy numbers determined by n real numbers and constants c_i , $0 \leq c_i \leq 1$, $i=1, \dots, k$, and we used the fuzzy comparison methods mentioned in [2] and [6] to compare between them. The method is based on the fuzzy number representation in fuzzy arithmetic and fuzzy number division in comparison used in [1].

Rao and Shankar [8] proposed a new method which is based on centroid of centroids to rank fuzzy quantities. A trapezoidal fuzzy number was split into three parts, two triangles and a rectangle. Then the centroids of these three parts were calculated followed by the calculation of the centroid of these centroids. Then a ranking procedure was defined depending on area and using mode and spreads in some cases.

In this paper, we generalize the method introduced in [8] and use it to rank the fuzzy numbers introduced in [7].

In section 2, we introduce the basic concepts, definitions and particular types of fuzzy numbers. In section 3, we generalize the method of [8] to our fuzzy numbers. In section 4, we give numerical examples. Finally comparing and comments are given in section 5.

2. Basic Concepts

Definition 1: A fuzzy subset A of some set Ω is defined by its membership function written $A(x)$ which produces values in $[0,1]$ for all x in Ω . That is $A(x)$ is a function mapping Ω into $[0,1]$. We place a bar over a letter to denote a fuzzy set, that is \bar{A} . The term crisp means not fuzzy. A crisp set is a regular set.

Definition 2: Let $\Omega = R$. An α -cut of \bar{A} , written $\bar{A}[\alpha]$, is defined as $\{x : \bar{A}(x) \geq \alpha\}$, for $0 < \alpha \leq 1$. $\bar{A}[0]$, the support of \bar{A} is defined as the closure of the union of all the $\bar{A}[\alpha]$, for $0 < \alpha \leq 1$.

Definition 3: A confidence interval is an interval of real numbers that provides a representation for an imprecise numerical value by means of its sharpest enclosing range.

Definition 4: A presumption level is an estimated truth-value about some knowledge. Presumption levels belong to the $[0, 1]$ interval; the maximum of estimated truth-value is at level 1 and the minimum is at level 0.

Definition 5: A fuzzy number \bar{N} is a fuzzy subset of the real numbers satisfying:

- (1) $\exists x \in R : \bar{N}(x) = 1$
- (2) $\bar{N}[\alpha]$ is a closed and bounded interval for $0 \leq \alpha \leq 1$.

A special type of fuzzy numbers $\bar{\bar{M}}$ is called a triangular fuzzy number. $\bar{\bar{M}}$ is defined by three numbers $a_1 < a_2 < a_3$ where (1) $\bar{\bar{M}}(x) = 1$ at $x = a_2$ (2) The graph of $\bar{\bar{M}}(x)$ on $[a_1, a_2]$ is a straight line from $(a_1, 0)$ to $(a_2, 1)$ and also on $[a_2, a_3]$ the graph is a straight line from $(a_2, 1)$ to $(a_3, 0)$ (3) $\bar{\bar{M}}(x) = 0$ for $x \leq a_1$ or $x \geq a_3$. We write $\bar{\bar{M}} = (a_1 / a_2 / a_3)$ for triangular fuzzy number $\bar{\bar{M}}$. If at least one of the graphs described above is not a straight line (curve), then $\bar{\bar{M}}$ is called triangular shaped fuzzy number and we write $\bar{\bar{M}} \approx (a_1 / a_2 / a_3)$.

Another special type of fuzzy numbers $\bar{\bar{M}}$ is called a trapezoidal fuzzy number. $\bar{\bar{M}}$ is defined by four numbers $a_1 < a_2 < a_3 < a_4$ where (1) $\bar{\bar{M}}(x) = 1$ on $[a_2, a_3]$ (2) The graph of $\bar{\bar{M}}(x)$ on $[a_1, a_2]$ is a straight line from $(a_1, 0)$ to $(a_2, 1)$ and also on $[a_3, a_4]$ the graph is a straight line from $(a_3, 1)$ to $(a_4, 0)$ (3) $\bar{\bar{M}}(x) = 0$ for $x \leq a_1$ or $x \geq a_4$. We write $\bar{\bar{M}} = (a_1, a_2, a_3, a_4)$ for trapezoidal fuzzy number $\bar{\bar{M}}$. If at least one of the graphs described above is not a straight line (curve), then $\bar{\bar{M}}$ is called trapezoidal shaped fuzzy number and we write $\bar{\bar{M}} \approx (a_1, a_2, a_3, a_4)$

If $\bar{\bar{M}}(x) = w < 1$ on $[a_2, a_3]$, then it is called a generalized trapezoidal fuzzy number.

