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A Study On Fuzzy Coloring In Fuzzy Graphs

[1]JahirHussain R and [2]Kanzul Fathima K S

[1,2] PG and Research Department of Mathematics, Jamal Mohamed College, Trichy-620020, Tamil Nadu, India, Cell No.:[2]+91 9150198781, Email: hssn_jhr@yahoo.com and [2]kanzulfathima@gmail.com.

ABSTRACT

Fuzzy coloring is an assignment of colors to nodes in which no two strong adjacent nodes have same color. In this paper, properties of fuzzy coloring are discussed. On which fuzzy graphs, fuzzy chromatic number (χ_f (G)) equals chromatic number ($\chi(G^*)$) of its corresponding underlying crisp graph are studied. The necessary and sufficient condition of a fuzzy graph on complete graph of 4 nodes to be regular can be found. The sufficient conditions of a fuzzy graph on complete graph of odd nodes and of even nodes to be regular are found. The fuzzy chromatic number of regular fuzzy graph on complete graph with same and distinct membership values is also found.

Key words: fuzzy coloring, fuzzy chromatic number, regular fuzzy graph, complete graph, Hamiltonian cycle and strong arc.

1 Introduction:

Fuzzy graph theory has numerous applications in real life situations. In particular, fuzzy coloring is one of the most important concepts in fuzzy graph theory. It is applicable in almost all applications like Traffic light control, Exam scheduling, Register allocation, tournament problem, banquet problem etc... In almost all problems, our aim is to find fuzzy chromatic number of the corresponding fuzzy graph model. The concept of fuzzy coloring was introduced by Eslahchi and Onagh [2]. The definition of fuzzy chromatic number defined by Eslahchi and Onagh [2] was modified by JahirHussain and KanzulFathima [4]. NagoorGani and Radha published a paper on Regular fuzzy graph which contains some properties of regular fuzzy graph on cycles, characterization of regular fuzzy graphs on a cycle etc. In this paper, some properties of fuzzy coloringare discussed. Wecharacterize the fuzzy graph on which$\chi_f(G) = \chi(G^*)$ and fuzzy coloring of regular fuzzy graph on complete graph is also studied.

2 Preliminaries

Definition 2.1: A *fuzzy graph* G is a pair of functions G=(σ,μ) where σ: V→ [0,1], where V is a node(vertex) set and μ :V×V→ [0,1], a symmetric fuzzy relation on σ. The *underlying crisp graph* of G= (σ,μ) is G= (V,E) and is denoted as G*, where E ⊆ V×V.

Definition 2.2: *Strength of a path* in fuzzy graph G is the weight of the weakest arc in that path.A *weakest arc* is an arc of minimum weight in G.

Definition 2.3: A *strongest path* between two nodes u,v is a pathcorresponding to maximum strength between u and v.
The *strength* of the strongest path is denoted by μ^{∞} (u,v).

Definition 2.4: An arc (x,y) is said to be a *strong arc* if $\mu^{\infty}(x,y) = \mu(x,y)$.Otherwise it is weak arc

Definition 2.5: The *degree of a node* u is the sum of membership values of arcs incident on it. It is denoted by d (u).

Definition 2.6: A fuzzy graph G is said to be *regular* if d (v) = k, for all v∈V and k is a constant.

Definition 2.7: A cycle in a fuzzy graph is said to be *fuzzy cycle* if it contains more than one weakest arc.

Definition 2.8: A fuzzy graph G is said to be *strong* if $\mu(x,y)= \sigma(x) \wedge \sigma(y)$, $\forall (x,y)\in E$.

Definition 2.9: A fuzzy graph G is said to be *complete* if $\mu(x,y)= \sigma(x) \wedge\sigma(y), \forall x,y\in V$.

3Properties of fuzzy coloring

Definition 3.1: If (x,y) is strong arc then x and y are *strong adjacent*.

Note: A weakest arc is different from weak arc. A weakest arc need not be weak arc and a weak arc may be weakest arc.

Definition 3.2: *Fuzzy coloring* is an assignment of colors to nodes of a fuzzy graph G such that strong adjacent nodes have different colors.

