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Approximate Solution of Second-Order Linear Differential Equation

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Abstract

This paper presents a efficient approach for determining the solution of second-order linear differential equation. The second-order linear ordinary differential equation is first converted to a Volterra integral equation. By solving the resulting Volterra equation by means of Taylor's expansion, different approaches based on differentiation and integration methods are employed to reduce the resulting integral equation to a system of linear equation for the unknown and its derivatives the approximate solution of second-order linear differential equation is obtained. By studying the estimation of the error give the efficiency and high accuracy of the proposed method.

AMS Subject Classification:
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1. Introduction

Some second-order differential equations with variable coefficients can be solved analytically by various methods (see [4]). For general cases, one must appeal to numerical techniques or approximate approaches for getting its solutions [2-3]. The Adomian decomposition method for solving differential and integral equations, linear or nonlinear, has been subject of extensive analytical and numerical studies. In particular Adomian's decomposition method has been proposed for solving second-order linear differential equation.

This paper presents a efficient approach for determining the solution of second-order linear differential equation. The second-order linear ordinary differential equation is first converted to a Volterra integral equation. By solving the resulting Volterra equation by means of Taylor's expansion, different approaches based on differentiation and integration methods are employed to reduce the resulting integral equation to a system of linear equation for the unknown and its derivatives the approximate solution of second-order linear differential equation is obtained. By studying the estimation of the error give the efficiency and high accuracy of the proposed method.

2. Volterra integrals equations

We consider the differential equation (E) following:

$$(E) : y''(t) + p(t)y'(t) + q(t)y(t) = g(t) \tag{2.1}$$

with $p, q\ et\ g$ are infinitely differential functions in open interval $I \subset \mathbb{R}$.

We fix a point a of the interval I. We have, $\forall\, x \in I$,

$$\int_a^x \frac{(t-x)^{i+1}}{(i+1)!} y''(t)dt = y(a)\frac{(a-x)^i}{i!} - y'(a)\frac{(a-x)^{i+1}}{(i+1)!} + \int_a^x \frac{(t-x)^{i-1}}{(i-1)!} y(t)dt \tag{2.2}$$

and

$$\int_a^x \frac{(t-x)^{i+1}}{(i+1)!} p(t)y'(t)dt = -p(a)y(a)\frac{(a-x)^{i+1}}{(i+1)!} - \int_a^x [\frac{(t-x)^i}{i!} p(t) + \frac{(t-x)^{i+1}}{(i+1)!} p'(t)]y(t)dt \tag{2.3}$$

The differential equation (E) equivalent at integral equation:

$$(E_i) : \ \forall\, x \in I, \ \int_a^x h_{i,x}(t)y(t)dt = f_i(x) \tag{2.4}$$

with

$$h_{i,x}(t) = \frac{(t-x)^{i-1}}{(i-1)!} - \frac{(t-x)^i}{i!} p(t) + \frac{(t-x)^{i+1}}{(i+1)!} (q(t) - p'(t)) \tag{2.5}$$

$$f_i(x) = -y(a)\frac{(a-x)^i}{i!} + [y'(a) + p(a)y(a)]\frac{(a-x)^{i+1}}{(i+1)!} + \int_a^x \frac{(t-x)^{i+1}}{(i+1)!} g(t)dt \tag{2.6}$$

3. Linear system and approximate solution

Next following the method suggested in [1,2]. We fix one positive integer $n \geq 1$. We have:

$$y(t) = \sum_{k=0}^{n-1} y^{(k)}(x)\frac{(t-x)^k}{k!} + R_{n,x}(t) \tag{3.1}$$

where $R_{n,x}(t)$ denotes integral remainder

$$R_{n,x}(t) = \int_x^t \frac{(t-s)^{n-1}}{(n-1)!} y^{(n)}(s)ds \tag{3.2}$$

In particular, if the desired solution $y(t)$ is a polynomial of degree equal to or less than n, then $R_{n,x}(t) = 0$. We put for all i and j positive integer ≥ 1:

$$b_{ij}(x) = \int_a^x h_{i,x}(t) \frac{(t-x)^{j-1}}{(j-1)!} dt \tag{3.3}$$

For an integer $i \geq 1$, the function y is solution of (E) if and only if we have:

