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SP-separation axioms in L-topological space[1]

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Abstract

In this paper, the SP-T_{-1}, SP-T_0, sub-SP-T_0 axioms are introduced in L-topological space. SP-T_0 implies SP-T_{-1} and sub-SP-T_0. We give some characterizations of SP-T_{-1}, SP-T_0, sub-SP-T_0.

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1. Introduction

There have been all kinds of studies on separation axioms in L-topology (see [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]). Some among them were based on the extension of separation axiom in general topology. The other were based on the relations between them and

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compactness, uniformity, metric, convergence, etc.. All these separation axioms have oneself characteristics.

In [1], the author introduced the concepts of strongly preopen and strongly preclosed sets, In this paper, we introduce strongly preclosed remote set and strongly preopen neighborhood set in terms of strongly preclosed L-sets and strongly preopen L-sets. We give the concepts of the SP-T_{-1}, SP-T_0, sub-SP-T_0 axioms by means of strongly preclosed remote set and strongly preopen neighborhood set in L-topological spaces. We will prove that SP-T_0 implies SP-T_{-1} and sub-SP-T_0.

2. Preliminaries

For a nonempty set X, L^X denotes the set of all L-fuzzy subsets (or L-subsets for short) on X. $\underline{0}$ and $\underline{1}$ respectively denote the smallest element and the largest element in L^X. It is easy to see that $M(L^X) = \{x_\alpha \mid x \in X, a \in M(L)\}$ is exactly the set of all nonzero $\vee$-irreducible elements in L^X. An L-topological space is a pair (X, τ), where τ is a subfamily L^X which contains $\underline{0}$, $\underline{1}$ and is closed for any suprema and finite infima. τ is called an L-topology on X. Each member of τ is called an open L-set and its quasi-complementation is called a closed L-set.

In [1], the concepts of strongly preopen and strongly preclosed sets were introduced in [0, 1]-fuzzy set theory by Biljana Krateska. They can easily be extended to L-sets as follows:

Definition 2.1. Let (X, τ) be an L-topological space, $A \in L^X$. Then

(1) A is called a strongly preopen L-set if and only if $A \leq int(cl_p(A))$;

(2) A is called a strongly preclosed L-set if and only if $cl(int_p(A)) \leq A$.
The set of all strong preopen L-sets and the set of all strong preclosed L-sets in L^X are respectively denoted as **SPO**(X) and **SPC**(X).

Remark 2.2. Every fuzzy open (closed) set is strongly preopen (strongly preclosed) L-set, but converts is not true.

3. SP-T_{-1}, SP-T_0 and sub-SP-T_0 separation axioms

Definition 3.1. Let (L^X, τ) be an L-topological space, $P, A \in L^X$ and $x_\lambda \in M(L^X)$.

(1) P is called a remote set of x_λ if $x_\lambda \not\leq P$. Remote set P of x_λ is called a strong preclosed remote set, if P is strong preclosed.

(2) A is called a neighborhood set of x_λ if $x_\lambda \leq A$. Neighborhood set A of x_λ is called a strong preopen neighborhood set, if A is strong preopen.

(3) The set of all strong preclosed remote set of x_λ will be denoted by $\eta_{sp}(x_\lambda)$. The set of all strong preopen neighborhood set of x_λ will be denoted by $\xi_{sp}(x_\lambda)$.

Definition 3.2. Let (X, τ) be an L-topological space and $x_\lambda, x_\mu \in M(L^X)$.

(1) If for any two molecules x_λ, x_μ and $x_\lambda \leq x_\mu$, there exists $P \in \eta_{sp}(x_\mu)$ such that $x_\lambda \leq P$, then (X, τ) is called SP-T_{-1};

(2) If for any two molecules x_λ, x_μ and $x_\lambda \neq x_\mu$, there exists $P \in \eta_{sp}(x_\mu)$ such that $x_\lambda \leq P$, or there exists $R \in \eta_{sp}(x_\lambda)$ such that $x_\mu \leq R$, then (X, τ) is called SP-T_0;

(3) If for ant two crisp points $x, y \in X$, there exists $\lambda \in M(L)$ such that $P \in \eta_{sp}(x_\lambda)$ and $y_\lambda \leq P$ or there exists $Q \in \eta_{sp}(y_\lambda)$ and $x_\lambda \leq Q$, then (L^X, τ) is called sub-SP-T_0.

