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Constructions on Bi-Cubic Subgroup Structures

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Abstract

In this paper, we introduce the notion of bi-cubic subgroup and related properties are investigated. Using S_G-idempotent interval t-conorm, we study the characterizations of a
bi-cubic subgroups are established and how the images or inverse images of bi-cubic subgroups become bi-cubic subgroups.

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1. Introduction

The theory of fuzzy set was first developed by Zadeh[9] and has been applied to many branches in mathematics. Later fuzzification of the "Group" concept into "Fuzzy subgroup" was made by Rosenfeld[8]. This work was the first fuzzification of any algebraic structures and thus opened a new direction, new exploration, new path of thinking to mathematicians, engineers, computer scientists and many others in various tests. Also, Zadeh[9] have introduced the concept of fuzzy set by an interval-valued fuzzy set (i.e. a fuzzy set with an interval-valued membership function) and he constructed a method of approximate inference using his interval-valued set, interval-valued fuzzy subgroup where first defined and studied by Biswas[1]. Based on the interval-valued fuzzy set, Jun etal[3] introduced the notion of cubic sub algebras/ideals in BCK/BCI- algebra and then they investigated several properties. They discussed relationship between a cubic sub algebra and a cubic ideal. Also they provided characterization of cubic sub algebras/ideals, and considered a method to make a new of cubic sub algebra from old method. Jun etal[5] introduced the notion of cubic o- sub algebras and cloud cubic ideals in BCK/BCI algebra , and then they investigated to several properties. They also investigated a condition for a cubic set in a BCK – algebra with condition(s) to be a cubic ideal. Finally, they dealt with a characterization of a cubic ideal in a BCK/BCI- algebra.

Jun et al [4]introduced the notion of cubic q-ideals in BCI-algebras. They discovered relation between a cubic ideal and cubic q-ideal and provided conditions for a cubic ideal to be a cubic q-ideal. They also established characterizations of q-ideals and considered the cubic extension property for a cubic q-ideal.

2. Preliminaries

In this section, we recall some basic definitions for the sake of completeness.

In what follows let G denote a group unless otherwise specified.

2.1 Definition [9]

Let G be a non-empty set . A fuzzy subset μ on G is defined by $\mu : G \to [0,1]$ for all $x \in G$.

2.2 Definition [8]

Let μ be a fuzzy subset in a group G. Then μ is called a fuzzy subgroup of G if (i) $\mu(xy)) \geq \min\{ \mu(x), \mu(y)\}$ for all $x, y \in G$ (ii) $\mu(x^{-1}) \geq \mu(x)$ for all $x \in G$.

An interval number on [0, 1], say $\tilde{a}$, is a closed subinterval of [0, 1], that is

$\tilde{a}= [a^-, a^+]$ where $0 \leq a^- \leq a^+ \leq 1$. Let D[0,1] denotes the family of all closed sub intervals of [0,1], $\tilde{0}$=[0,0] and $\tilde{1}$=[1,1]. Let us define what is known as refined minimum (briefly rmin) of two elements in D[0,1]. Now we define "≤", "≥", "=", "rmin", "rmax" in case of two elements in D[0,1]. Consider two elements $\tilde{a}= [a^-, a^+]$ and $\tilde{b}= [b^-, b^+]$ in D[0,1], then

(i) $\tilde{a} \leq \tilde{b}$ if and only if $a^- \leq b^-$ and $a^+ \leq b^+$,
(ii) $\tilde{a} \geq \tilde{b}$ if and only if $a^- \geq b^-$ and $a^+ \geq b^+$,
(iii) $\tilde{a} = \tilde{b}$ if and only if $a^- = b^-$ and $a^+ = b^+$,
(iv) rmin$\{ \tilde{a}, \tilde{b}\}$ = [min$\{ a^-, b^- \}$, min$\{ a^+, b^+ \}$],
(v) rmax$\{\tilde{a}, \tilde{b}\}$ = [max$\{ a^-, b^- \}$, max$\{ a^+, b^+ \}$].