Definition 3.3: *Fuzzy chromatic number* of a fuzzy graph G is a minimum number of colors needed for fuzzy coloring of G. It is denoted by χ_f(G).
Since in any fuzzy graph model our aim is to find fuzzy chromatic number of that

model, here fuzzy coloring means proper fuzzy coloring with minimum number of colors.
In crisp graph theory, nodes of an arc must have different color in proper coloring. But in fuzzy graph, arcs are classified as strong arc and weak arc and nodes of strong arc always have different color whereas nodes of weak arc have the same color in every fuzzy coloring. Other than these two arcs, in this section coloring the nodes of a weakest arc and fuzzy bridge are discussed.

Property 3.4: Nodes of a weakest arc may or may not have same color in every fuzzy coloring.

Proof: Let G be a fuzzy graph and (x,y) be a weakest arc of G. Then the strength of connectedness between x and y can be at least $\mu(x,y)$ i.e., $\mu^{\infty}(x,y) \geq \mu(x,y)$. If $\mu(x,y) = \mu^{\infty}(x,y)$ then (x,y) is strong arc which implies x and y cannot have same color in a fuzzycoloring. If $\mu(x,y) < \mu^{\infty}(x,y)$, then (x,y) is not strong arc and so x and y can have same color.

Property 3.5: Nodes of a weakest arc will have same color if it is unique and lies on a cycle.

Proof: Let G be a fuzzy graph such that it has unique weakest arc (x, y). Assume that (x,y) lies on a cycle. Then strength of connectedness between x and y is greater than $\mu(x,y)$ since (x,y) lies on a cycle and it is unique. Therefore $\mu(x,y) \neq \mu^{\infty}(x,y)$. Hence x and y must have same color in fuzzy coloring.

Note: In general, a weakest arc need not be weak arc. But it is weak arc if it is unique and lies on a cycle.

Property 3.6: In a fuzzy graph G, nodes of a fuzzy bridge need not have same color inproper fuzzycoloring.

Proof: This property follows from that "fuzzy bridges need not to be a strong arc".

4 Fuzzy Graphs with $\chi_f(G) = \chi(G^*)$

Theorem 4.1: If G is a strong fuzzy graph then $\chi_f(G) = \chi(G^*)$.

Proof: Let G be a strong fuzzy graph. Then all arcs of G are strong arc. Now construct a graph$G_1 = (V, E^1)$ such that an arc (x,y) is strong arc in G iff it is an arc in G_1. Clearly G_1 is a crisp graph. Since all arcs of G are strong and there is no weak arc, every arc of G is also an arc of G_1 and so $\chi_f(G) = \chi(G_1)$. Clearly G is same as G^*. Hence $\chi_f(G) = \chi(G_1) = \chi(G^*)$.

Corollary4.2: If G is a complete fuzzy graph then$\chi_f(G) = \chi(G^*)$.

Proof: Let G be a complete fuzzy graph. Since every complete fuzzy graph is strong, by theorem 4.1 $\chi_f(G) = \chi(G^*)$.

Theorem 4.3: If G is a fuzzy cycle then $\chi_f(G) = \chi(G^*)$.

ProofLet G be a fuzzy cycle of length n. Then either n is odd or even.

Case i: Assume that n is odd. We know that fuzzy chromatic number of fuzzy cycle of odd length is 3. The underlying crisp graph G^* is the crisp cycle of odd length whose chromatic number is 3. Thus $\chi_f(G) = \chi(G^*)$.

Case ii: If n is even then fuzzy chromatic number of G is 2. Since G^* is the cycle of even length, we have $\chi_f(G) = \chi(G^*)$.

Corollary 4.4: Let G be a regular fuzzy graph such that G^* is a cycle. Then $\chi_f(G) = \chi(G^*)$.

Proof: If G is a regular fuzzy graph where $G^*$is a cycle then G is a fuzzy cycle. Therefore by theorem 4.3, $\chi_f(G) = \chi(G^*)$.

Theorem 4.5: Let G be a fuzzy graph such that every node is strong adjacent to all other nodes. Then $\chi_f(G) = n$. In this case $\chi_f(G) = \chi(G^*)$.

Proof: Let G be a fuzzy graph such that every node is strong adjacent to all other nodes. Clearly this is a fuzzy graph in which all arcs are strong. In this type of fuzzy of, each node will receive a unique color. Since there are n nodes in G, $\chi_f(G) = n$. Clearly the underlying crisp graph of this fuzzy graph is a complete graph. So $\chi_f(G) = \chi(G^*)$.

