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The Entropy of Fuzzy Dynamical Systems

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Abstract

Some Fuzzy modifications of the Kolmogorov-Sinaj entropy has been studied by B. Riecan, D. Markechovfi.In this paper, we show that proposition 3.3 of their paper has a counter example, and the Propositions 3.3 and 3.4 and Corollaries 3.5 and 3.6 need to be modified.

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1. Introduction

By a fuzzy partition we mean a finite collection $\mathcal{A} = \{f_1, \dots, f_n\}$ of fuzzy subset of Ω ,$f_i : \Omega \longrightarrow [0, 1]$ such that

$$\sum_{i=1}^{n} f_i(\omega) = 1, \quad \forall \omega \in \Omega$$

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There are many possibilities how to define operations with fuzzy sets. The first model, studied mainly by Markechovfi was based on the Zadeh connectives. Instead of a probability measure a function $m : \mathcal{F} \longrightarrow [0,1]$ on the space of Fuzzy sets has been considered such that $m(\bigvee_{i=1}^{k} g_i) = \sum_{i=1}^{k} m(g_i)$ whenever $g_i \leq 1 - g_i (i \neq j)$.

The second model, studied by Dumitrescu, (but about ten years ago implicitly by Rie an) the Lukasziewicz connectives have been used. Instead of a probability measure, a function $m : \mathcal{F} \longrightarrow [0,1]$ has been considered such that

$$m(\bigvee_{i=1}^{k} g_i) = \sum_{i=1}^{k} m(g_i),$$

whenever, $\sum_{i=1}^{k} g_i \leq 1$.

2. Counter example

In this section, we give a counter example of proposition 2.1 in [1].

Proposition 2.1. [Proposition 3.3 of [1]] Let $(B_n)_n$ be an increasing sequence of crisp partitions such that $\sigma\left(\cup_{n=1}^{\infty} B_n\right) = \phi$. Let $\mathcal{A} = \{g_1, g_2, \ldots, g_k\}$ be a fuzzy partition. Then

$$\lim_{n\to\infty} H(A|B_n) = \sum_{j=1}^{k} \int \varphi(g_j) dP = \int \left(\sum_{j=1}^{k} \varphi(g_j)\right) dP. \tag{1}$$

Example 2.2. [counter example] Let $B_1 = \{\mathbb{R}\}$, and

$$\begin{aligned}
B_2 &= \{(-\infty, 0), (1, +\infty), [0, 1]\}, \\
B_3 &= \{(-\infty, 0), (1, +\infty), [0, 1/2), [1/2, 1]\}, \\
B_4 &= \{(-\infty, 0), (1, +\infty), [0, 1/4), [1/4, 1/2), [1/2, 3/4), [3/4, 1]\}, \\
&\vdots \\
B_n &= \{(-\infty, 0), (1, +\infty), [0, 1/n), [1/n, 2/n), [2/n, 3/n), \ldots, [(n-1)/n, 1]\}, \\
&\vdots
\end{aligned}$$

So $\sigma\left(\cup_{n=1}^{\infty} B_n\right) = \phi$ has infinite atoms.

Now we define $E(g_j|\phi) = \sup_{\phi_0 \subseteq \phi} E(g_j|\phi_0)$ that ϕ_0 is a finite σ-algebra. If $g_1 = 1/4$ and $g_2 = 3/4$ be constant functions on $\mathbb{R}$, then $\mathcal{A} = \{1/4, 3/4\}$ is a fuzzy partition.

We show that, $E(1/4|\phi) \neq 1/4$ and $E(3/4|\phi) \neq 3/4$. Let $\phi_1 = \{\mathbb{R}, \emptyset, (-\infty, 0), [0, +\infty)\} \subseteq \phi$ be a finite σ-algebra, then

$$\begin{aligned} E(1/4|\phi_1) &= \sum_{i=1}^{4} \left(\frac{1}{P(u_i)} \int_{u_i} \frac{1}{4} dP \right) \chi_{u_i} \\ &= \sum_{i=1}^{4} \left(\frac{1}{4} \chi_{u_i} \right) \\ &= \frac{1}{4} \left(\chi_{\mathbb{R}} + \chi_{\emptyset} + \chi_{(-\infty,0)} + \chi_{[0,+\infty)} \right) \\ &= \frac{1}{4} (1 + 0 + 1) = \frac{1}{2}. \end{aligned}$$

Hence, $E(1/4|\phi) = \sup_{\phi_0 \subseteq \phi} E(1/4|\phi_0) \geq E(1/4|\phi_1) = 1/2 > 1/4$, i.e $E(1/4|\phi) \geq 1/2 > 1/4$, therefore $E(1/4|\phi) \neq 1/4$.

