Advances in Fuzzy Mathematics.
ISSN 0973-533X Volume 8, Number 1 (2013), pp. 1–16

http://www.ripublication.com/afm.htm

Fixed and variable-basis fuzzy closure operators

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Abstract

Closure operators are very useful tools in several areas of classical mathematics and in general category theory. In fuzzy set theory, fuzzy closure operators have been studied by G. Gerla (1996). These works generally define a fuzzy subset as a mapping from a set X to the real unit interval, as a complete and complemented lattice. More recently, Y. C. Kim (2003), F. G. Shi (2009), J. Fang and Y. Yue (2010) propose theories of fuzzy closure systems and fuzzy closure operators in a more general settings, but still using complemented lattices.

The aim of this paper is to propose a more general theory of fixed and variable-basis fuzzy closure operators, employing both categorical tools and the lattice theoretical fundations investigated by S. E. Rodabaugh (1999), where the lattices are usually non-complemented. Besides, we construct topological categories in both cases.

AMS subject classification: 06B05, 18B35, 54A40, 54B30.
Keywords: Fixed-basis closure operator, variable-basis closure operator, lattices and sets categories, topological category, closed and dense fuzzy sets.

Introduction

It is well-known that the associated closure and interior operators provide equivalent descriptions of set-theoretic topology; but this is not generally true in other categories, consequently it makes sense to define and study the notion of closure operators C in the context of fuzzy set theory, where we can find categories in a lattice-theoretical context.

Closure operators are very useful tools in several areas of classical mathematics, and particularly in category theory. In fuzzy set theory, fuzzy closure operators have been studied by Gerla and others, (see e.g. [4]). These works generally define a fuzzy subset as a mapping from a set X to the real unit interval $[0, 1]$, as a complete and complemented lattice.

More recently, [3], [6] and [9] propose theories of fuzzy closure systems and fuzzy closure operators in a more general settings, but using complemented lattices.

The aim of this paper is to propose a more general theory of fixed and variable-basis fuzzy closure operators, employing both categorical tools and the lattice theoretical fundations investigated in [7] and [5], where the lattices are usually non-complemented. The type of closure operator we use is similar to that used in [2].

The paper is organized as follows: Following [7] and [5] we introduce, in section 1, the basic lattice theoretical fundations. In section 2, we present the concept of fixed-basis fuzzy closure operators and then we construct a topological category (FBCO-SET, U), next in section 3 we present the concept of variable-basis fuzzy closure operators and then we construct a topological category (VBCO-SET, U).

In section 4, we study some additional stability properties of closure operators, that is idempotent and additive closure operators. Finally in section 5, we present some examples of various classes of closure maps.

1. From Lattice Theoretic Foundations

Let $(L, \leq)$ be a complete, infinitely distributive lattice, i.e. $(L, \leq)$ is a partially ordered set such that for every subset $A \subset L$ the join $\bigvee A$ and the meet $\bigwedge A$ are defined, moreover $(\bigvee A) \wedge \alpha = \bigvee\{a \wedge \alpha) \mid a \in A\}$ and $(\bigwedge A) \vee \alpha = \bigwedge\{a \vee \alpha) \mid a \in A\}$ for every $\alpha \in L$. In particular, $\bigvee L = \top$ and $\bigwedge L = \perp$ are respectively the universal upper and the universal lower bounds in L. We assume that $\perp \neq \top$, i.e. L has at least two elements.

1.1. Complete quasi-monoidal lattices

The definition of complete quasi-monoidal lattices introduced by S. E. Rodabaugh in [7] is the following:

A $cqm-$lattice (short for complete quasi-monoidal lattice) is a triple $(L, \leqslant, \otimes)$ provided with the following properties

(1) $(L, \leqslant)$ is a complete lattice with upper bound $\top$ and lower bound $\perp$.

(2) $\otimes : L \times L \to L$ is a binary operation satisfying the following axioms:

  (a) $\otimes$ is isotone in both arguments, i.e. $\alpha_1 \leqslant \alpha_2$, $\beta_1 \leqslant \beta_2$ implies $\alpha_1 \otimes \beta_1 \leqslant \alpha_2 \otimes \beta_2$;

  (b) $\top$ is idempotent, i.e. $\top \otimes \top = \top$.

The category $CQML$ comprises the following data:

(a) **Objects**: Complete quasi-monoidal lattices.

(b) **Morphisms**: All SET morphisms, between the above objects, which preserve $\otimes$ and $\top$ and arbitrary $\bigvee$.

(c) Composition and identities are taken from SET.

The category $LOQML$ is the dual of $CQML$, i.e. $LOQML = CQML^{op}$.

1.2. $GL-$monoids

A $GL-$monoid (see [5]) is a complete lattice enriched with a further binary operation $\otimes$, i.e. a triple $(Ł, \leq, \otimes)$ such that:

(1) $\otimes$ is isotone, commutative and associative;

(2) $(Ł, \leq, \otimes)$ is integral, i.e. $\top$ acts as the unity: $\alpha \otimes \top = \alpha, \forall \alpha \in Ł$;

(3) $\perp$ acts as the zero element in $(Ł, \leq, \otimes)$, i.e. $\alpha \otimes \perp = \perp, \forall \alpha \in Ł$;

(4) $\otimes$ is distributive over arbitrary joins, i.e. $\alpha \otimes (\bigvee_\lambda \beta_\lambda) = \bigvee_\lambda (\alpha \otimes \beta_\lambda)$, $\forall \alpha \in Ł, \forall \{\beta_\lambda : \lambda \in I\} \subset Ł$;

(5) $(Ł, \leq, \otimes)$ is divisible, i.e. $\alpha \leq \beta$ implies the existence of $\gamma \in Ł$ such that $\alpha = \beta \otimes \gamma$.