2.3 Definition [7]

Let G be a set. An interval-valued fuzzy set A defined on G is given by

A= $\{(x, \mu_A{}^-(x), \mu_A{}^+(x)\}$, for all $x \in G$.Briefly denote A by A= $[\mu_A{}^-, \mu_A{}^+]$ where $\mu_A{}^-$ and $\mu_A{}^+$ are Lower and Upper fuzzy sets in G such that $\mu_A{}^-(x) \leq \mu_A{}^+(x)$ for all $x \in G$.

2.4 Definition [2]

An interval-valued fuzzy set 'A' in G is called an interval-valued fuzzy subgroup of G if

(i) $\tilde{\mu}_A(xy) \geq$ rmin $\{\tilde{\mu}_A(x)\tilde{\mu}_A(y)\}$ for all $x, y \in G$,
(ii) $\tilde{\mu}_A(x^{-1}) \geq \tilde{\mu}_A(x)$ for all $x \in G$.

2.5 Definition [2]

A mapping S: [0,1] × [0,1] → [0,1] is called a t-conorm if for every $x, y, z \in [0,1]$, it satisfies the following conditions :

(i) $S(x,0) = x$,

(ii) $S(x, y) = S(y, x)$,

(iii) $S(S(x, y), z) = S(x,S(y, z))$,

(iv) $S(x, y) \leq S(x, z)$, if $y \leq z$.

Let 'S' be a t-conorm, if for arbitrary $x \in [0,1]$, it satisfies $S(x, x) = x$, then S is called an idempotent t-conorm.

2.6 Definition [2]

Let S be an idempotent t-conorm. Define the mapping S_G: D[0,1] × D[0,1] → D[0,1] by $(\tilde{a}, \tilde{b}) \to S_G(\tilde{a}, \tilde{b}) = [\, S(a^-, b^-), S(a^+, b^+)\,]$, then S_G is called an idempotent interval t-conorm.

2.7 Definition [2]

Let G be a group and S_G be an idempotent interval t-conorm. An interval-valued fuzzy set A in G is called an S_G-interval-valued fuzzy subgroup of G if the following condition hold,

(i) $\tilde{\mu}_A(xy) \leq S_G\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$ for all $x, y \in G$,

(ii) $\tilde{\mu}_A(x^{-1}) \leq \tilde{\mu}_A(x)$ for all $x \in G$.

2.8 Definition

Let G be a non-empty set. A bi-cubic set A in a set G is a structure $A = \{(x, \tilde{\mu}_A(x), V_A(x)) : x \in G\}$ which is briefly denoted by $A = <\tilde{\mu}_A, V_A>$ where $\tilde{\mu}A=[\mu_A^-, \mu_A^+]$ is an interval-valued fuzzy set in G,V_A is a vague set in G. Denote by C(G) the family of bi-cubic in a set G.

2.9 Definition

Let G be a group and S_G be an idempotent interval t-conorm. A bi-cubic set $A = (\tilde{\mu}_A, V_A)$ in G is called a bi-cubic subgroup of G if it satisfies: for all $x, y \in G$,

(i) $\tilde{\mu}A(xy) \leq S_G\{\tilde{\mu}A(x), \tilde{\mu}A(y)\}$

(ii) $\tilde{\mu}A(x^{-1}) \leq \tilde{\mu}A(x)$

(iii) $V_A(xy) \geq \min\{V_A(x), V_A(y)\}$

(iv) $V_A(x^{-1}) \geq V_A(x)$

Example: Let G be the Klein's four group. We have G={e, a, b, ab} where $a^2 = e = b^2$ and $ab = ba$. We define $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ and $V_A = [t_A, f_A]$ by

$$\tilde{\mu}_A = \begin{pmatrix} e & a & b & ab \\ [0.2,0.6] & [0.3,0.7] & [0.5,0.8] & [0.3,0.7] \end{pmatrix} \text{ and}$$

$$V_A = \begin{pmatrix} e & a & b & ab \\ (0.4,0.8) & (0.1,0.6) & (0.2,0.8) & (0.3,0.5) \end{pmatrix}.$$

Then $A = (\tilde{\mu}_A, V_A)$ is a bi-cubic group.

2.10 Definition [2]

Let $A = (\tilde{\mu}_A, V_A)$ be a bi-cubic set in a set $G(\gamma,\delta) \in [0,1]$ and $[\alpha,\beta] \in D[0,1]$ the set $\cup(A\ ;\ [\alpha,\beta]\ ,\ (\gamma,\delta)\ ) = \{\ x \in G\ /\ \tilde{\mu}_A(x) \leq [\alpha,\beta]\ ,\ V_A(x) \geq (\gamma,\delta)\ \}$ is called the cubic level set of A. .