$$\sum_{j=1}^{n} b_{ij}(x) y^{(j-1)}(x) = f_i(x) \tag{3.4}$$

We consider the matrix:

$$B_n(x) = (b_{ij}(x))_{i,j=1,\dots,n} \tag{3.5}$$

and the column

$$F_n(x) = \begin{pmatrix} f_1(x) \\ . \\ . \\ f_n(x) \end{pmatrix}, \ Y_n(x) = \begin{pmatrix} y(x) \\ y'(x) \\ . \\ . \\ y^{(n-1)}(x) \end{pmatrix} \tag{3.6}$$

For all positive integer $n \geq 2$, the equation (E) equivalent at

$$\forall x \in I, \quad B_n(x)Y_n(x) = F_n(x) \tag{3.7}$$

Application of Cramer's rule to the resulting system yields an approximate solution of equation (18). It is also noted that not only $y(x)$ but also $y^i(x)$ $(1 \leq 1 \leq n-1)$ are determined via solving the resulting system. The solution $y(x)$ can be approximately obtained to be:

$$y_n(x) = \frac{\det(B_1(x))}{\det(B_n(x))} \tag{3.8}$$

where

$$B_1(x) = \begin{pmatrix} f_1(x) & b_{12}(x) & \cdots & b_{1n}(x) \\ f_2(x) & b_{22}(x) & \cdots & b_{2n}(x) \\ . & . & & . \\ . & . & & . \\ f_n(x) & b_{n2}(x) & \cdots & b_{nn}(x) \end{pmatrix} \tag{3.9}$$

4. Error analysis

4.1. Approximation of b_{ij}

We are going to use the explore Taylor approximations, in a of b_{ij} at order $i+j+n-1$. Let: $i \geq 1$ et $j \geq 1$ For all $k = 0, \ldots, i+j-2$, we have:

$$\begin{aligned} b_{ij}^{(k)}(x) &= (-1)^{i+j} \int_a^x [C_{i+j-2}^{j-1} \frac{(x-t)^{i+j-k-2}}{(i+j-k-2)!} - C_{i+j-1}^{j-1} \frac{(x-t)^{i+j-k-1}}{(i+j-k-1)!} p(t) \\ &+ C_{i+j}^{j-1} \frac{(x-t)^{i+j-k}}{(i+j-k)!} (p'(t)+q(t))]dt \end{aligned} \tag{4.1}$$

In particular, we have:

$$\begin{aligned} b_{ij}^{(i+j-2)}(x) &= (-1)^{i+j} [C_{i+j-2}^{j-1}(x-a) - C_{i+j-1}^{j-1} \int_a^x (x-t)p(t)dt \\ &+ C_{i+j}^{j-1} \int_a^x \frac{(x-t)^2}{2} (p'(t)+q(t))dt] \end{aligned} \tag{4.2}$$

Furthermore, we have:

$$\begin{aligned} b_{ij}^{(i+j-1)}(x) &= (-1)^{i+j} C_{i+j-2}^{j-1} + (-1)^{i+j-1} C_{i+j-1}^{j-1} \int_a^x p(t)dt \\ &+ (-1)^{i+j} C_{i+j}^{j-1} \int_a^x (x-t)(p'(t)+q(t))dt \end{aligned} \tag{4.3}$$

$$\begin{aligned} b_{ij}^{(i+j)}(x) &= (-1)^{i+j-1} C_{i+j-1}^{j-1} p(x) + (-1)^{i+j} C_{i+j}^{j-1} \int_a^x (p'(t)+q(t))dt \\ &= (-1)^{i+j} [C_{i+j}^{j-1} - C_{i+j-1}^{j-1}] p(x) - (-1)^{i+j} C_{i+j}^{j-1} p(a) \\ &\quad + (-1)^{i+j} C_{i+j}^{j-1} \int_a^x q(t)dt \end{aligned} \tag{4.4}$$

$$b_{ij}^{(i+j+1)}(x) = (-1)^{i+j} [C_{i+j}^{j-1} - C_{i+j-1}^{j-1}] p'(x) + C_{i+j}^{j-1} q(x)] \tag{4.5}$$