5 Fuzzy coloring of regular fuzzy graph on complete graph

Every complete graph K_n is the union of cycle of length n and its complementi.e. $K_n = C_n \cup \overline{C_n}$.

Theorem 5.1: Let G be a fuzzy graph such that G^* is complete graph on 4 nodes ($K_4 = C_4 \cup \overline{C_4}$). Then G is regular fuzzy graph iff alternate arcs of C_4 have same membership values and arcs of $\overline{C_4}$have same membership values.

Proof: Assume that G is a fuzzy graph on complete graph with 4 nodes v_1, v_2, v_3 and v_4. Let$e_1 = (v_1, v_2), e_2 = (v_2, v_3), e_3 = (v_3, v_4)$and $e_4 = (v_4, v_1)$ are arcs of C_4 and$e_5 = (v_1, v_3), e_6 = (v_2, v_4)$ are arcs of$\overline{C_4}$. Clearly each node has degree 3 in G^* i.e. 3 arcs are incident on each node. By our assumption, e_1, e_4 and e_5 are incident on v_1, e_1, e_2 and e_6

are incident on v_2, e_2,e_3 ande_5 are incident on v_3 ande_3,e_4, e_6 are incident on v_4. Now G is regular iff the following equations hold
$\mu(e_1) + \mu(e_4) + \mu(e_5) = a$
$\mu(e_1) + \mu(e_2) + \mu(e_6) = a$
$\mu(e_2) + \mu(e_3) + \mu(e_5) = a$
$\mu(e_3) + \mu(e_4) + \mu(e_6) = a$

Solving these equations, we get $\mu(e_1) = \mu(e_3)$, $\mu(e_2) = \mu(e_4)$ and $\mu(e_5) = \mu(e_6)$. Thus G is regular iff$\mu(e_1) = \mu(e_3)$, $\mu(e_2) = \mu(e_4)$ and $\mu(e_5) = \mu(e_6)$.

Corollary 5.2: In the above theorem, if the three membership values are in strictly increasing (strictly decreasing) sequence then χ_f (G) = 2.

Proof: Let $\mu(e_1) = \mu(e_3) = a$, $\mu(e_2) = \mu(e_4) = b$ and $\mu(e_5) = \mu(e_6) = c$ and assume that $a<b<c$. then$\mu^{\infty}(e_i) = c$, if i=5, 6 and$\mu^{\infty}(e_i) = b$ if i=1, 2, 3, 4. Thus arcs e_2, e_4, e_5, e_6 are strong arcs and e_1, e_3 are weak arcs. Now assign color 1 to v_1, v_2 and color 2 to v_3, v_4. Clearly this is fuzzy coloring of minimum cardinality. Hence χ_f (G) = 2. Similarly we can prove this result for strictly decreasing sequence.

Corollary 5.3: Let G be a fuzzy graph given in the theorem 5.1. If all arcs of G have same membership values, then χ_f (G) = 4. In this case, $\chi_f(G) = \chi(G^*)$.

Proof: If all membership values of arcs are same, then G should be strong fuzzy graph. By theorem 4.1, each node will receive unique color. Hence $\chi_f(G) = 4$.

Theorem 5.4: The complete graph K_{2n+1}, $n \geq 1$ have n edge-disjoint Hamiltonian cycles.

Proof: Let G be a complete graph on odd nodes. Name the nodes of G as v, 0, 1, 2…2n-1. Then G contains the following edge-disjoint Hamiltonian cycles.
C_1: v, 0, 2n-1, 1, 2n-2, 2, 2n-3…n-1, n, v.
C_2: v, 1, 0, 2, 2n-1, 3, 2n-2…n, n+1, v.
C_3: v, 2, 1, 3, 0, 4, 2n-1…n+1, n+2, v.
.
.
.
C_n: v, n-1, n-2, n, n-3, n+1, n-4…2n-2, 2n-1, v.

For example name the nodes of K_9 as v, 0, 1, 2, 3, 4, 5, 6, and 7. Then K_9 is decomposable into following four cycles

C_1: v	0	7	1	6	2	5	3	4	v
C_2: v	1	0	2	7	3	6	4	5	v
C_3: v	2	1	3	0	4	7	5	6	v
C_4: v	3	2	4	1	5	0	6	7	v

Theorem 5.5: Let G be a fuzzy graph such that G^* is a complete graph on odd nodes (K_{2n+1}). Then G is regular fuzzy graph if arcs of every Hamiltonian cycles in G^* have same membership values in G.