It is well known that every $GL-$monoid is residuated, i.e. there exists a further binary operation "$\longmapsto$" (implication) on Ł satisfying the following condition:

$$\alpha \otimes \beta \leq \gamma \iff \alpha \leq (\beta \longmapsto \gamma) \qquad \forall \alpha, \beta, \gamma \in Ł.$$

Explicitly the implication is given by

$$\alpha \longmapsto \beta = \bigvee \{\lambda \in Ł \mid \alpha \otimes \lambda \leq \beta\}.$$

If X is a set and L is a GL-monoid (or a complete quasi-monoidal lattice), then the fuzzy powerset L^X in an obvious way can be pointwise endowed with a structure of a GL-monoid (or of a complete quasi-monoidal lattice). In particular the L-sets 1_X and 0_X defined by $1_X(x) = \top$ and $0_X(x) = \perp$ $\forall x \in X$ are respectively the universal upper and lower bounds in L^X.

1.3. Powerset operator foundations

We give the powerset operators, developed and justified in detail by S.E. Rodabaugh in [7] and [8]. Let $f \in SET(X, Y)$, $L, M \in |CQML|$, $\phi \in LOQML(L, M)$, and $\wp(X)$, $\wp(Y)$, L^X, M^Y be the classical powerset of X, the classical powerset of Y, the L-powerset of X, and the M-powerset of Y, respectively. Then the following powerset operators are defined:

1. $f^{\rightarrow} : \wp(X) \rightarrow \wp(Y)$ by $f^{\rightarrow}(A) = \{f(x) \mid x \in A\}$

2. $f^{\leftarrow} : \wp(Y) \to \wp(X)byf^{\leftarrow}(B) = \{x \in X \mid f(x) \in B\}$
3. $f_L^{\rightarrow} : L^X \to L^Ybyf_L^{\rightarrow}(a)(y) = \bigvee_{f(x)=y} a(x)$
4. $f_L^{\leftarrow} : L^Y \to L^Xbyf_L^{\leftarrow}(b) = b \circ f$
5. $^*\phi : L \to M$by$^*\phi(\alpha) = \bigwedge\{\beta \in M \mid \alpha \leqslant \phi^{op}(\beta)\}$
6. $\langle {}^*\phi \rangle : L^X \to M^Xby\langle {}^*\phi \rangle(a) = {}^*\phi \circ a$
7. $\langle \phi^{op} \rangle : M^X \to L^Xby\langle \phi^{op} \rangle(b) = \phi^{op} \circ b$
8. $(f, \Phi)^{\rightarrow} : L^X \to M^Y$by $(f, \Phi)^{\rightarrow}(a) = \bigwedge\{b \mid f_L^{\rightarrow}(a) \leqslant \left(\Phi^{op}\right)(b)\}$,
9. $(f, \Phi)^{\leftarrow} : M^Y \to L^X$by $(f, \Phi)^{\leftarrow}(b) = \Phi^{op} \circ b \circ f$, in other words, that diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "(f,\Phi)^{\leftarrow}(b)"'] & Y \arrow[d, "b"] \\ L & M \arrow[l, "\Phi^{op}"] \end{tikzcd}$$

is commutative.

Note that these operators were defined taking into account the Adjoint functor theorem. Consequently, we have that $f^{\rightarrow}$, $f_L^{\rightarrow}$, and $(f, \Phi)^{\rightarrow}$ are left adjoints of $f^{\leftarrow}$, $f_L^{\leftarrow}$, and $(f, \Phi)^{\leftarrow}$, respectively.

2. Basic properties of fixed-basis fuzzy closure operators

Definition 2.1. Given L an object of the category $CQML$, a fuzzy closure operator C of the category SET (of sets and functions between them) with respect to L is given by a family $C = (c_X)_{X \in |SET|}$ of maps $c_X : L^X \to L^X$ such that for every set X:

(C_1) (Extension) $u \leqslant c_X(u)$ for all $u \in L^X$;

(C_2) (Monotonicity) if $u \leqslant v$ in L^X, then $c_X(u) \leqslant c_X(v)$;

(C_3) (Lower bound) $c_X(0_X) = 0_X$.

Definition 2.2. A fuzzy C-space is a pair (X, c_X) where X is a set and c_X is a fuzzy closure map on X.

Definition 2.3. A function $f : X \to Y$ in SET is said to be C-continuous if

$$f_L^{\rightarrow}(c_X(u)) \leqslant c_Y(f_L^{\rightarrow}(u)) \text{for all} u \in L^X. \tag{1}$$

Proposition 2.4. The C-continuity condition can equivalently be expressed as

$$c_X(f_L^{\leftarrow}(v)) \leqslant f_L^{\leftarrow}(c_Y(v)) \text{for all} v \in L^Y. \tag{2}$$

Proof. Condition (1) gives that, for $v \in L^Y$ and $u = f_L^{\leftarrow}(v)$,

$$f_L^{\rightarrow}(c_X(f_L^{\leftarrow}(v))) \leqslant c_Y(f_L^{\rightarrow}(f_L^{\leftarrow}(v))) \leqslant c_Y(v),$$

then

$$c_X(f_L^{\leftarrow}(v)) \leqslant f_L^{\leftarrow}(c_Y(v)).$$

On the other hand, condition (2) gives that, for $u \in L^X$ and $v = f_L^{\rightarrow}(u)$,

$$c_X(u) \leqslant c_X(f_L^{\leftarrow}(f_L^{\rightarrow}(u))) \leqslant f_L^{\leftarrow}(c_Y(f_L^{\rightarrow}(u)))),$$

then

$$f_L^{\rightarrow}(c_X(u)) \leqslant c_Y(f_L^{\rightarrow}(u)).$$

■

Proposition 2.5. Let $f : X \to Y$ and $g : Y \to Z$ be two C-continuous functions then the function $g \circ f$ is C-continuous.