A fuzzy number is represented as an ordered set of confidence intervals, each of them provides the related numerical value at a given presumption level $\alpha \in [0, 1]$. These confidence intervals should comply with the relation if $\alpha_1 > \alpha_2$ then $A_{\alpha_1} \subset A_{\alpha_2}$, where $\alpha_1, \alpha_2 \in [0, 1]$ and $A_{\alpha_1}, A_{\alpha_2}$ are the confidence intervals at presumption levels α_1, α_2 respectively. More details, properties and operations, can be found in [3], [4] and [12].

In [5], we introduced other types of fuzzy numbers. We considered a fuzzy number that is determined by five real numbers a_1, a_2, a_3, a_4 and c such that $a_1 < a_2 < a_3 < a_4$ and $0 < c < 1$, denoted by $\bar{\bar{N}}_c = (a_1 / a_2 / a_3 / a_4)$ or $(a_1, a_2, a_3, a_4)_c$ whose membership function is given by

$$\bar{\bar{N}}_c(x) = \begin{cases} 0, x \leq a_1 \\ \frac{c}{a_2 - a_1}(x - a_1), a_1 \leq x \leq a_2 \\ 1 - \frac{1-c}{(a_2 - a_3)^2}(2x - a_2 - a_3)^2, a_2 \leq x \leq a_3 \\ \frac{-c}{a_4 - a_3}(x - a_4), a_3 \leq x \leq a_4 \\ 0, x \geq a_4 \end{cases}$$

1. $\bar{\bar{N}}_c(x)$ is a line from $(a_1, 0)$ to (a_2, c) .
2. $\bar{\bar{N}}_c(x)$ is a parabola from (a_2, c) to (a_3, c) whose vertex is $(\frac{a_2 + a_3}{2}, 1)$ and focus is $(\frac{a_2 + a_3}{2}, 1 - \frac{(a_3 - a_2)^2}{16(1-c)})$.
3. $\bar{\bar{N}}_c(x)$ is a line from (a_3, c) to $(a_4, 0)$.

We called such a fuzzy number a Trapezoidal-Parabolic Fuzzy Number and if it is determined by five numbers and it is not of this form, we called it a Trapezoidal-Parabolic Shaped Fuzzy Number.

If we put $c = 0$ in the membership function of the trapezoidal-parabolic fuzzy number, then we get another fuzzy number. We called it parabolic fuzzy number, and is defined by two real numbers a_1 & a_2 with $a_1 < a_2$, denoted by $\bar{\bar{N}} = (a_1 / a_2)$ or $\bar{\bar{N}} = (a_1, a_2)$. The membership function $\bar{\bar{N}}(x)$ is given by

$$\bar{\bar{N}}(x) = \begin{cases} 0, x \leq a_1 \\ 1 - \frac{1}{(a_1 - a_2)^2}(2x - a_1 - a_2)^2, a_1 \leq x \leq a_2 \\ 0, x \geq a_2 \end{cases}$$

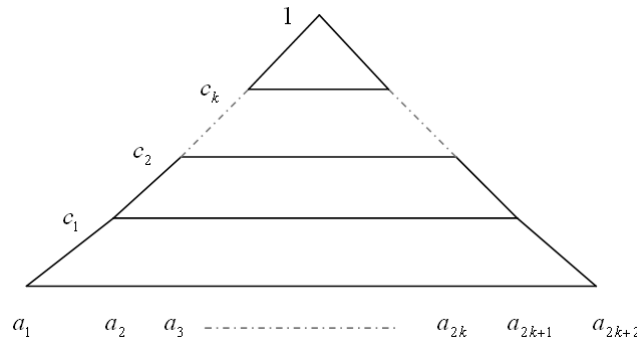
1. $\bar{\bar{N}}(x)$ is parabolic in $[a_1, a_2]$ whose vertex is $(\frac{a_1 + a_2}{2}, 1)$ and focus $(\frac{a_1 + a_2}{2}, 1 - \frac{(a_1 - a_2)^2}{16})$.
2. If $\bar{\bar{N}}(x)$ is not parabolic in $[a_1, a_2]$ then it is called a parabolic shaped fuzzy number and is denoted by $\bar{\bar{N}} \approx (a_1 / a_2)$ or $\bar{\bar{N}} \approx (a_1, a_2)$.
3. $\bar{\bar{N}}[\alpha] = [\frac{-(a_2 - a_1)}{2}\sqrt{1-\alpha} + \frac{a_2 + a_1}{2}, \frac{(a_2 - a_1)}{2}\sqrt{1-\alpha} + \frac{a_2 + a_1}{2}]$ for $0 \leq \alpha \leq 1$.
4. $\bar{\bar{N}} \approx (a_1 / \frac{a_1 + a_2}{2} / a_2)$ is a triangular shaped fuzzy number.