Obviously we have the following proposition.

Proposition 3.3. SP-T_0 implies SP-T_{-1} and sub-SP-T_0.

Remark 3.4. T_i implies SP-$T_i (i = -1, 0)$, but inverses are not true, sub-T_0 implies sub-SP-T_0, but inverse is not true, We only give an example that explain an L-space is SP-T_1, but it is not T_1-space:

Example 3.5. Let $X = \{x_1, x_2\}$, $L = \{0, a, b, c, d, 1\}$, where $a' = a, b' = b, c' = d, d' = c, 1' = 0, 0' = 1$; $0 < d < a < c < 1, 0 < d < b < c < 1$, a and b are incomparable. $\forall \lambda, \mu \in L$ we define fuzzy set $C(\lambda, \mu) : X \to L$ such that

$$C(\lambda, \mu)(x) = \begin{cases} \lambda, & \text{if} \quad x = x_1, \\ \mu, & \text{if} \quad x = x_2. \end{cases}$$

Let (X, τ) be an L-topological space, where

$$\tau = \{C(0, 0), C(a, c), C(b, d), C(d, d), C(c, c), C(1, 1)\}$$

and Let Φ be the set of all strongly preclosed L-sets, then

$$\Phi = \{C(0, 0), C(a, d), C(b, c), C(c, c), C(d, a), C(d, b), C(d, c), C(d, d), C(1, 1)\}.$$

We also know that

$$M(L^X) = \{C(0, a), C(0, b), C(0, c), C(0, d), C(0, 1), C(a, 0), C(b, 0), C(c, 0), C(d, 0), (1, 0)\}.$$

So that we take any $x_\lambda, x_\mu \in M(L^X), \mu < \lambda$, there exists $P \in \eta_{sp}(x_\lambda)$ such that $x_\mu \leq P$. Hence (X, τ) is a SP-T_{-1} space, but we can prove that (X, τ) is not T_{-1}, in fact, we take molecules $C(0, a), C(0, c)$, strongly preclosed remote set of $C(0, c)$ are only $C(a, d), C(d, d)$, but $C(0, a) \not\leq C(a, d), C(0, a) \not\leq C(d, d)$. This implies that (X, τ) is not T_{-1}.

Theorem 3.6. Let (X, τ) be an L-topological space, Then it is SP-T_{-1} if and only if for any $x_\lambda \in M(L^X)$, x_λ is a component of $cl_{sp}(x_\lambda)$.

Proof. Let $x_\lambda \in M(L^X)$, x_λ is not a component of $cl_{sp}(x_\lambda)$, then there exists a $x_\mu \in M(L^X)$ such that $x_\lambda < x_\mu \leq cl_{sp}(x_\lambda)$. Let $P \in \eta_{sp}(x_\mu)$, then $x_\lambda \not\leq P$, because if $x_\lambda \leq P$, then $cl_{sp}(x_\lambda) \leq P$, so $x_\mu \leq P$, a contradiction! Hence (X, τ) is not SP-T_{-1}.

Conversely, Suppose that (X, τ) is not SP-T_{-1}, then there exist $\lambda, \mu \in M(L)$ and $x \in X, \lambda < \mu$, for any $P \in \eta_{sp}(x_\mu)$, $x_\lambda \leq P$. By $x_\lambda \leq cl_{sp}(x_\lambda)$ we have $x_\mu \leq cl_{sp}(x_\lambda)$, because if $x_\mu \not\leq cl_{sp}(x_\lambda)$, then $cl_{sp}(x_\lambda) \in \eta_{sp}(x_\mu)$, so $x_\lambda \not\leq cl_{sp}(x_\lambda)$, a contradiction. Hence x_λ is not a component of $cl_{sp}(x_\lambda)$. ■

Theorem 3.7. Let (X, τ) be an L-topological space, where $L = [0, 1]$. Then (X, τ) is SP-T_{-1} if and only if for each fuzzy point x_λ, x_λ is a component of a certain strongly preopen L-set.