Proof. Since $f : X \to Y$ is C-continuous, we have

$$f_L^{\rightarrow}(c_X(u)) \leqslant c_Y(f_L^{\rightarrow}(u)) \text{for all} u \in L^X,$$

it follows that

$$g_L^{\rightarrow}\big(f_L^{\rightarrow}(c_X(u))\big) \leqslant g_L^{\rightarrow}\big(c_Y(f_L^{\rightarrow}(u))\big),$$

now, by the C-continuity of g,

$$g_L^{\rightarrow}\big(c_Y(f_L^{\rightarrow}(u))\big) \leqslant c_Z\big(g_L^{\rightarrow}(f_L^{\rightarrow}(u))\big)$$

therefore

$$(g \circ f)_L^{\rightarrow}(c_X(u)) \leqslant c_Z(g \circ f)_L^{\rightarrow}(u)$$

■

as a consequence we obtain

Definition 2.6. The category FBCO-SET of C-spaces comprises de following data:

(1) **Objects**: pairs (X, c_X), where X is a set and c_X is a clousure map on X.

(2) **Morphisms**: Functions which are C-continuous.

2.1. The lattice structure of all closure operators

We consider the collection

$$C(SET, L)$$

of all closure operators on SET with respect to the complete quasi-monoidal lattice L. It is ordered by

$$C \leqslant D \Leftrightarrow c_X(u) \leqslant d_X(u), \text{for all set } X, \text{ and for all} u \in L^X.$$

This way $C(SET, L)$ inherents a lattice structure from L:

Proposition 2.7. Every family $(C_\lambda)_{\lambda\in\Lambda}$ in $C(SET, L)$ has a join $\bigvee_{\lambda\in\Lambda} C_\lambda$ and a meet $\bigwedge_{\lambda\in\Lambda} C_\lambda$ in $C(SET, L)$. The discrete closure operator

$$C_D = (c_{D_X})_{X\in|SET|}$$

is the least element in $C(SET, L)$, and the trivial closure operator

$$C_T = (c_{T_X})_{X\in|SET|} \qquad \text{with} \qquad (c_{T_X})(u) = \begin{cases} 1_X \text{ for all} u \neq 0 \\ 0_X \text{ if} u = 0_X \end{cases}$$

is the largest one.

Proof. For $\Lambda \neq \emptyset$, let $\tilde{C} = \bigwedge_{\lambda\in\Lambda} C_\lambda$, then

$$\tilde{C}_X = \bigwedge_{\lambda\in\Lambda} C_{\lambda_X},$$

where X is an arbitrary set, satisfies

- $u \leqslant \tilde{c}_X(u)$, because $u \leqslant c_{\lambda_X}(u)$ for all $u \in L^X$ and for all $\lambda \in \Lambda$.
- If $u_1 \leqslant u_2$ in L^X then $c_{\lambda_X}(u_1) \leqslant c_{\lambda_X}(u_2)$ for all $\lambda \in \Lambda$, therefore $\tilde{c}_X(u_1) \leqslant \tilde{c}_X(u_2)$.
- Since $c_{\lambda_X}(0_X) = 0_X$ for all $\lambda \in \Lambda$, we have that $\tilde{c}_X(0_X) = 0_X$.

Similary $\bigvee_{\lambda\in\Lambda} C_{\lambda_X}$, C_{T_X} and C_{D_X} are closure operators. ■

Consequently,

Corollary 2.8. For every set X

$$CL(X) = \{c_X \mid c_X \text{is a closure map on X}\}$$

is a complete lattice.

2.2. Initial closure operators

Let (Y, c_Y) be an object of the category FBCO-SET, and let X be a set. For each function $f : X \to Y$ we define on X the map

$$c_{X_f} : L^X \to L^X \qquad \text{by} \qquad c_{X_f} = f_L^{\leftarrow} \circ c_Y \circ f_L^{\rightarrow} \tag{3}$$

i.e. the following diagram is conmutative

$$\begin{tikzcd}
L^X \arrow[r, "f_L^{\rightarrow}"] \arrow[d, dotted, "c_{X_f}"'] & L^Y \arrow[d, "c_Y"] \\
L^X & L^Y \arrow[l, "f_L^{\leftarrow}"]
\end{tikzcd}$$

Proposition 2.9. The map (3) is a closure map on X for which the function f is C-continuous.

Proof.