Also,

$$\begin{aligned} E(3/4|\phi_1) &= \sum_{i=1}^{4} \left(\frac{1}{P(u_i)} \int_{u_i} \frac{3}{4} dP \right) \chi_{u_i} \\ &= \sum_{i=1}^{4} \left(\frac{3}{4} \chi_{u_i} \right) \\ &= \frac{3}{4} \left(\chi_{\mathbb{R}} + \chi_{\emptyset} + \chi_{(-\infty,0)} + \chi_{[0,+\infty)} \right) \\ &= \frac{3}{4} (1 + 0 + 1) = \frac{3}{2}. \end{aligned}$$

Hence, $E(3/4|\phi) = \sup_{\phi_0 \subseteq \phi} E(3/4|\phi_0) \geq E(3/4|\phi_1) = 3/2 > 3/4$, i.e $E(3/4|\phi) \geq 3/2 > 3/4$, therefore $E(3/4|\phi) \neq 3/4$.

On the other hand, $\sigma(B_n) \nearrow \phi$. By the martingale convergence theorem,

$$E(1/4|\sigma(B_n)) \nearrow E(1/4|\phi) \neq 1/4, \qquad (n \to \infty)$$

and

$$E(3/4|\sigma(B_n)) \nearrow E(3/4|\phi) \neq 3/4, \qquad (n \to \infty)$$

Now, if $\phi_2 = \{\mathbb{R}, \emptyset, (-\infty, 0), [0, +\infty), (1, +\infty), (-\infty, 1], [0, 1], (-\infty, 0) \cup (1, +\infty)\}$

be a finite σ-algebra, then it is clear that $\phi_2 \subseteq \phi$ and we have

$$\begin{aligned} E(1/4|\phi) &= \sup_{\phi_0 \subseteq \phi} E(1/4|\phi_0) \geq E(1/4|\phi_2) \\ &= \frac{1}{4}\big(\chi_{\mathbb{R}} + \chi_{\emptyset} + \chi_{(-\infty,0)} + \chi_{[0,+\infty)} + \chi_{(1,+\infty)} \\ &+ \chi_{(-\infty,1]} + \chi_{[0,+1]} + \chi_{(-\infty,0)\cup(1,+\infty)}\big) \\ &= \frac{1}{4}(1+0+1+1+1) = 1. \end{aligned}$$

Hence, $E(1/4|\phi) \geq 1$. Since,

$$\varphi(x) = \begin{cases} -x\log(x) & x > 0 \\ 0 & x = 0 \end{cases} \tag{2}$$

is strictly decreasing on $(1/e, +\infty)$, so $\varphi(E(1/4|\phi)) \leq \varphi(1) = -\log(1) = 0$. Hence

$$\varphi(E(1/4|\phi)) \leq 0, \qquad \varphi(1/4) = \frac{\log(4)}{4} > 0. \tag{3}$$

Also, since $E(3/4|\phi) > 3/4$ and $3/4 \in (1/e, +\infty)$, we have $\varphi(E(3/4|\phi)) < \varphi(3/4)$. So, by 3,

$$\varphi(E(1/4|\phi)) + \varphi(E(3/4|\phi)) < 0 + \varphi(3/4) < \varphi(1/4) + \varphi(3/4). \tag{4}$$

Now, by definition and LDCT theorem we can say

$$\begin{aligned} \lim_{n\to\infty} H(A|B_n) &= \lim_{n\to\infty} \int_{\mathbb{R}} \varphi(E(1/4|\sigma(B_n)))dP + \lim_{n\to\infty} \int_{\mathbb{R}} \varphi(E(3/4|\sigma(B_n)))dP \\ &= \int_{\mathbb{R}} \varphi(E(1/4|\phi))dP + \int_{\mathbb{R}} \varphi(E(3/4|\phi))dP \\ &= \int_{\mathbb{R}} (\varphi(E(1/4|\phi)) + \varphi(E(3/4|\phi)))\, dP \\ &\neq \int_{\mathbb{R}} (\varphi(1/4) + \varphi(3/4))\, dP, \end{aligned}$$

where, the last follows by 4.

3. Modifications

In this section we press Modifications of Propositions 3.3 and 3.4 and Corollaries 3.5 and 3.6 (see [1]). Also we prove the Propositions and Corollaries that said in these results.