Thus, for all $k = 0, \ldots, i+j-2$, we have:

$$b_{ij}^{(k)}(a) = 0 \tag{4.6}$$

So we have:

$$b_{ij}^{(i+j-1)}(a) = (-1)^{i+j} C_{i+j-2}^{j-1} \tag{4.7}$$

$$b_{ij}^{(i+j)}(a) = (-1)^{i+j-1} C_{i+j-1}^{j-1} p(a) \tag{4.8}$$

The explore Taylor approximations, in a of $b_{ij}^{(i+j+1)}$ at order $n-2$, give that of b_{ij} at order $n+i+j-1$ following:

$$b_{ij}(x) = b_{(n)ij}(x) + o((x-a)^{n+i+j-1}) \tag{4.9}$$

with

$$b_{(n)ij}(x) = -\frac{(a-x)^{i+j-1}}{(i+j-1)(i-1)!(j-1)!} - p(a)\frac{(a-x)^{i+j}}{(i+j)i!(j-1)!} + \sum_{k=0}^{n-2}(-1)^{i+j}[(C_{i+j}^{j-1} - C_{i+j+1}^{j-1})p^{(k+1)}(a) + C_{i+j}^{j-1}q^{(k)}(a)]\frac{(x-a)^{i+j+1+k}}{(i+j+1+k)!} \tag{4.10}$$

For $i \geq 1$ and $j \geq 1$, we put:

$$b_{ij}(x) = -\frac{(a-x)^{i+j-1}}{(i-1)!(j-1)!}\widetilde{b}_{ij}(x) \tag{4.11}$$

We have:

$$\widetilde{b}_{ij}(x) = \tilde{b}_{(n)ij}(x) + o((x-a)^n) \tag{4.12}$$

with:

$$\begin{aligned} \tilde{b}_{(n)ij}(x) &= \frac{1}{i+j-1} - \frac{p(a)}{(i+j)i}(x-a) \\ &+ \frac{j-1}{i(i+1)(i+j)}\sum_{l=2}^{n}\left[\frac{1}{C_{i+j+l-1}^{l-1}}\right]\frac{p^{(l-1)}(a)}{(l-1)!}(x-a)^l \\ &+ \frac{1}{i(i+1)(i+j+1)}\sum_{l=2}^{n}\left[\frac{1}{C_{i+j+l-1}^{l-2}}\right]\frac{q^{(l-2)}(a)}{(l-1)!}(x-a)^l \end{aligned} \tag{4.13}$$

So the approximation in a, at order n, of $\widetilde{b}_{ij}$ use the approximation at order $n-1$ of p and the approximation at order $n-2$ of q.

We put: $\tilde{B}_n(x) = (\tilde{b}_{ij}(x))_{i,j=1,\dots,n}$ and $\tilde{B}_{(n)}(x) = (\tilde{b}_{(n)ij}(x))_{i,j=1,\dots,n}$

Let $D_n(x)$ the diagonal matrix having $\dfrac{(a-x)^{i-1}}{(i-1)!}$ as i-th diagonal coefficient. For all : $i = 1, \dots, n$.

$$D_n(x) = \begin{pmatrix} 1 & 0 & . & . & 0 \\ 0 & (a-x) & 0 & . & . \\ . & . & . & . & . \\ . & . & . & . & 0 \\ 0 & . & . & 0 & \dfrac{(a-x)^{n-1}}{(n-1)!} \end{pmatrix} \tag{4.14}$$

We have:

$$B_n(x) = (x-a)D_n(x)\widetilde{B}_n(x)D_n(x) \tag{4.15}$$

$$B_{(n)}(x) = (x-a)D_n(x)\widetilde{B}_{(n)}(x)D_n(x) \tag{4.16}$$

$$\widetilde{B}_n(x) - \widetilde{B}_{(n)}(x) = o((x-a)^n) \tag{4.17}$$

4.2. Approximation of F_n

We have:

$$f_i(x) = -y(a)\frac{(a-x)^i}{i!} + [y'(a) + p(a)y(a)]\frac{(a-x)^{i+1}}{(i+1)!} + \int_a^x \frac{(t-x)^{i+1}}{(i+1)!}g(t)dt \tag{4.18}$$