1) (Extension) $c_{X_f}(u) = f_L^{\leftarrow}\big(c_Y(f_L^{\rightarrow}(u))\big) \geqslant f_L^{\leftarrow}(f_L^{\rightarrow}(u)) \geqslant u$,
therefore $u \leqslant c_{X_f}(u)$

2) (Monotonicity) $u_1 \leqslant u_2$ in L^X implies $f_L^{\rightarrow}(u_1) \leqslant f_L^{\rightarrow}(u_2)$
then $c_Y(f_L^{\rightarrow}(u_1)) \leqslant c_Y(f_L^{\rightarrow}(u_2))$, consequently

$$c_{X_f}(u_1) = \Big(f_L^{\leftarrow}\big(c_Y(f_L^{\rightarrow}(u_1))\big)\Big) \leqslant \Big(f_L^{\leftarrow}\big(c_Y(f_L^{\rightarrow}(u_2))\big)\Big) = c_{X_f}(u_2).$$

3) (Lower bound) $c_{X_f}(0_X) = f_L^{\leftarrow}\big(c_Y(f_L^{\rightarrow}(0_X))\big) = 0_X$.
Finally

$$f_L^{\rightarrow}\big(c_{X_f}(u)\big) = f_L^{\rightarrow}\Big(f_L^{\leftarrow}\big(c_Y(f_L^{\rightarrow}(u))\big)\Big) \leqslant c_Y\big(f_L^{\rightarrow}(u)\big) \text{ for all } u \in L^X.$$

■

It is clear that c_{X_f} is the finner map on L^X for which the function f is C-continuous, more precisaly.

Proposition 2.10. Let (Z, c_Z) and (Y, c_Y) be objects of FBCO-SET, and let X be a set. For each function $g : Z \to X$ and for $f : (X, c_{X_f}) \to (Y, c_Y)$ a C-continuous function, g is C-continuous if and only if $f \circ g$ is C-continuous.

Proof. Suppose that $f \circ g$ is C-continuous, i. e.

$$c_Z((f \circ g)_L^{\leftarrow}(v)) \leqslant ((f \circ g)_L^{\leftarrow} c_Y(v))$$

for all $v \in L^Y$. Then, for all $u \in L^X$, we have

$$\begin{aligned} g_L^{\leftarrow}(c_{X_f}(u)) &= g_L^{\leftarrow}((f_L^{\leftarrow} \circ c_Y \circ f_L^{\rightarrow})(u)) = (f \circ g)_L^{\leftarrow}(c_Y((f_L^{\rightarrow}(u))) \\ &\geqslant c_Z((f \circ g)_L^{\leftarrow} f_L^{\rightarrow}(u))) = c_Z((g_L^{\leftarrow} \circ f_L^{\leftarrow} \circ f_L^{\rightarrow})(u)) \geqslant c_Z(g_L^{\leftarrow}(u)). \end{aligned}$$

■

Proposition 2.11. Let X be a set, let (Y_j, c_{Y_j}) be a family of fuzzy C-spaces, where $j \in J$ for some indexed set J, and let $f_j : X \to Y_j$ be functions. Then the structured source $(X, f_j : X \to Y_j)$ w.r.t the forgetful functor U from FBCO-SET to SET has a unique initial lift $((X, \hat{c}_X) \to (Y_j, c_{Y_j}))$, where $\hat{c}_X$ is the join $\bigvee_{\lambda \in \Lambda} c_{f_j}$ of all initial closure maps c_{f_j} w.r.t. f_j, where $j \in J$.

Proof. We must show that for every object (Z, c_Z) of FBCO-SET, each function $g : Z \to X$ is C-continuous iff $f_j \circ g$ is C-continuous, for all $j \in J$. In fact,

$$\begin{aligned} g_L^{\leftarrow}\big(\bigvee_{j \in J} c_{f_j}(u)\big) &= g_L^{\leftarrow}\big(\bigvee_{j \in J} (f_j^{\leftarrow} \circ c_{f_j} \circ f_j^{\rightarrow})(u)\big) \\ &= \bigvee_{j \in J} g_L^{\leftarrow}\big((f_j^{\leftarrow} \circ c_{f_j} \circ f_j^{\rightarrow})(u)\big) \\ &\geqslant (f_j \circ g)^{\leftarrow}(c_{Y_j}(f_j^{\rightarrow}(u))) \\ &\geqslant c_Z\big((f_j \circ g)^{\leftarrow}(f_j^{\rightarrow}(u))\big) \\ &\geqslant c_Z(g_L^{\leftarrow}(u)). \end{aligned}$$

■

Now, remember that

Definition 2.12. (Adámek [1] and Rodabaugh [8]) Category $\mathcal{A}$ is topological with regard to $\mathcal{X}$ and functor $V : \mathcal{A} \to \mathcal{X}$ iff eah V-structured source in $\mathcal{X}$ has a unique, initial V-lift in $\mathcal{A}$. We may also say that $\mathcal{A}$ is topological over $\mathcal{X}$ with regard to functor V.

As a consequence of corollary (2.8), proposition (2.9) and proposition (2.11), we obtain

Theorem 2.13. The concrete category (FBCO-SET, $\mathcal{O}$) over SET is a topological category.