Proof: Let G be a fuzzy graph such that G^* is K_{2n+1}. Now we make some observations about G^*. By theorem 5.4, for any $n\geq1$ K_{2n+1}has an n Hamiltonian edge-disjoint cycle. They are given below

C_1: v, 0, 2n-1, 1, 2n-2, 2, 2n-3…n-1, n, v.
C_2: v, 1, 0, 2, 2n-1, 3, 2n-2…n, n+1, v.
C_3: v, 2, 1, 3, 0, 4, 2n-1…n+1, n+2, v.
.
.
.
C_n: v, n-1, n-2, n, n-3, n+1, n-4…2n-2, 2n-1, v.

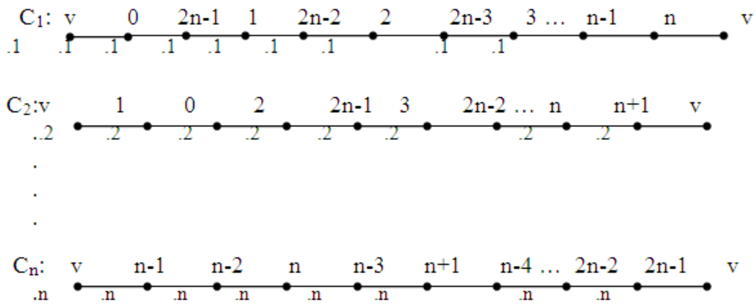
From these cycles, every node is adjacent to exactly two nodes of every cycle. That is if $v\in V$, then $d_i^*(v)=2$, i=1, 2…n. ($d_i^*(v)$ denotes the degree of v in i^{th} cycle of G^{*-}) so that$d^*(v) = d_1^*(v) + d_2^*(v) + d_3^*(v) + \ldots + d_n^*(v) = 2n$, for all $v\in V$.
In fuzzy graph G, assume that the arcs of n Hamiltonian edge-disjoint cycles have same membership values.
Let a_1, a_2…a_n be the membership values of C_1, C_2…C_n respectively. We know that every node of G is adjacent to 2n nodes and exactly two arcs of every Hamiltonian cycle are incident with each node. Since there are n Hamiltonian edge-disjoint cycles, the degree of v in G is $d(v) = 2a_1+ 2a_2+\ldots+2a_n = 2a$, where $a= a_1+ a_2+\ldots+a_n$. similarly $d(0) = d(1) = \ldots = d(2n-1) = 2a$. Thus degree of every node is 2a, for some a. Hence G is regular.

Corollary 5.6: In the above theorem, if the values of a_1, a_2… $a_n$are same then $\chi_f(G) = 2n+1$. In this case $\chi_f(G) = \chi(G^*)$.

Proof: Let us assume that $a_1= a_2= \ldots=a_n= a$. then $\mu(x,y) = \mu^\infty(x,y)$, for all $(x,y) \in E$ since strength of connectedness between any pair of nodes must be a. Thus all arcs of G are strong. Since G^* is complete, every node is adjacent to every other node in G^*. Also we have every node is strong adjacent to every other node in G. Each node will have a unique color and hence$\chi_f(G) = 2n+1$.

Corollary 5.7: In the above theorem, if the values of $a_1,a_2\ldots a_n$are strictly increasing (or strictly decreasing) sequence then $\chi_f(G) = 3$. In this case, $\chi_f(G) = \chi_f(C)$, where C is the fuzzy cycle of odd length.

Proof: Since G^* is complete, G has n (2n+1) arcs. So every Hamiltonian cycle has 2n+1 arc. Without loss of generality, let us take a_1=.1, a_2=.2 … a_n=.n. then



For any arc (x,y) in C_1, it has at least one path in C_n.Therefore strength of connectedness between x and y is.n which implies $\mu(x,y)<\mu^{\infty}(x,y)$, for all $(x,y)\in C_1$. Since any arc (x,y) in C_2 has at least one x-y path in C_n, $\mu(x,y)<\mu^{\infty}(x,y)$. Similarly for any arc (x,y) in C_{n-1} $\mu(x,y)<\mu^{\infty}(x,y)$.
Thus for any arc (x,y)in $C_1, C_2,\ldots C_{n-1}$, $\mu(x,y) \neq \mu^{\infty}(x,y)$ which implies all arcs of $C_1, C_2,\ldots C_{n-1}$ is not strong arc. But every arc in C_n satisfies $\mu(x,y) = \mu^{\infty}(x,y)$. Therefore every arc of C_n is strong.
Since every arc of G except arcs of C_n are not strong, χ_f (G) is equal to χ_f (C) where C is a fuzzy cycle on (2n+1) nodes (here we denote it as C_n). We know that fuzzy chromatic number of odd fuzzy cycle is 3. Therefore χ_f (G) = 3.
Similarly we can prove the theorem for strictly decreasing sequence of $a_1, a_2 \ldots a_n$.