In [7] we generalized the fuzzy numbers in [5] and considered a fuzzy number that is determined by n real numbers $a_1, a_2, \dots, a_{2k+2}$ and $c_i, 0 \leq c_i \leq 1, i = 1, 2, \dots, k$ where n is a multiple of 3 and $a_1 < a_2 < \dots < a_{2k+2}$ (see figure 1). Denote it by $\bar{A}_{(c_1, \dots, c_k)} = (a_1, a_2, \dots, a_{2k+2})$ and whose membership function is given by

$$\mu_{\bar{A}_{(c_1, \dots, c_k)}}(x) = \begin{cases} 0, & x \leq a_1 \\ \frac{c_1(x - a_1)}{a_2 - a_1}, & a_1 \leq x \leq a_2 \\ \cdot \\ \cdot \\ \frac{-c_1(x - a_{2k+2})}{a_{2k+2} - a_{2k+1}}, & a_{2k+1} \leq x \leq a_{2k+2} \\ 0, & x \geq a_{2k+2} \end{cases}$$

- 1- $\mu_{\bar{A}_{(c_1, \dots, c_k)}}(x)$ is a line from $(a_1, 0)$ to (a_2, c_1) and from (a_{2k+1}, c_1) to $(a_{2k+2}, 0)$
- 2- $\mu_{\bar{A}_{(c_1, \dots, c_k)}}(x)$ is a line from (a_{i+1}, c_i) to $(a_{i+2}, c_{i+1}), i=1, 2, \dots, k-1$.
- 3- $\mu_{\bar{A}_{(c_1, \dots, c_k)}}(x)$ is a line from (a_{k+3+i}, c_{k-i}) to $(a_{k+4+i}, c_{k-i-1}), i=0, 1, 2, \dots, k-2$.
- 3- $\mu_{\bar{A}_{(c_1, \dots, c_k)}}(x)$ is a line from (a_{k+1}, c_k) to $(a_{k+2}, 1)$ and from $(a_{k+2}, 1)$ to (a_{k+3}, c_k)
- 4- $\mu_{\bar{A}_{(c_1, \dots, c_k)}}(x) = 0$ for $x \leq a_1, x \geq a_{2k+2}$.

We called such a fuzzy number a k-trapezoidal-triangular fuzzy number. If k=0, a k-trapezoidal-triangular Fuzzy Number is a triangular Fuzzy Number.



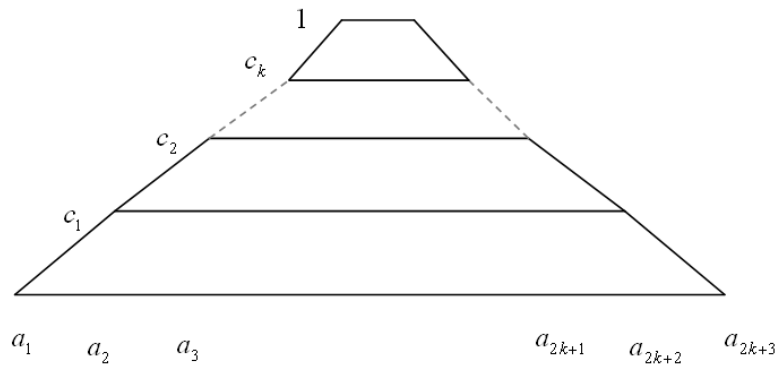
(figure 1)

We also considered a fuzzy number that is determined by n real numbers $a_1, a_2, \dots, a_{2k+3}$ and $c_i, 0 \leq c_i \leq 1, i=1, 2, \dots, k$ where n=4, 7, 10, ...and $a_1 < a_2 < \dots < a_{2k+3}$ (see figure 2). Denote it by $\bar{A}_{(c_1, \dots, c_k)} = (a_1, \dots, a_{2k+3})$ and whose membership function is given by

$$\mu_{\bar{A}(c_1, \dots, c_k)}^-(x) = \begin{cases} 0, x \leq a_1 \\ \frac{c_1(x - a_1)}{a_2 - a_1}, a_1 \leq x \leq a_2 \\ \cdot \\ \cdot \\ 1, a_{k+2} \leq x \leq a_{k+3} \\ \cdot \\ \cdot \\ \frac{-c_1(x - a_{n-k})}{a_{n-k} - a_{n-k-1}}, a_{2k+2} \leq x \leq a_{2k+3} \\ 0, x \geq a_{2k+3} \end{cases}$$