3. Basic properties of variable-basis closure operators

In this section we consider a subcategory $\mathcal{D}$ of CQML in order to construct fuzzy variable-basis closure operators on the category $SET \times \mathcal{D}$ that has as objects all pairs (X, L), where X is a set and L is an object of $\mathcal{D}$, as morphisms from (X, L) to (Y, M) all pairs of maps (f,ϕ) with $f \in SET(X,L)$ and $\phi \in CQML(L,M)$, identities given by $id_{(X,L)} = (id_X, id_L)$, and composition defined by

$$(f,\phi)\circ(g,\psi) = (f\circ g, \phi\circ\psi).$$

Definition 3.1. A closure operator of the category $SET \times \mathcal{D}$ is given by a family $C = (c_{XL})_{(X,L)\in|SET\times\mathcal{D}|}$ of maps $c_{XL} : L^X \longrightarrow L^X$ that satisfies the requeriment:

(C_1) (Extension) $u \leqslant c_X(u)$ for all $u \in L^X$;

(C_2) (Monotonicity) if $u \leqslant v$ in L^X, then $c_X(u) \leqslant c_X(v)$;

(C_3) (Lower bound) $c_X(0_X) = 0_X$.

Definition 3.2. A fuzzy variable-basis C-space is a triple (X, L, c_{XL}), where (X, L) is an object of $SET \times \mathcal{D}$ and c_{XL} is a closure map on (X, L).

Definition 3.3. A morphism $(f,\phi) : (X,L) \longrightarrow (Y,M)$ in $SET \times \mathcal{D}$ is said to be fuzzy c-continuous if

$$(f,\phi)^{\rightarrow}(c_{XL}(u)) \leqslant c_{YM}\big((f,\phi)^{\rightarrow}(u)\big) \text{ for all } u \in L^X. \tag{4}$$

Proposition 3.4. Condition (4) is equivalent to

$$c_{XL}\big((f,\phi)^{\leftarrow}(v)\big) \leqslant (f,\phi)^{\leftarrow}(c_{YM}(v)) \quad \text{for all} \quad v \in M^Y. \tag{5}$$

Proof. Condition (4) gives that, for $v \in M^Y$ and $u = (f,\phi)^{\leftarrow}(v)$,

$$\begin{aligned}(f,\phi)^{\rightarrow}(c_{XL}(u)) &= (f,\phi)^{\rightarrow}(c_{XL}(f,\phi)^{\leftarrow}(v)) \\ &\leqslant c_{YM}\big((f,\phi)^{\rightarrow}(f,\phi)^{\leftarrow}(v)\big) \leqslant c_{YM}(v),\end{aligned}$$

then

$$c_{XL}((f,\phi)^{\leftarrow}(v)) \leqslant (f,\phi)^{\leftarrow}(c_{YM}(v)).$$

On the other hand, condition (5) gives that, for $u \in L^X$

$$\begin{aligned}c_{XL}(u) &\leqslant c_{XL}\big((f,\phi)^{\leftarrow}(f,\phi)^{\rightarrow}(u)\big) \\ &\leqslant (f,\phi)^{\leftarrow}\big(c_{YM}((f,\phi)^{\rightarrow}(u))\big),\end{aligned}$$

then $(f,\phi)^{\rightarrow}(c_{XL}(u)) \leqslant c_{YM}((f,\phi)^{\rightarrow}(u))$. ■

Proposition 3.5. Consider two fuzzy c-continuous morphisms $(f,\phi) : (X,L) \longrightarrow (Y,M)$ and $(g,\psi) : (Y,M) \longrightarrow (Z,N)$ be , then the morphism $(g,\psi)\circ(f,\phi)$ is fuzzy c-continuous.

Proof. Since $(f,\phi) : (X,L) \longrightarrow (Y,M)$ is c-continuous we have

$$(f,\phi)^{\rightarrow}(c_{XL}(u)) \leqslant c_{YM}((f,\phi)^{\rightarrow}(u)) \quad \text{for all} \quad u \in L^X,$$

it follows that

$$(g,\psi)^{\rightarrow}\big((f,\phi)^{\rightarrow}(c_{XL}(u))\big) \leqslant (g,\psi)^{\rightarrow}(c_{YM}((f,\phi)^{\rightarrow}(u))),$$

now, by the fuzzy c-continuity of (g,ψ),

$$(g,\psi)^{\rightarrow}\big(c_{YM}((f,\phi)^{\rightarrow}(u))\big) \leqslant c_{ZN}\big((g,\psi)^{\rightarrow}((f,\phi)^{\rightarrow}(u))\big),$$

therefore

$$\big((g,\psi)\circ(f,\phi)\big)^{\rightarrow}(c_{XL}(u)) \leqslant c_{ZN}\big(((g,\psi)\circ(f,\phi))^{\rightarrow}(u)\big).$$

■

As a consequence we obtain

Definition 3.6. The category VBCO-SET that has as objects all triples (X, L, c_{XL}) where (X, L) is an object of $SET \times \mathcal{D}$ and $c_{XL} : L^X \longrightarrow L^X$ is a fuzzy closure map, as morphisms from (X, L, c_{XL}) to (Y, M, c_{YM}) all pairs of fuzzy c-continuous functions $(f,\phi) : (X, L, c_{XL}) \longrightarrow (Y, M, c_{YM})$, identities and composition as in $SET \times D$.

3.1. Initial variable-basis closure operator

Let (Y, M, c_{YM}) be an object of the category VBCO-SET and let (X, L) be an object of the category $SET \times \mathcal{D}$. For each morphism $(f,\phi) : (X, L) \longrightarrow (Y, M)$ in $SET \times \mathcal{D}$ we define on (X, L) the map $\hat{c}_{XL} : L^X \longrightarrow L^X$ by

$$\hat{c}_{XL} = (f,\phi)^{\leftarrow} \circ c_{YM} \circ (f,\phi)^{\rightarrow} \tag{6}$$

i.e the following diagram is conmutative

$$\begin{tikzcd} L^X \arrow[r, "(f,\phi)^{\rightarrow}"] \arrow[d, dotted, "\hat{c}_{XL}"'] & M^Y \arrow[d, "c_{YM}"] \\ L^X & M^Y \arrow[l, "(f,\phi)^{\leftarrow}"] \end{tikzcd}$$

Proposition 3.7. The map (6) is a closure map on (X, L) for which the morphism (f,ϕ) is fuzzy c-continuous.