We are going to use the explore Taylor approximations, in a of $f_i(x)$ at order $i+n$. For $k \leq i+1$, we have:

$$\frac{d^k}{dx^k}\int_a^x \frac{(t-x)^{i+1}}{(i+1)!}g(t)dt = (-1)^k \int_a^x \frac{(t-x)^{i+1-k}}{(i+1-k)!}g(t)dt \tag{4.19}$$

and

$$f_i^{(i+2)}(x) = (-1)^{i+1}g(x) \tag{4.20}$$

Thus we have:

$$f_i(x) = f_{(n)i}(x) + o((x-a)^{i+n}) \tag{4.21}$$

with

$$\begin{aligned} f_{(n)i}(x) = {} & -y(a)\frac{(a-x)^i}{i!} + [y'(a) + p(a)y(a)]\frac{(a-x)^{i+1}}{(i+1)!} \\ & + \frac{(-1)^{i+1}}{(i+2)!}\sum_{k=0}^{n-2}\frac{1}{C_{k+i+2}^k}\frac{g^{(k)}(a)}{k!}(x-a)^{k+i+2} \end{aligned} \tag{4.22}$$

We put:

$$F_{(n)}(x) = \begin{pmatrix} f_{(n)1}(x) \\ f_{(n)2}(x) \\ . \\ . \\ f_{(n)n}(x) \end{pmatrix} \tag{4.23}$$

We have:

$$F_{(n)}(x) = (x-a)D_n(x)\widetilde{F}_{(n)}(x) \tag{4.24}$$

with:

$$\widetilde{F}_{(n)}(x) = \begin{pmatrix} \widetilde{f}_{(n)1}(x) \\ \widetilde{f}_{(n)2}(x) \\ . \\ . \\ \widetilde{f}_{(n)n}(x) \end{pmatrix} \tag{4.25}$$

For $i = 1 \ldots, n$, we have:

$$\widetilde{f}_{(n)i}(x) = \frac{1}{i}y(a) + \frac{1}{i(i+1)}[y'(a) + p(a)y(a)] + \frac{1}{i(i+1)(i+2)} \sum_{k=2}^{n} \frac{1}{C_{k+i}^{k-2}} \frac{g^{(k-2)}(a)}{(k-2)!}(x-a)^k \tag{4.26}$$

The construction of $F_{(n)}$ uses $p(a)$, the approximate of g in a at order $n-2$, as well as the initial condition of $y(a)$ and $y'(a)$. We have:

$$F_n(x) = (x-a)D_n(x)[\widetilde{F}_{(n)}(x) + o((x-a)^n)] \tag{4.27}$$

Lemma 4.1.

1. The matrix $\left(\dfrac{1}{i+j-1}\right)_{i,j=1,\ldots,n}$ has positive determinant.
2. We have:

$$\frac{\det\left(\frac{1}{i+j-1}\right)_{i,j=1,\ldots,n}}{\det\left(\frac{1}{i+j}\right)_{i,j=1,\ldots,n}} = C_{2n}^{n} \tag{4.28}$$

Proof. We provide $\mathbb{C}[X]$ by scalar product: $\langle v, w\rangle = \int_{-1}^{1} v(x)\overline{w(x)}dx$

1. Let $v_0, v_1, \ldots$ the orthogonal basis given by:

$$v_k = \frac{\sqrt{2k+1}}{\sqrt{2}} L_k$$

   with $L_k = \dfrac{1}{2^k k!}\dfrac{d^k}{dx^k}(x^2-1)^k$ Legendre polynomial of degree k.

   We put: $e_k = \dfrac{(x+1)^k}{2^k\sqrt{2}}$. The Gramm Schmidt matrix of linearly independent system $(e_0, \ldots, e_{n-1})$ is $\widetilde{B}_n(a) = \left(\dfrac{1}{i+j-1}\right)_{i,j=1,\ldots,n}$.

   In particular we have: $\det(\widetilde{B}_n(a)) > 0$.