- 1- $\mu_{\bar{A}(c_1, \dots, c_k)}^-(x)$ is a line from $(a_1, 0)$ to (a_2, c_1) and from (a_{2k+2}, c_1) to $(a_{2k+3}, 0)$
- 2- $\mu_{\bar{A}(c_1, \dots, c_k)}^-(x)$ is a line from (a_{i+1}, c_i) to (a_{i+2}, c_{i+1}) , $i=1, 2, \dots, k-1$.
- 3- $\mu_{\bar{A}(c_1, \dots, c_k)}^-(x)$ is a line from (a_{k+4+i}, c_{k-i}) to (a_{k+5+i}, c_{k-i-1}) , $i=0, 1, 2, \dots, k-2$.
- 4- $\mu_{\bar{A}(c_1, \dots, c_k)}^-(x)$ is a line from (a_{k+1}, c_k) to $(a_{k+2}, 1)$ and from $(a_{k+3}, 1)$ to (a_{k+4}, c_k)
- 5- $\mu_{\bar{A}(c_1, \dots, c_k)}^-(x) = 1$ if $a_{k+2} \leq x \leq a_{k+3}$.
- 6- $\mu_{\bar{A}(c_1, \dots, c_k)}^-(x) = 0$ for $x \leq a_1, x \geq a_{2k+3}$.

We called such a fuzzy number a k+1-trapezoidal fuzzy number. If k=0, a k+1-trapezoidal Fuzzy Number is a trapezoidal Fuzzy Number.



(figure 2)

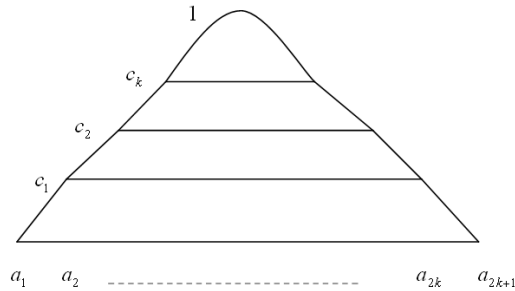
We also Considered a fuzzy number that is determined by n real numbers $a_1, a_2, \dots, a_{2k+1}$ and $c_i, 0 \leq c_i \leq 1, i=1, 2, \dots, k$ where $n=5, 8, 11, \dots$ and

$a_1 < a_2 < \dots < a_{2k+1}$ (see figure 3). Denote it by $\bar{A}_{(c_1, \dots, c_k)} = (a_1, \dots, a_{2k+1})$ and whose membership function is given by

$$\mu_{\bar{A}_{(c_1, \dots, c_k)}}(x) = \begin{cases} 0, x \leq a_1 \\ \frac{c_1(x - a_1)}{a_2 - a_1}, a_1 \leq x \leq a_2 \\ \cdot \\ \cdot \\ 1 - \frac{(1 - c_k)}{(a_{k+1} - a_{k+2})^2} (2x - a_{k+1} - a_{k+2})^2, a_{k+1} \leq x \leq a_{k+2} \\ \cdot \\ \cdot \\ \frac{-c_1(x - a_{2k+1})}{a_{2k+1} - a_{2k}}, a_{2k} \leq x \leq a_{2k+1} \\ 0, x \geq a_{2k+1} \end{cases}$$

- 1- $\mu_{\bar{A}_{(c_1, \dots, c_k)}}(x)$ is a line from $(a_1, 0)$ to (a_2, c_1) and from (a_{2k}, c_1) to $(a_{2k+1}, 0)$
- 2- $\mu_{\bar{A}_{(c_1, \dots, c_k)}}(x)$ is a line from (a_{i+1}, c_i) to (a_{i+2}, c_{i+1}) , $i=1, 2, \dots, k-1$.
- 3- $\mu_{\bar{A}_{(c_1, \dots, c_k)}}(x)$ is a line from (a_{k+2+i}, c_{k-i}) to (a_{k+3+i}, c_{k-i-1}) , $i=0, 1, 2, \dots, k-2$.
- 4- $\mu_{\bar{A}_{(c_1, \dots, c_k)}}(x)$ is a parabola from (a_{k+1}, c_k) to (a_{k+2}, c_k) whose vertex $(\frac{a_{k+1} + a_{k+2}}{2}, 1)$ and focus is $(\frac{a_{k+1} + a_{k+2}}{2}, 1 - \frac{(a_{k+2} - a_{k+1})^2}{16(1 - c_k)})$
- 6- $\mu_{\bar{A}_{(c_1, \dots, c_k)}}(x) = 0$ for $x \leq a_1, x \geq a_{2k+1}$.

We called such a fuzzy number a k-trapezoidal –parabolic fuzzy number. If $k=0$, a k- trapezoidal – parabolic Fuzzy Number is a parabolic Fuzzy Number. If $k=1$, a k- trapezoidal– parabolic Fuzzy Number is a trapezoidal–parabolic Fuzzy Number.