Proof.

1) (Extension) For every $u \in L^X$,

$$\hat{c}_{XL}(u) = (f,\phi)^{\leftarrow}\big(c_{YM}((f,\phi)^{\rightarrow}(u))\big) \geqslant (f,\phi)^{\leftarrow}(f,\phi)^{\rightarrow}(u) \geqslant u,$$

therefore $u \leqslant \hat{c}_{XL}(u)$.

2) (Monotonicity) $u_1 \leqslant u_2$ in L^X implies $(f,\phi)^{\rightarrow}(u_1) \leqslant (f,\phi)^{\rightarrow}(u_2)$, then $c_{YM}\big((f,\phi)^{\rightarrow}(u_1)\big) \leqslant c_{YM}\big((f,\phi)^{\rightarrow}(u_2)\big)$,
consequentently

$$\begin{aligned}\hat{c}_{XL}(u_1) &= (f,\phi)^{\leftarrow}\big(c_{YM}((f,\phi)^{\rightarrow}(u_1))\big)\\ &\leqslant (f,\phi)^{\leftarrow}\big(c_{YM}((f,\phi)^{\rightarrow}(u_2))\big) = \hat{c}_{XL}(u_2)\end{aligned}$$

3) (Lower bound)

$$\hat{c}_{XL}(0_X) = (f,\phi)^{\leftarrow}\big(c_{YM}((f,\phi)^{\rightarrow}(0_X))\big) = 0_X.$$

Finally,

$$\begin{aligned}(f,\phi)^{\rightarrow}(\hat{c}_{XL}(u)) &= (f,\phi)^{\rightarrow}\big((f,\phi)^{\leftarrow}(c_{YM}((f,\phi)^{\rightarrow}(u)))\big)\\ &\leqslant c_{YM}((f,\phi)^{\rightarrow}(u))), \text{ for all} u \in L^X.\end{aligned}$$

■

It is clear that $\hat{c}_{XL}$ is the finest map on L^X for which the morphism (f,ϕ) es fuzzy c-continuous, more precisely.

Proposition 3.8. Let (Z,N,c_{ZN}) and (Y,M,c_{YM}) be objects of the category VBCO-SET and let (X,L) be an object of $SET\times\mathcal{D}$. For each morphism $(g,\psi):(Z,N)\longrightarrow(X,L)$ of $SET\times\mathcal{D}$ and for $(f,\phi):(X,\hat{c}_{XL})\longrightarrow(Y,c_{YN})$ a fuzzy c-continuous morphism, (g,ψ) is a fuzzy c-continuous morphism if and only if $(f,\phi)\circ(g,\psi)$ is fuzzy c-continuous.

Proof. Suppose that $(f,\phi)\circ(g,\psi)$ is fuzzy c-continuous, that is

$$c_{ZN}\big(((f,\phi)\circ(g,\psi))^{\leftarrow}(v)\big) \leqslant ((f,\phi)\circ(g,\psi))^{\leftarrow}(c_{YN}(v))\text{for all} v \in L^Y.$$

Then for all $u \in L^X$, we have

$$\begin{aligned}(g,\psi)^{\leftarrow}(\hat{c}_{XL}(u)) &= (g,\psi)^{\leftarrow}\big((f,\phi)^{\leftarrow}(c_{YM}((f,\phi)^{\rightarrow}(u)))\big)\\ &= ((f,\phi)\circ(g,\psi))^{\leftarrow}(c_{YM}((f,\phi)^{\rightarrow}(u)))\\ &\geqslant c_{ZN}((f,\phi)\circ(g,\psi))^{\leftarrow}((f,\phi)^{\rightarrow}(u))\\ &= c_{ZN}(g,\psi)^{\leftarrow}((f,\phi)^{\leftarrow}(f,\phi)^{\rightarrow}(u))\\ &\geqslant ((g,\psi)^{\leftarrow}(u))\end{aligned}$$

■

As in theorem (2.13), we have

Theorem 3.9. The concrete category (VBCO-SET, U) over $SET\times\mathcal{D}$ is a topological category.

3.2. Closed and dense fuzzy sets

Definition 3.10. An L-fuzzy subset u of X is called c-closed in (X, L) if it is equal to it is closure, i.e, $c_{XL}(u) = u$. The fuzzy c-continuity condition (4) implies that c-closedness is preserved by inverse images.

Proposition 3.11. Let $(f, \phi) : (X, L, c_{XL}) \longrightarrow (Y, M, c_{YM})$ be a morphism in VBCO-SET. If $v \in M^Y$ is c-closed then $(f, \phi)^{\leftarrow}(v)$ is c-closed in (X, L).

Proof. If $v = c_{YM}(v)$, for $v \in M^Y$, then $(f, \phi)^{\leftarrow}(v) = (f, \phi)^{\leftarrow}(c_{YM}(v)) \geqslant c_{XL}((f, \phi)^{\leftarrow}(v))$, so $c_{XL}((f, \phi)^{\leftarrow}(v)) = (f, \phi)^{\leftarrow}(v)$. ■

Definition 3.12. An L-fuzzy subset u of X is called c-dense in (X, L) if it is c-closure is 1_X.