Proposition 3.1. [The modification of Proposition 3.3(see [1])] Let $(B_n)_n$ be an increasing sequence of crisp partitions such that $\sigma\left(\cup_{n=1}^{\infty} B_n\right) = \phi$. Let $\mathcal{A} = \{g_1, g_2, \ldots, g_k\}$ be a fuzzy partition. Then

$$\lim_{n\to\infty} H(A|B_n) = \sum_{j=1}^{k} \int \varphi(E(g_j|\phi))dP = \int \left(\sum_{j=1}^{k} \varphi(E(g_j|\phi))\right) dP. \tag{5}$$

Proof. Since $\sigma(B_n) \nearrow \phi$, by the martingale convergence theorem,

$$E(g_j|\sigma(B_n)) \nearrow E(g_j|\phi), \qquad (n \to \infty)$$

Since g_j are ϕ-measurable. Since ϕ is a continuous function, we obtain

$$\lim_{n\to\infty} \varphi(E(g_j|\sigma(B_n))) = \varphi(E(g_j|\phi)).$$

Finally, by the Lebesgue-dominated convergence theorem, the linearity of the integral and proposition 3.2 (see [1]),

$$\begin{aligned}
\lim_{n\to\infty} H(A|B_n) &= \lim_{n\to\infty} \sum_{j=1}^{k} \int_{\Omega} \varphi(E(g_j|\sigma(B_n)))dP \\
&= \sum_{j=1}^{k} \int_{\Omega} \varphi(E(g_j|\phi))dP \\
&= \int_{\Omega} \left(\sum_{j=1}^{k} \varphi(E(g_j|\phi))\right) .dP
\end{aligned}$$

■

Proposition 3.2. [The modification of Proposition 3.4(see [1])] Let $\mathcal{C} = \{c_1, \ldots, c_k\}$ be a measurable partition of Ω being a generator, i.e. $\sigma\left(\cup_{i=0}^{\infty} U^i\mathcal{C}\right) = \phi$. Then for every fuzzy partition $\mathcal{A} = \{g_1, \ldots, g_k\}$ there holds.

$$h(\mathcal{A}, U) \leq h(\mathcal{C}, U) + \int_{\Omega} \left(\sum_{j=1}^{k} \varphi(E(g_j|\phi))\right) dP.$$

Proof. Put $B_n = \bigvee_{i=0}^{n} U^i\mathcal{C}$. Since $\sigma\left(\bigcup_{i=0}^{\infty} U^i\mathcal{C}\right) = \phi$, we obtain, $\sigma(B_n) \nearrow \phi$. By Theorem 2.12 and proposition 2.10 (see [1]),

$$h(\mathcal{A}, U) \leq h(\mathcal{C}, U) + H(A|B_n).$$

Hence, proposition 3.1 (see [1]) implies

$$h(\mathcal{A}, U) \leq h(\mathcal{C}, U) + \int_{\Omega} \left(\sum_{j=1}^{k} \varphi(E(g_j|\phi)) \right).$$

■

Corollary 3.3. [The modification of Corollary 3.5(see [1])] Let $\mathcal{C} = \{c_1, \ldots, c_k\}$ be a generator,

$$K_G = \sup \left\{ \int_{\Omega} \left(\sum_{j} \varphi(E(g_j|\phi)) \right) dP : \quad g_j \in G, \{g_1, \ \ldots, g_n\} \text{ is a fuzzy partition} \right\}$$

Then $h(\mathcal{A}, U) \leq h(\mathcal{C}, U) + K_G$, for every partition $\mathcal{A} \subseteq G$.

Proof. Suppose $\mathcal{A} = \{g_1, \ldots, g_k\} \subseteq \sigma$ be a fuzzy partition. By Proposition 3.2 and definition of G, we have

$$h(\mathcal{A}, U) \leq h(\mathcal{C}, U) + \int_{\Omega} \left(\sum_{j=1}^{k} \varphi(E(g_j|\phi)) \right) dP \leq h(\mathcal{C}, U) + K_G.$$

■

Corollary 3.4. [The modification of Corollary 3.6(see [1])] If $\mathcal{C}$ is a generator of dynamical system (Ω, ϕ, P, T), then $h(T) = h(\mathcal{C}, T)$.

Proof. Put $G = \{\chi_A : \ A \in \phi\} \subset \mathcal{F}$. By Proposition 3.2,

$$h(\mathcal{A}, T) \leq h(\mathcal{C}, T) + \sum_{j} \int_{\Omega} \varphi(E(g_j|\phi)) dP,$$

whenever, $\mathcal{A} = \{g_1, \ldots, g_r\}$. Of course, $g_j = \chi_{A_j}$ and $\varphi(E(g_j|\phi)) = 0$, because

$$E(g_j|\phi) = \sum_{j=1}^{r} \left(\frac{1}{P(A_j)} \int_{A_j} \chi_{A_j} dP \right) \chi_{A_j} = \sum_{j=1}^{r} \chi_{A_j} = \sum_{j=1}^{r} g_j = 1.$$

where, the last equality follows by definition of fuzzy partition. Therefore $\varphi(E(g_j|\phi)) = \varphi(1) = -\log(1) = 0$. Hence, $h(\mathcal{A}, T) \leq h(\mathcal{C}, T)$ for every crisp partition $\mathcal{A}$ and therefore

$$h(T) = \sup \{h(\mathcal{A}, T) : \ \mathcal{A} \text{ partition}\} \leq h(\mathcal{C}, T).$$

■

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