(figure 3)

Definition 6

Let $T_1 = (a_1, a_2, a_{2k+1}, a_{2k+2}; c_1), \dots, T_i = (a_i, a_{i+1}, a_{2k+2-i}, a_{2k+2-(i-1)}; c_i - c_{i-1}), i = 2, \dots, k$ be generalized trapezoidal fuzzy number and $T_{k+1} = (a_{k+1}, a_{k+2}, a_{k+3}; 1 - c_k)$ be a generalized triangular fuzzy number. If

$\bar{A} = (T_1, T_2, \dots, T_k, T_{k+1})$ & $\bar{B} = (T_{1^-}, T_{2^-}, \dots, T_{k^-}, T_{k+1^-})$ are two k-trapezoidal-triangular fuzzy numbers. Then:

$$1. \quad \bar{A} \oplus \bar{B} = (T_1 \oplus T_{1^-}, T_2 \oplus T_{2^-}, \dots, T_k \oplus T_{k^-}, T_{k+1} \oplus T_{k+1^-})$$

Where

$$T_i \oplus T_{i^-} = (a_i + a_{i^-}, a_{i+1} + a_{i+1^-}, a_{2k+2-i} + a_{2k+2-i^-}, a_{2k+2-(i-1)} + a_{2k+2-(i-1)^-}; c_i - c_{i-1})$$

And

$$T_{k+1} \oplus T_{k+1^-} = (a_{k+1} + a_{k+1^-}, a_{k+2} + a_{k+2^-}, a_{k+3} + a_{k+3^-}; 1 - c_k)$$

$$2. \quad \bar{A} \ominus \bar{B} = (T_1 \ominus T_{k+1^-}, T_2 \ominus T_{k^-}, \dots, T_k \ominus T_{2^-}, T_{k+1} \ominus T_{1^-})$$

Where

$$T_i \ominus T_{i^-} = (a_i - a_{2k+2-(i-1)^-}, a_{i+1} - a_{2k+2-i^-}, a_{2k+2-i} - a_{i+1^-}, a_{2k+2-(i-1)} - a_{i^-}; c_i - c_{i-1})$$

And

$$T_{k+1} \ominus T_{k+1^-} = (a_{k+1} - a_{k+3^-}, a_{k+2} - a_{k+2^-}, a_{k+3} - a_{k+1^-}; 1 - c_k)$$

3. If $M > 0$, then

$$M \bar{A} = (MT_1, MT_2, \dots, MT_k, MT_{k+1}) \text{ where}$$

$$MT_i = (Ma_i, Ma_{i+1}, Ma_{2k+2-i}, Ma_{2k+2-(i-1)}; c_i - c_{i-1}) \text{ and}$$

$$MT_{k+1} = (Ma_{k+1}, Ma_{k+2}, Ma_{k+3}; 1 - c_k)$$

And if $M < 0$, then

$$M \bar{A} = (MT_{k+1}, MT_k, \dots, MT_2, MT_1) \text{ where}$$

$$MT_i = (Ma_{2k+2-(i-1)}, Ma_{2k+2-i}, Ma_{i+1}, Ma_i; c_i - c_{i-1}) \text{ and}$$

$$MT_{k+1} = (Ma_{k+1}, Ma_{k+2}, Ma_{k+3}; 1 - c_k)$$

Notice that the same operations can be defined similarly for k+1-trapezoidal fuzzy numbers and all fuzzy numbers mentioned above.

In [6-7] we used the method in [1-2] to compare between such numbers. In section 3, we will generalize and use the method in [8] for comparison.

3. Ranking method

In [8], the introduced method is based on centroid of centroids to rank fuzzy quantities. A trapezoidal fuzzy number was split into three parts, two triangles and a rectangle. Then the centroids of these three parts were calculated followed by the calculation of the centroid of these centroids. They considered two generalized

trapezoidal fuzzy numbers, $\bar{A} = (a_1, b_1, c_1, d_1; w_1)$ & $\bar{B} = (a_2, b_2, c_2, d_2; w_2)$. After defining the ranking function (R), the mode, the spread, the left spread and the right spread of the generalized trapezoidal fuzzy numbers, the procedure to compare \bar{A} & \bar{B} was given.

Now, we will apply a similar method for other types of fuzzy numbers. Consider a k -trapezoidal-triangular fuzzy number $\bar{A}_{(c_1, \dots, c_k)} = (a_1, a_2, \dots, a_{2k+2})$ whose membership function is

$$\mu_{\bar{A}_{(c_1, \dots, c_k)}}(x) = \begin{cases} 0, & x \leq a_1 \\ \frac{c_1(x - a_1)}{a_2 - a_1}, & a_1 \leq x \leq a_2 \\ \dots \\ \frac{-c_1(x - a_{2k+2})}{a_{2k+2} - a_{2k+1}}, & a_{2k+1} \leq x \leq a_{2k+2} \\ 0, & x \geq a_{2k+2} \end{cases}$$