3. Properties of Bi-Cubic Groups

3.1 Proposition

Let $A = (\tilde{\mu}_A, V_A)$ be a bi-cubic subgroup of G. Then $\tilde{\mu}_A(x^{-1}) = \tilde{\mu}_A(x)$ and $V_A(x^{-1}) = V_A(x)$ for all $x \in G$.

Proof: For all $x \in G$, we have $\tilde{\mu}_A(x) = \tilde{\mu}_A((x^{-1})^{-1}) \leq \tilde{\mu}_A(x^{-1}) \leq \tilde{\mu}_A(x)$ and $V_A(x) = V_A((x^{-1})^{-1}) \geq V_A(x^{-1}) \geq V_A(x)$. Hence $\tilde{\mu}_A(x^{-1}) = \tilde{\mu}_A(x)$ and $V_A(x^{-1}) = V_A(x)$.

3.2 Proposition

A bi-cubic set $A = (\tilde{\mu}_A, V_A)$ in G is a bi-cubic subgroup of G if and only if it Satisfies

(i) $\tilde{\mu}_A(xy^{-1}) \leq S_G\ \{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$

(ii) $V_A(xy^{-1}) \geq \min\{V_A(x), V_A(y)\}$ for all $x, y \in G$.

Proof: Assume that $A = (\tilde{\mu}_A, V_A)$ is a bi-cubic subgroup of G and let $x, y \in G$.
Then $\tilde{\mu}_A(xy^{-1}) \leq S_G\{\tilde{\mu}_A(x), \tilde{\mu}_A(y^{-1})\} = S_G\ \{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$ proposition 3.1 and
$V_A(xy^{-1}) \geq \min\ \{V_A(x), V_A(y^{-1})\} = \min\{\ V_A(x), V_A(y)\}$ by proposition 3.1.
Conversely, suppose that (I)and (II)are valid. If we take y = x in (I) and (II),
then $\tilde{\mu}_A(e) = \tilde{\mu}_A(xx^{-1}) \leq S_G\ \{\tilde{\mu}_A(x), \tilde{\mu}_A(x)\} = \tilde{\mu}_A(x)$ and
$V_A(e) = V_A(xx^{-1}) \geq \min\{V_A(x), V_A(x)\} = V_A(x)$. It follows from (I) and (II) that

$\tilde{\mu}_A(y^{-1}) = \tilde{\mu}_A(ey^{-1}) \leq S_G\ \{\tilde{\mu}_A(e), \tilde{\mu}_A(y)\} = \tilde{\mu}_A(y)$ and

$V_A(y^{-1}) = V_A(ey^{-1}) \geq \min\ \{\ V_A(e), V_A(y)\} = V_A(y)$

So that $\tilde{\mu}_A(xy) = \tilde{\mu}_A(x(y^{-1})^{-1}) \leq S_G\{\tilde{\mu}_A(x), \tilde{\mu}_A(y^{-1})\} \leq S_G\ \{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$ and $V_A(xy) = V_A(x(y^{-1})^{-1}) \geq \min\ \{V_A(x), V_A(y^{-1})\} \geq \min\{\ V_A(x), V_A(y)\}$.

Therefore $A = (\tilde{\mu}_A, V_A)$ is a bi-cubic subgroup of G.

3.3 Proposition

Let $A = (\tilde{\mu}_A, V_A)$ be a bi-cubic subgroup of G, then $\tilde{\mu}_A(e) \leq \tilde{\mu}_A(x)$ and $V_A(e) \geq V_A(x)$ for all $x \in G$, where e is the identity element in G.

Proof: Let $x \in G$, Using proposition-3.2 we have

$\tilde{\mu}_A(e) = \tilde{\mu}_A(xx^{-1}) \leq S_G\{\tilde{\mu}_A(x), \tilde{\mu}_A(x)\} = \tilde{\mu}_A(x)$ and

$V_A(e) = V_A(xx^{-1}) \geq \min\{V_A(x), V_A(x)\} = V_A(x)$, this complete the proof.