Proposition 3.13. Let $(f, \phi) : (X, L, c_{XL}) \longrightarrow (Y, M, c_{YM})$ be an epimorphism in VBCO-SET. If $u \in L^X$ is $(f, \phi)^{\rightarrow}(u)$ is c-dense in (Y, M).

Proof. It $c_{XL}(u) = 1_X$ then

$$1_Y = (f, \phi)^{\rightarrow}(1_X) = (f, \phi)^{\rightarrow}(c_{XL}(u)) \leqslant c_{YM}((f, \phi)^{\rightarrow}(u))$$

■

3.3. C-closed morphisms

Definition 3.14. A morphism $(f, \phi) : (X, L, c_{XL}) \longrightarrow (Y, M, c_{YM})$ between fuzzy variable-basis c-spaces is fuzzy c-closed if

$$c_{YM}((f, \phi)^{\rightarrow}(u)) \leqslant (f, \phi)^{\rightarrow}(c_{XL}(u)) \quad \text{for all} \quad u \in L^X. \tag{7}$$

Proposition 3.15. Let $(f, \phi) : (X, L, c_{XL}) \longrightarrow (Y, M, c_{YM})$ and $(g, \psi) : (Y, M, c_{YM}) \longrightarrow (Z, N, c_{ZN})$ be two fuzzy c-closed morphisms, then the morphism $(f, \phi) \circ (g, \psi)$ is fuzzy c-closed.

Proof. Since $(f, \phi) : (X, L, c_{XL}) \longrightarrow (Y, M, c_{YM})$ is fuzzy c-closed, we have

$$c_{YM}((f, \phi)^{\rightarrow}(u)) \leqslant (f, \phi)^{\rightarrow}(c_{XL}(u)) \quad \text{for all} \quad u \in L^X,$$

it follows that

$$(g, \psi)^{\rightarrow}\big(c_{YM}((f, \phi)^{\rightarrow}(u))\big) \leqslant (g, \psi)^{\rightarrow}\big((f, \phi)^{\rightarrow}(c_{XL}(u))\big)$$

now, by the fuzzy c-closedness of (g, ψ),

$$c_{ZN}((g, \psi)^{\rightarrow}(v)) \leqslant (g, \psi)^{\rightarrow}(c_{YM}(v)) \quad \text{for all} \quad v \in L^Y,$$

in particular for $v = (f, \phi)^{\rightarrow}(u)$,

$$c_{ZN}((g, \psi)^{\rightarrow}((f, \phi)^{\rightarrow}(u))) \leqslant (g, \psi)^{\rightarrow}(c_{YM}((f, \phi)^{\rightarrow}(u)))$$

therefore

$$c_{ZN}\big(((g,\psi)\circ(f,\phi))^{\rightarrow}(u)\big) \leqslant (((g,\psi)\circ(f,\phi))^{\rightarrow}(c_{XL}(u))$$

■

If we replace in the category VBCO-SET fuzzy c-continuous morphisms by fuzzy c-closed morphisms, we obtain another topological category. The morphisms (f,ϕ) : $(X,L,c_{XL}) \longrightarrow (Y,M,c_{YM})$ between fuzzy variable-basis c-spaces which are bijective, fuzzy c-continuous and c-closed, forms a group. We can say that a way of seeing fuzzy variable-basis topology is studying invariants of the action of these groups aver the category $SET \times \mathcal{D}$.

4. Idempotent and additive closure operators

Definition 4.1. The closure operator $C = (c_X)_{X\in|SET|}$ of definition (2.1) is called idempotent if the condition

$$c_X(c_X(u)) = c_X(u) \quad \text{for all} \quad u \in L^X$$

holds for every set X.

Proposition 4.2. Let $C = (c_Y)_{Y\in|SET|}$ be an idempotent closure operator. Then the initial closure operator $C = (c_{X_f})_{X\in|SET|}$ defined by

$$c_{X_f} = f_L^{\leftarrow}\circ c_Y \circ f_L^{\rightarrow} \qquad \text{for each function} \qquad f : X \to Y$$

is also idempotent.

Proof. Suppose that $C = (c_Y)_{Y\in|SET|}$ is an idempotent closure operator and let $f : X \to Y$ be a function. Then

$$\begin{aligned} c_{X_f}\circ c_{X_f} &= (f_L^{\leftarrow}\circ c_Y\circ f_L^{\rightarrow})\circ(f_L^{\leftarrow}\circ c_Y\circ f_L^{\rightarrow}) \\ &= (f_L^{\leftarrow}\circ c_Y)\circ(f_L^{\rightarrow}\circ f_L^{\leftarrow})\circ(c_Y\circ f_L^{\rightarrow}) \\ &\leqslant f_L^{\leftarrow}\circ(c_Y\circ c_Y)\circ f_L^{\rightarrow} \\ &= f_L^{\leftarrow}\circ c_Y\circ f_L^{\rightarrow} \\ &= c_{X_f}. \end{aligned}$$

On the other hand, the monotonicity condition of closure operators implies that

$$c_{X_f} \leqslant c_{X_f}\circ c_{X_f}.$$

■

By using similar arguments, we can proof that

Proposition 4.3. Let $C = (c_{XL})_{(X,L)\in|SET\times\mathcal{D}|}$ be an idempotent closure operator. Then the initial closure operator $\hat{c}_{XL} : L^X \longrightarrow L^X$, defined by

$$\hat{c}_{XL} = (f,\phi)^{\leftarrow} \circ c_{YM} \circ (f,\phi)^{\rightarrow},$$

for each morphism $(f,\phi) : (X,L) \longrightarrow (Y,M)$ in $SET \times \mathcal{D}$ is also idempotent.