Divide it into k -trapezoidal fuzzy numbers $\bar{A}_1, \dots, \bar{A}_k$ and a triangular fuzzy number \bar{A}_{k+1} . The centroid of each trapezoidal fuzzy number \bar{A}_i is given by:

$$G_{\bar{A}_i}(x_i, y_i) = \left(\frac{2a_i + 7a_{i+1} + 7a_{2k+2-i} + 2a_{2k+2-(i-1)}}{18}, \frac{7(c_i - c_{i-1})}{18} + c_{i-1} \right) \text{ with } c_0 = 0 \text{ and}$$

the centroid of the triangular fuzzy number \bar{A}_{k+1} is given by

$$G_{\bar{A}_{k+1}}(x_{k+1}, y_{k+1}) = \left(\frac{a_{k+1} + 7a_{k+2} + a_{k+3}}{9}, \frac{7(1 - c_k)}{18} + c_k \right).$$

This can be found in [8] with adding c in the second coordinate to shift the fuzzy number up according to its position in the plane. Now, we consider the average point to be the reference point of the k -trapezoidal-triangular fuzzy number, that is:

$$G_{\bar{A}}(x, y) = \left(\frac{\sum_{i=1}^{k+1} x_i}{k+1}, \frac{\sum_{i=1}^{k+1} y_i}{k+1} \right) \text{ and the ranking function of } k\text{-trapezoidal-triangular}$$

fuzzy numbers is $R(\bar{A}) = x \cdot y$.

Now

$$\begin{aligned} \text{mode}(\bar{A}) &= \frac{1}{2} \int_0^{c_1} (a_2 + a_{2k+1}) dx + \frac{1}{2} \sum_{i=2}^k \int_{c_{i-1}}^{c_i} (a_{i+1} + a_{2k+2-i}) dx + \frac{1}{2} \int_{c_k}^1 (a_{k+2} + a_{k+2}) dx \\ &= \frac{c_1}{2} (a_2 + a_{2k+1}) + \frac{(1 - c_k)}{2} (a_{k+2}) + \frac{1}{2} \sum_{i=2}^k (a_{i+1} + a_{2k+2-i})(c_i - c_{i-1}) \end{aligned}$$

$$\begin{aligned}
\text{spread}(\bar{A}) &= \int_0^{c_1} (a_{2k+2} - a_1) dx + \sum_{i=2}^k \int_{c_{i-1}}^{c_i} (a_{2k+2-(i-1)} - a_i) dx + \int_{c_k}^1 (a_{k+3} - a_{k+1}) dx \\
&= c_1(a_{2k+2} - a_1) + (1 - c_k)(a_{k+3} - a_{k+1}) + \sum_{i=2}^k (a_{2k+2-(i-1)} - a_i)(c_i - c_{i-1}) \\
\text{leftspread}(\bar{A}) &= \int_0^{c_1} (a_2 - a_1) dx + \sum_{i=2}^k \int_{c_{i-1}}^{c_i} (a_{i+1} - a_i) dx + \int_{c_k}^1 (a_{k+2} - a_{k+1}) dx \\
&= c_1(a_2 - a_1) + (1 - c_k)(a_{k+2} - a_{k+1}) + \sum_{i=2}^k (a_{i+1} - a_i)(c_i - c_{i-1}) \\
\text{rightspread}(\bar{A}) &= \int_0^{c_1} (a_{2k+2} - a_{2k+1}) dx + \sum_{i=2}^k \int_{c_{i-1}}^{c_i} (a_{2k+2-(i-1)} - a_{2k+2-i}) dx + \int_{c_k}^1 (a_{k+3} - a_{k+2}) dx \\
&= c_1(a_{2k+2} - a_{2k+1}) + (1 - c_k)(a_{k+3} - a_{k+2}) + \sum_{i=2}^k (a_{2k+2-(i-1)} - a_{2k+2-i})(c_i - c_{i-1})
\end{aligned}$$

Finally, the procedure to compare \bar{A} & \bar{B} is as follows:

Step1: Find $R(\bar{A})$ and $R(\bar{B})$. Case1) If $R(\bar{A}) > R(\bar{B})$ then $\bar{A} \succ \bar{B}$. Case 2) If $R(\bar{A}) < R(\bar{B})$ then $\bar{A} \prec \bar{B}$. Case 3) If $R(\bar{A}) = R(\bar{B})$ then comparison is not possible, then go to step 2.

Step2: Find $\text{mode}(\bar{A})$ and $\text{mode}(\bar{B})$. Case1) If $\text{mode}(\bar{A}) > \text{mode}(\bar{B})$ then $\bar{A} \prec \bar{B}$. Case2) If $\text{mode}(\bar{A}) < \text{mode}(\bar{B})$ then $\bar{A} \succ \bar{B}$. Case3) If $\text{mode}(\bar{A}) = \text{mode}(\bar{B})$ then comparison is not possible, then go to step 3.