3.4 Proposition

If $A = (\tilde{\mu}_A, V_A)$ be a bi-cubic subgroup of G, then the set

$S = \{ x \in G / \tilde{\mu}_A(x) = \tilde{\mu}_A(e), V_A(x) = V_A(e)\}$ is a subgroup of G.

Proof: Let $x, y \in G$, then $\tilde{\mu}_A(x) = \tilde{\mu}_A(e) = \tilde{\mu}_A(y)$ and $V_A(x) = V_A(e) = V_A(y)$. It follows from

proposition-3.2 that $\tilde{\mu}_A(xy^{-1}) \leq S_G\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} = \tilde{\mu}_A(e)$ and

$V_A(xy^{-1}) \geq \min\{ V_A(x), V_A(y)\} = V_A(e)$ so from proposition-3.3 that $\tilde{\mu}_A(xy^{-1}) = \tilde{\mu}_A(e)$ and $V_A(xy^{-1}) = V_A(e)$. Hence $xy^{-1} \in S$, and so S is a sub group of G.

3.5 Proposition

Let $A = (\tilde{\mu}_A, V_A)$ be a bi-cubic subgroup of G,then the following conditions are equivalent:

(i) $A = (\tilde{\mu}_A, V_A)$ is a bi-cubic subgroup of G.

(ii) The non empty cubic level set of $A = (\tilde{\mu}_A, V_A)$ is a sub group of G.

Proof: Assume that $A = (\tilde{\mu}_A, V_A)$ is a bi-cubic subgroup of G. Let $x, y \in \cup (A: [\alpha, \beta], (\gamma, \delta))$ for all $(\gamma, \delta) \in [0,1]$ and $[\alpha, \beta] \in D[0,1]$, then $\tilde{\mu}_A(x) \leq [\alpha, \beta]$, $V_A(x) \geq (\gamma, \delta)$ and $\tilde{\mu}_A(y) \leq [\alpha, \beta]$, $V_A(y) \geq (\gamma, \delta)$. It follows from the proposition-3.2 that $\tilde{\mu}_A(xy^{-1}) \leq S_G\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} \leq [\alpha, \beta]$ and $V_A(xy^{-1}) \geq \min\{ V_A(x), V_A(y)\} \geq (\gamma, \delta)$ so that $xy^{-1} \in \cup (A : [\alpha, \beta], (\gamma, \delta))$. Therefore the non empty cubic level set of $A = (\tilde{\mu}_A, V_A)$ is a subgroup of G.

Conversely, let $(\gamma, \delta) \in [0,1]$ and $[\alpha, \beta] \in D[0,1]$ be such that $\cup (A: [\alpha, \beta], (\gamma, \delta)) \neq \emptyset$, and $\cup (A: [\alpha, \beta], (\gamma, \delta))$ is a subgroup of G. Suppose that the Proposition 3.2 (i) is not true and proposition 3.2 (ii) is valid. Then there exists $[\alpha_0, \beta_0] \in D[0,1]$ and $a, b \in G$ such that $\tilde{\mu}_A(ab^{-1}) \geq [\alpha_0, \beta_0] \geq S_G \{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}$ and $V_A(ab^{-1}) \geq \min\{V_A(a), V_A(b)\}$. It follows that $a, b \in \cup (A : [\alpha_0, \beta_0], \min\{ V_A(a), V_A(b)\})$ but $ab^{-1} \in \cup (A : [\alpha_0, \beta_0], \min\{ V_A(a), V_A(b)\})$. This is contradiction of Proposition 3.2 (i) is true and proposition 3.2 (ii) is not valid, then $\tilde{\mu}_A(ab^{-1}) \leq S_G \{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}$ and $V_A(ab^{-1}) \leq (\gamma_0, \delta_0) \leq \min\{ V_A(a), V_A(b)\}$ for some $(\gamma_0, \delta_0) \in [0,1]$ and $a, b \in G$. Thus $a, b \in \cup (A: S_G \{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}, (\gamma_0, \delta_0))$ but $ab^{-1} \in \cup (A: S_G \{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}, (\gamma_0, \delta_0))$ which is contradiction. Assume that there exists $[\alpha_0, \beta_0] \in D[0,1]$, $(\gamma_0, \delta_0) \in [0,1]$ and $a, b \in G$ such that $\tilde{\mu}_A(ab^{-1}) \geq [\alpha_0, \beta_0] \geq S_G \{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}$ and $V_A(ab^{-1}) \leq (\gamma_0, \delta_0) \leq \min\{ V_A(a), V_A(b)\}$ then $a, b \in \cup (A: [\alpha_0, \beta_0], (\gamma_0, \delta_0))$ but $ab^{-1} \in \cup (A: [\alpha_0, \beta_0], (\gamma_0, \delta_0))$. This is also a contradiction. Hence (i) and (ii) of proposition-3.2 are true. Therefore A is a bi-cubic subgroup of G.