Note: Theorem 5.4 is not true for a fuzzy graph G such that G^* is K_{2n}. But it is regular in the following case.

Theorem 5.8: Let G be a fuzzy graph such that G^* is K_{2n} $(C_{2n} \cup \overline{C_{2n}})$ then G is regular ifμ (x,y) = a,$\forall$ (x,y) $\in C_{2n}$ and μ (x,y) = b,$\forall$ (x,y) $\in \overline{C_{2n}}$, where a, b $\in$(0,1].

Proof: Let G be a fuzzy graph such that G^* K_{2n} which is the union of C_{2n} (cycle of length 2n, n≥2) and its complement.
Now suppose that μ (x,y) = a, if (x,y) $\in C_{2n}$ and μ (x,y) =b, if (x,y) $\in \overline{C_{2n}}$.
Since $K_{2n} = C_{2n} \cup \overline{C_{2n}}$ exactly two arcs of C_{2n} and 2n-3 arcs of $\overline{C_{2n}}$ are incident to each node of G so that degree of every node in G^* becomes 2n-1. So $d_1^*(v_i)$ =2 (degree of every node in C_{2n}), for all i and $d_2^*(v_i)$ =2n-3 (degree of every node in$\overline{C_{2n}}$), for all i.
Therefore d (v) = 2a+ (2n-3) b
= 2a+kb, where k =2n-3
= k^1, where k^1 = 2a+kb, for all v $\in$V.
Thus G is regular.

Note: The converse of the above theorem is not true.

Corollary 5.9: In the above theorem, if a=b then χ_f (G) = 2n. In this case χ_f (G) = $\chi(G^*)$.

Proof: Let a=b. then μ (x,y) = μ^{∞} (x,y), for all (x,y) in G. so all arcs of G are strong. Since every node is strong adjacent to every other node, each node will receive a unique color. Henceχ_f (G) = 2n

Corollary5.10: In the above theorem, if a<b then χ_f (G) = n. in this case χ_f (G) = $\chi(\overline{C_{2n}})$.

Proof: Assume that a<b. For any arc (x,y) in C_{2n}, $\overline{C_{2n}}$ has at least one x-y path. Therefore μ^{∞}(x,y) =b andμ(x,y) =a<b = μ^{∞}(x,y) i.e. μ(x,y) <μ^{∞} (x,y) if (x,y)$\in C_{2n}$. so all arcs of C_{2n} are not strong. Similarly for any arc (x,y) in $\overline{C_{2n}}$, μ(x,y) = b= μ^{∞} (x,y) and so all arcs of $\overline{C_{2n}}$ are strong. Thus χ_f (G) equals $\chi(\overline{C_{2n}})$ (since every arcs of C_{2n} are not strong, without loss of generality we omit these arcs when we find fuzzy chromatic number of G).
Now we have to determine chromatic number of$\overline{C_{2n}}$. In$\overline{C_{2n}}$, each node is adjacent to every other node except two nodes (two neighbouring nodes). Let v_1, v_2 …v_{2n} be the nodes of$\overline{C_{2n}}$. Then every node of $\overline{C_{2n}}$ is not adjacent to exactly two nodes i.e. v_i(i=1, 2…2n, $v_0$means v_{2n} and v_{2n+1}means v_1) is not adjacent tov_{i-1} and v_{i+1}. We can give same color to v_i, v_{i-1} and v_{i+1}, i=1, 2…2n. But v_{i-1} is adjacent to v_{i+1}. So we choose any one from v_{i-1} and v_{i+1}. Choose v_{i+1}. Now assigncolor$c_1$to v_1 and v_2, c_2 to $v_3$and v_4 … c_n to v_{2n-1}, v_{2n}. Hence$\chi(\overline{C_{2n}})$ =n. Thusχ_f (G) = n.

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