Step3: Find $\text{spread}(\bar{A})$ and $\text{spread}(\bar{B})$. Case1) If $\text{spread}(\bar{A}) > \text{spread}(\bar{B})$ then $\bar{A} \succ \bar{B}$. Case2) If $\text{spread}(\bar{A}) < \text{spread}(\bar{B})$ then $\bar{A} \prec \bar{B}$. Case3) If $\text{spread}(\bar{A}) = \text{spread}(\bar{B})$ then comparison is not possible, then go to step 4.

Step4: Find left spread(\bar{A}) and left spread(\bar{B}). Case1) If left spread(\bar{A}) > left spread(\bar{B}) then $\bar{A} \succ \bar{B}$. Case2) if left spread(\bar{A}) < left spread(\bar{B}) then $\bar{A} \prec \bar{B}$. Case3) if left spread(\bar{A}) = left spread(\bar{B}) then comparison is not possible, then go to step 5.

Step5: Examine $w_1 = (\sum_{i=1}^k c_i)/k$ & $w_2 = (\sum_{i=1}^k c_i)/k$ (corresponding to \bar{A} & \bar{B} respectively). Case1) If $w_1 > w_2$ then $\bar{A} \succ \bar{B}$. Case2) If $w_1 < w_2$ then $\bar{A} \prec \bar{B}$. Case3) If $w_1 = w_2$ then $\bar{A} \approx \bar{B}$.

Notice that the same steps and procedure are valid for k+1-trapezoidal fuzzy number (we will have k+1 trapezoidal fuzzy numbers instead of k trapezoidal and a

triangular fuzzy number) and all other types (after using approximation if needed [6-7]).

Moreover, propositions 4.1, 4.2 and 4.3 in [8] still valid since we have separated k generalized trapezoidal fuzzy numbers and a generalized triangular fuzzy number (or according to the considered fuzzy number).

4.Numerical Examples

Example 4.1

Let $\bar{A} = (\bar{T}_1 = (0, .5, 1.5, 2; 0.5)$, $\bar{T}_2 = (0.5, 1, 1.5; 0.5)$ and

$$\bar{B} = (\bar{T}_{1^-} = (\frac{1}{5}, \frac{3}{5}, \frac{5}{4}, \frac{7}{4}; .5)$$
, $\bar{T}_{2^-} = (\frac{3}{5}, 1, \frac{5}{4}; 0.5)$

Step 1 :

$$G_{\bar{T}_1}(x_1, y_1) = (1, 0.1944) \ \& \ G_{\bar{T}_2}(x_2, y_2) = (1, 0.694) \ \text{then} \ G_{\bar{A}}(x, y) = (1, 0.444)$$

$$G_{\bar{T}_{1^-}}(x_1, y_1) = (0.936, 0.1944) \quad \text{and} \quad G_{\bar{T}_{2^-}}(x_2, y_2) = (0.983, 0.694) \ \text{then}$$

$$G_{\bar{B}}(x, y) = (0.96, 0.444)$$

$$R(\bar{A}) = 0.444 \ \text{and} \ R(\bar{B}) = 0.426$$

since $R(\bar{A}) > R(\bar{B})$ then $\bar{A} \succ \bar{B}$

Example 4.2

Let

$$\bar{A} = (\bar{T}_1 = (0, 0.5, 1.5, 2; 0.3), \bar{T}_2 = (0.5, 0.7, 1.2, 1.5; 0.3), \bar{T}_3 = (0.7, 1, 1.2; 0.4))$$

and

$$\bar{B} = (\bar{T}_{1^-} = (\frac{1}{5}, \frac{3}{5}, \frac{5}{4}, \frac{7}{4}; 0.3), \bar{T}_{2^-} = (\frac{3}{5}, \frac{4}{5}, 1.2, \frac{5}{4}; 0.3), \bar{T}_{3^-} = (\frac{4}{5}, 1, 1.2; 0.4))$$

$$G_{\bar{T}_1}(x_1, y_1) = (1, 0.167), \ G_{\bar{T}_2}(x_2, y_2) = (0.961, 0.417), \ G_{\bar{T}_3}(x_3, y_3) = (0.989, 0.756)$$