3.6 Proposition

Let $f: G \to G'$ be a homomorphism of groups. If $A = (\tilde{\mu}_A, V_A)$ is a bi-cubic subgroup of G' then $A^f = (\tilde{\mu}_A^f, V_A^f)$ is bicubic sub group of G.

Proof: Let $x, y \in G$,

$\tilde{\mu}_A^f(xy) = \tilde{\mu}_A(f(xy))$

$= \tilde{\mu}_A(f(x).f(y))$

$\leq S_G\{\tilde{\mu}_A(f(x)), \tilde{\mu}_A(f(y))\}$

$= S_G\{\tilde{\mu}_A{}^f(x), \tilde{\mu}_A{}^f(y)\}$

$\tilde{\mu}_A{}^f(x^{-1}) = \tilde{\mu}_A f(x^{-1})$.

$\leq \tilde{\mu}_A(f(x))$

$= \tilde{\mu}_A{}^f(x)$ and

$V_A{}^f(xy) = V_A(f(xy))$

$= V_A(f(x).f(y))$

$\geq \min\{V_A(f(x)), V_A(f(y))\}$

$= \min\{V_A{}^f(x), V_A{}^f(y)\}$

$V_A{}^f(x^{-1}) = V_A f(x^{-1})$.

$\geq V_A{}^f(x)$

$A^f = (\tilde{\mu}_A{}^f, V_A{}^f)$ is bicubic sub group of G.

3.7 Proposition

Let A be a bi-cubic set in G. If $(\tilde{\mu}_A, V_A)$ be a S_G bi-cubic subgroup of G, then $(\tilde{\mu}_A{}^r, V_A{}^r)$ is T_G bi-cubic subgroup of G.

Proof: Let A be a bi-cubic subgroup of G, then for all $x, y \in G$.

$\tilde{\mu}_A{}^r(xy) = {}^G/_{\tilde{\mu}_A(xy)}$

$\leq \left({}^G/_{S_G\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}}\right)$

$= T_G\left\{\left({}^G/_{\tilde{\mu}_A(x)}\right), \left({}^G/_{\tilde{\mu}_A(y)}\right)\right\}$

$= T_G\{\tilde{\mu}_A{}^r(x), \tilde{\mu}_A{}^r(y)\}$ where T_G is the idempotent t-norm in G.

$\tilde{\mu}_A{}^r(x^{-1}) = {}^G/_{\tilde{\mu}_A(x^{-1})}$

$\leq {}^G/_{\tilde{\mu}_A(x)}$

$= \tilde{\mu}_A{}^r(x)$

$V_A{}^r(xy) = {}^G/_{V_A(xy)}$

$\geq \left({}^G/_{\min\{V_A(x), V_A(y)\}}\right)$

$= \max\left\{\left({}^G/_{V_A(x)}\right), \left({}^G/_{V_A(y)}\right)\right\}$

$= \max\{V_A{}^r(x), V_A{}^r(y)\}$

$V_A{}^r(x^{-1}) = {}^G/_{V_A(x^{-1})}$

$\geq {}^G/_{V_A(x)}$

$= V_A{}^r(x)$

$(\tilde{\mu}_A{}^r, V_A{}^r)$ is anti bi-cubic subgroup of G.

Conclusion

Jun etal introduced the notion of cubic ideal in BCK Algebra and cubic-o sub algebras. In this paper, we have to investigate the concept of bi-cubic group and its characterization.

Future Work

One can obtain the similar results by changing soft sets or rough sets instead of vague set.

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