2. We put: $V_n = \left(\dfrac{1}{i+j}\right)_{i,j=1,\ldots,n}$. For: $j = 1, \ldots, n-1$, the j-th column of (V_n) and $(j+1)$-th column of $(\widetilde{B}_n(a))$ equal $\begin{pmatrix} \langle e_0, e_j\rangle \\ . \\ . \\ \langle e_{n-1}, e_j\rangle \end{pmatrix}$

The n- column of (V_n) is $\begin{pmatrix} \langle e_0, e_n\rangle \\ \cdot \\ \cdot \\ \langle e_{n-1}, e_n\rangle \end{pmatrix}$

The first column of $(\widetilde{B}_n(a))$ is $\begin{pmatrix} \langle e_0, e_0\rangle \\ \cdot \\ \cdot \\ \langle e_{n-1}, e_0\rangle \end{pmatrix}$.

Let $P = e_n - \langle e_n, v_n\rangle v_n = \sum_{j=0}^{n-1} x_j e_j$ orthogonal projection of e_n on $\mathbb{C}_{n-1}[X]$. We have : $\forall k = 0, \ldots, n-1, \quad \langle e_k, e_n\rangle = \langle e_k, P\rangle = \sum_{j=0}^{n-1} x_j \langle e_k, e_j\rangle$.

Thus, n-th column of (V_n) equal:

$$x_0 \begin{pmatrix} \langle e_0, e_0\rangle \\ \cdot \\ \cdot \\ \langle e_{n-1}, e_0\rangle \end{pmatrix} + \sum_{j=1}^{n-1} x_j \begin{pmatrix} \langle e_0, e_j\rangle \\ \cdot \\ \cdot \\ \langle e_{n-1}, e_j\rangle \end{pmatrix}$$

Thus: $\det(V_n) = (-1)^{n-1} x_0 \det(\widetilde{B}_n(a))$. We have:

$$P(-1) = -\langle e_n, v_n\rangle v_n(-1) = x_0 \frac{1}{\sqrt{2}}$$

n-th order vector entry of e_n equal n-th order vector entry of $\langle e_n, v_n\rangle v_n$. One arrives at:

$$\begin{aligned} x_0 \frac{1}{\sqrt{2}} &= -\frac{\text{n-th order vector entry of } e_n}{\text{n-th order vector entry of } v_n} v_n(-1) \\ &= -\frac{\text{n-th order vector entry of } e_n}{\text{n-th order vector entry of } Q} Q(-1) \end{aligned}$$

such as $Q = \frac{1}{n!} \frac{d^n}{dx^n} (x^2 - 1)^n$. Taylor's expansion: $Q(-1) = (-2)^n$. Thus :

$$x_0 \frac{1}{\sqrt{2}} = -\frac{\frac{1}{2^n \sqrt{2}}}{\frac{(2n)!}{n!n!}} (-2)^n = (-1)^{n-1} \frac{1}{\sqrt{2}} \frac{1}{C_{2n}^n}$$

Thus:

$$\frac{\det(\frac{1}{i+j-1})_{i,j=1,\ldots,n}}{\det(\frac{1}{i+j})_{i,j=1,\ldots,n}} = C_{2n}^n$$

■

We have: $\widetilde{B}_{(n)}(a) = \widetilde{B}_n(a) = \left(\dfrac{1}{i+j-1}\right)_{i,j=1,\ldots,n}$.
As, $\det(\widetilde{B}_{(n)}(a)) \neq 0$, thus taking x close of a, $\det(\widetilde{B}_{(n)}(x)) \neq 0$.
Hence, taking x close of a; $x \neq a$ $B_{(n)}(x) \neq 0$. Similarly, defines to be :

$$Y_{(n)}(x) = B_{(n)}(x)^{-1} F_{(n)}(x) \tag{4.29}$$

We put :

$$Y_{(n)}(x) = \begin{pmatrix} y_{(n)0}(x) \\ y_{(n)1}(x) \\ . \\ . \\ y_{(n)n-1}(x) \end{pmatrix} \tag{4.30}$$

We have:

$$\widetilde{B}_{(n)}(x) D_n(x) Y_{(n)}(x) = \widetilde{F}_{(n)}(x) \tag{4.31}$$

The first element of $D_n(x)Y_{(n)}(x)$ is $y_{(n)0}$ juste what we look for.

Theorem 4.2.