$$\text{Then} \ G_{\bar{A}}(x, y) = (0.983, 0.447)$$

$$G_{\bar{T}_{1^-}}(x_1, y_1) = (0.936, 0.167),$$

$$G_{\bar{T}_{2^-}}(x_2, y_2) = (0.983, 0.417)$$

$$\ \& \ G_{\bar{T}_{3^-}}(x_3, y_3) = (1, 0.756)$$

$$\text{Then} \ G_{\bar{B}}(x, y) = (0.973, 0.447)$$

$$R(\bar{A}) = 0.439 \ \text{and} \ R(\bar{B}) = 0.435$$

Since $R(\bar{A}) > R(\bar{B})$ then $\bar{A} \succ \bar{B}$

Example 4.3

Let

$$\bar{A} = (\bar{T}_1 = (0, 0.5, 1.5, 2; 0.3), \bar{T}_2 = (0.5, 0.7, 1.2, 1.5; 0.2), \bar{T}_3 = (0.7, 0.9, 1.1, 1.2; 0.2), \bar{T}_4 = (0.9, 1.1, 1.1; 0.3))$$

and

$$\bar{B} = (\bar{T}_{1^-} = (\frac{1}{5}, \frac{3}{5}, \frac{5}{4}, \frac{7}{4}; 0.3), \bar{T}_{2^-} = (\frac{3}{5}, \frac{4}{5}, 1.2, \frac{5}{4}; 0.2), \bar{T}_{3^-} = (\frac{4}{5}, 0.9, 1.2, 1.1; 0.2), \bar{T}_{4^-} = (0.9, 1.1, 1.1; 0.3))$$

$$G_{\bar{T}_1}(x_1, y_1) = (1, 0.117), G_{\bar{T}_2}(x_2, y_2) = (0.961, 0.378), G_{\bar{T}_3}(x_3, y_3) = (0.989, 0.578)$$

$$G_{\bar{T}_4}(x_4, y_4) = (1, 0.817) \quad \text{so,} \quad G_{\bar{A}}(x, y) = (0.9875, 0.4725)$$

$$G_{\bar{T}_{1^-}}(x_1, y_1) = (0.936, 0.117), G_{\bar{T}_{2^-}}(x_2, y_2) = (0.983, 0.378), G_{\bar{T}_{3^-}}(x_3, y_3) = (1, 0.578)$$

$$G_{\bar{T}_{4^-}}(x_4, y_4) = (1, 0.817) \quad \text{so} \quad G_{\bar{B}}(x, y) = (0.97975, 0.4725)$$

$$R(\bar{A}) = 0.467 \quad \text{and} \quad R(\bar{B}) = 0.463$$

Since $R(\bar{A}) > R(\bar{B})$ then $\bar{A} \succ \bar{B}$

Example 4.4Let $\bar{A} = (\bar{T}_1 = (1, 2, 3, 4; 0.5), \bar{T}_2 = (2, 2.5, 3; 0.5))$ and

$$\bar{B} = (\bar{T}_{1^-} = (2, 2.3, 2.7, 3; 0.5), \bar{T}_{2^-} = (2.3, 2.5, 2.7; 0.5))$$

Step 1 :

$$G_{\bar{T}_1}(x_1, y_1) = (2.5, 0.194) \quad \text{and} \quad G_{\bar{T}_2}(x_2, y_2) = (2.5, 0.694) \quad \text{then} \quad G_{\bar{A}}(x, y) = (2.5, 0.444)$$

$$G_{\bar{T}_{1^-}}(x_1, y_1) = (2.5, 0.194) \quad \text{and} \quad G_{\bar{T}_{2^-}}(x_2, y_2) = (2.5, 0.694) \quad \text{then} \quad G_{\bar{B}}(x, y) = (2.5, 0.444)$$

$$R(\bar{A}) = 1.11 \quad \text{and} \quad R(\bar{B}) = 1.11 \Rightarrow R(\bar{A}) = R(\bar{B}), \quad \text{so go to step 2.}$$

Step 2 :

$$\text{mode}(\bar{A}) = 2.5 \quad \text{and} \quad \text{mode}(\bar{B}) = 2.5, \quad \text{so go to step 3.}$$

step 3 :

$$\text{spread}(\bar{A}) = 2 \quad \text{and} \quad \text{spread}(\bar{B}) = 0.7$$

$$\text{since} \quad \text{spread}(\bar{A}) > \text{spread}(\bar{B}) \Rightarrow \bar{A} \prec \bar{B}$$

5. Conclusions

In conclusion, we have the following:

- The results obtained in example 4.4 agrees with the results of the same example given in [6-7].
- With minimal additional calculations and modifications, we were able to use the method of [8] for our particular fuzzy numbers.

- With little more computations, one can add several criteria steps for the ranking procedure like the left (right) and height of each sub-fuzzy number considered (which is a generalized trapezoidal fuzzy number). In the first step we exclude one, then two..etc (if we have equality).
- We assumed the average point of the centroids of all sub-fuzzy numbers as the reference point, similar left and right parts for each sub-fuzzy number.

For future work, it will be interesting to consider several cases and improve the ranking procedure (combined with a computer program) which leads to more accuracy.

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