Definition 4.4. The closure operator $C = (c_X)_{X\in|SET|}$ of definition (2.1) is called

1. Additive if the condition

$$c_X(u \vee v) = c_X(u) \vee c_X(v) \quad \text{for all} \quad u, v \in L^X$$

   holds for every set X.

2. Fully additive if the condition

$$c_X\Big(\bigvee_{\lambda\in\Lambda} u_\lambda\Big) = \bigvee_{\lambda\in\Lambda} c_X(u_\lambda) \quad \text{for all} \quad \{u_\lambda \mid \lambda \in \Lambda\} \subseteq L^X$$

   holds for every set X.

Proposition 4.5. Let $C = (c_Y)_{Y\in|SET|}$ be a fully additive closure operator. Then the initial closure operator $C = (c_{X_f})_{X\in|SET|}$ defined by

$$c_{X_f} = f_L^{\leftarrow} \circ c_Y \circ f_L^{\rightarrow} \qquad \text{for each function} \qquad f : X \to Y$$

is also fully additive.

Proof. Suppose that $C = (c_Y)_{Y\in|SET|}$ is a fully additive closure operator and let $f : X \to Y$ be a function. Then, for all $\{u_\lambda \mid \lambda \in \Lambda\} \subseteq L^X$,

$$\begin{aligned} c_{X_f}\Big(\bigvee_{\lambda\in\Lambda} u_\lambda\Big) &= f_L^{\leftarrow}\Big(c_Y\big(f_L^{\rightarrow}\big(\bigvee_{\lambda\in\Lambda} u_\lambda\big)\big)\Big) \\ &= \bigvee_{\lambda\in\Lambda} \big(f_L^{\leftarrow}\big(c_Y(f_L^{\rightarrow}(u_\lambda))\big) \\ &= \bigvee_{\lambda\in\Lambda} c_{X_f}(u_\lambda). \end{aligned}$$

■

By using similar arguments, we can proof that

Proposition 4.6. Let $C = (c_{XL})_{(X,L)\in|SET\times\mathcal{D}|}$ be a fully additive closure operator. Then the initial closure operator $\hat{c}_{XL} : L^X \longrightarrow L^X$, defined by

$$\hat{c}_{XL} = (f,\phi)^{\leftarrow} \circ c_{YM} \circ (f,\phi)^{\rightarrow},$$

for each morphism $(f,\phi) : (X,L) \longrightarrow (Y,M)$ in $SET \times \mathcal{D}$ is also fully additive.

5. Some examples of closure operators

Example 5.1. Let $I = [0, 1]$ be the unit interval considered as a subespace of the real numbers $\mathbb{R}$.

(i) For each topological space X, we define $c_X : I^X \to I^X$ by

$$c_X(u) = \bigwedge\{v \in I^X \mid v \quad \text{is upper semi-continuous and} \quad u \leqslant v\}.$$

Clearly, the family $C = (c_X)_{X \in |TOP|}$ is a fuzzy closure operator of the category TOP. Since the fixed points of the restriction of c_X to 2^X produces the closed sets of X, this operator is an extension of the usual closure in TOP.

(ii) For each compact topological space Y, we define $d_Y : I^Y \to I^Y$ by

$$d_Y(v) = M_v \quad \text{where} \quad M_v = \max\{v(y) \mid y \in Y\}.$$

Undoubtedly, the family $D = (d_Y)_{Y \in |COMP|}$ is a fuzzy closure operator of the category $COMP$ (of compact topological spaces).

(iii) Every map $f : X \to Y$ from a topological space X to a compact space Y is fuzzy CD-continuous since each constant map is upper semi-coninuous.

(iv) On the other hand, the only fuzzy DC-continuous maps between compact spaces an topological spaces are the constant.

Example 5.2. Let $X = \{x\}$ be a single point set and $L = [0, 1]$ be the usual unit interval. The maps $c_n : L^X \to L^X$ defined by $c_n(t) = t^{\frac{1}{n}}$, for $n = 1, 2, \cdots$ are closure maps, from which just c_1 and $c_\infty = \lim_{n\to\infty} t^{\frac{1}{n}}$ are idempotent.

Example 5.3. For a $GL-$monoid Ł and for an Ł-topology $\tau \subseteq Ł^X$, we define

$$c_X(u) = \bigwedge_{v\in\tau}\{v \longmapsto 0_X \mid u \leqslant v\}.$$

These maps produce a closure operator of the category SET.

Example 5.4. Let μ be a fuzzy subgroup of a group G. Acording to Theorem 1.3 of [10], the fuzzy normalizer $N(\mu)$ of μ is a subgroup of G and μ is a fuzzy normal subgroup of the group $N(\mu)$. Therefore

$$c(\mu) = \bigwedge\{\eta \mid \eta \leqslant \mu G\}, \quad \text{for all } G \text{ in the category GRP of all groups}$$

define the fuzzy normal closure operator of groups, which is idempotent.

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