1. Taking $k = 0, \ldots, n-1$, we have:

$$y_{(n)k}(x) - y^{(k)}(x) = o((x-a)^{n-k-1}) \tag{4.32}$$

2. We have:

$$y(x) = y_{(n)0}(x) + \frac{1}{C_{2n}^n} \frac{y^{(n)}(a)}{n!}(x-a)^n + o((x-a)^n) \tag{4.33}$$

Proof. We have:

$$F_n(x) - Z_n(x) = F_{(n)}(x) - Z_{(n)}(x) + (x-a)D_n(x)o((x-a)^n)$$

Taking x close of a, $x \neq a$, we have:

$$B_n(x)Y_n(x) = B_{(n)}(x)Y_{(n)}(x) - Z_{(n)}(x) + (x-a)D_n(x)o((x-a)^n)$$

one arrives at,

$$\widetilde{B}_n(x)D_n(x)Y_n(x) = \widetilde{B}_{(n)}(x)D_n(x)Y_{(n)}(x) + \frac{y^{(n)}(a)}{n!}(x-a)^n(-1)^{n-1} \begin{pmatrix} \frac{1}{1+n} \\ \frac{1}{2+n} \\ . \\ . \\ \frac{1}{2n} \end{pmatrix} + o((x-a)^n)$$

Thus:

$$\widetilde{B}_n(x)[D_n(x)Y_n(x)-D_n(x)Y_{(n)}(x)] = \frac{y^{(n)}(a)}{n!}(x-a)^n(-1)^{n-1}\begin{pmatrix} \frac{1}{1+n} \\ \frac{1}{2+n} \\ \cdot \\ \cdot \\ \frac{1}{2n} \end{pmatrix} + o((x-a)^n)$$

Furthermore:

$$D_n(x)Y_n(x)-D_n(x)Y_{(n)}(x) = \frac{y^{(n)}(a)}{n!}(x-a)^n(-1)^{n-1}\widetilde{B}^{-1}_{(n)}(a)\begin{pmatrix} \frac{1}{1+n} \\ \frac{1}{2+n} \\ \cdot \\ \cdot \\ \frac{1}{2n} \end{pmatrix} + o((x-a)^n)$$

By looking at the first constituent and allowing that $\widetilde{B}_{(n)}(a) = \left(\frac{1}{i+j-1}\right)_{i,j=1,\ldots,n}$, we obtain:

$$y(x) - y_{(n)0}(x) = \frac{y^{(n)}(a)}{n!}(x-a)^n(-1)^{n-1}\frac{\begin{vmatrix} \frac{1}{1+n} & \frac{1}{2} & \cdot\ \cdot & \frac{1}{n} \\ \frac{1}{2+n} & \frac{1}{3} & & \frac{1}{1+n} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \frac{1}{2n} & \frac{1}{1+n} & \cdot\ \cdot & \frac{1}{2n} \end{vmatrix}}{\begin{vmatrix} 1 & \frac{1}{2} & \cdot\ \cdot & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & & \frac{1}{1+n} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \frac{1}{n} & \frac{1}{1+n} & \cdot\ \cdot & \frac{1}{2n} \end{vmatrix}} + o((x-a)^n)$$

Thus :

$$y(x) - y_{(n)0}(x) = \frac{1}{C^n_{2n}}\frac{y^{(n)}(a)}{n!}(x-a)^n + o((x-a)^n)$$

∎

Example 4.3. We consider the following second-order differential equation with variable coefficients

$$y''(x) + y'(x) + xy(x) = \frac{1}{6}x^4 + \frac{1}{2}x^2 + x \tag{4.34}$$

under the initial condition

$$y(0) = 0,\ y'(0) = 0 \tag{4.35}$$

The exact solution can be easily determined to be

$$y(x) = \frac{x^3}{6} \tag{4.36}$$

However, adopting the process described in the present paper, we can derive several lower-order approximations. For example, taking $n = 2$, one can arrive at the second-order approximate solution as

$$y_2(x) = \frac{11x^{13}}{907200} + \frac{x^{11}}{37800} + \frac{73x^{10}}{3202400} - \frac{x^9}{1728} - \frac{x^8}{126} - \frac{17x^7}{2880} - \frac{x^6}{144} - \frac{5x^5}{48} \tag{4.37}$$

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