Proof. Let (X, τ) be SP-T_{-1} and $x_\lambda \in M(L^X)$. If $\lambda = 1$, then x_λ is a component of the strongly preopen L-set $\underline{1}$. If $\lambda \neq 1$, then $x_{1-\lambda}$ is a component of $cl_{sp}(x_{1-\lambda})$ by Theorem 3.6. Hence $x_\lambda = x_{1-(1-\lambda)}$ is a component of strongly preopen L-set $(cl_{sp}(x_{1-\lambda}))'$.

Conversely, suppose that (X, τ) is not SP-T_{-1}, then there exists $x_\lambda \in M(L^X)$ such that x_λ is not a component of $cl_{sp}(x_\lambda)$. Since $cl_{sp}(x_\lambda)$ is intersection of all strongly preclosed L-sets containing x_λ, it is impossible for x_λ to be a component of any strongly preclosed L-set. In fact, if x_λ is a component of certain strongly preclosed L-set A, then $x_\lambda \leq A$, so that $A \in \{B \mid B \geq x_\lambda, B \text{ is strongly preclosed}\}$, hence $A \geq \bigwedge\{B \mid B \geq x_\lambda, B \text{ is strongly preclosed}\}$, i.e., $A \geq cl_{sp}(x_\lambda) \geq x_\lambda$, this contradicts that x_λ is a component of A. Thus x_λ is not a component of any strongly preclosed L-set. This implies $\lambda \neq 1$. Hence there exists a $x_{1-\lambda}$, it is not a component of any strongly preopen L-set. A contradiction. ■

Theorem 3.8. Let (X, τ) be an L-topological space, where $L = [0, 1]$. Then it is SP-T_{-1} if and only if for fuzzy point x_λ, $x_\lambda \leq \bigvee \eta_{sp}(x_\lambda)$.

Proof. Suppose that for any $x_\lambda \in M(L^X)$, $x_\lambda \leq \bigvee \eta_{sp}(x_\lambda)$ and $\mu \in [0, 1]$, $\mu < \lambda$. Put

$$\eta_{sp_1}(x_\lambda) = \{P : P \wedge a \neq \underline{0}, P \in \eta_{sp}(x_\lambda)\}$$

and $x_\lambda^P = x_\lambda \wedge P$, where $P \in \eta_{sp_1}(x_\lambda)$. Then $x_\lambda \leq \bigvee \eta_{sp_1}(x_\lambda)$, $x_\lambda^P \in M(L^X)$ and

$$x_\lambda = x_\lambda \wedge (\bigvee \eta_{sp_1}(x_\lambda)) = \bigvee\{x_\lambda^P : P \in \eta_{sp_1}(x_\lambda)\}$$

Since $x_\lambda^P \leq a$ and $\mu \leq \lambda$, $x_\lambda^P \wedge P \neq \underline{0}$. If for any $x_\lambda^P \in M(L^X)$, $x_\lambda^P < x_\mu$, then $x_\lambda \leq x_\mu$. This is a contradiction. Hence there exists $\eta_{sp_1}(x_\lambda)$ such that $x_\mu \leq x_\lambda^P$ and $x_\mu \leq P$, so that (X, τ) is SP-T_1.

Conversely, suppose that (X, τ) is SP-T_{-1}, $x_\lambda \in M(L^X)$. Put $M_1 = \{x_\mu : \mu < \lambda\}$, then $x_\lambda = \bigvee M_1$. Since (X, τ) is SP-T_{-1}, for any $x_\mu \in M_1$, there exists a $P_\mu \in \eta_{sp}(x_\lambda)$ such that $x_\mu \leq P_\mu$. Hence $x_\lambda \leq \bigvee\{P_\mu : x_\mu \in M_1\}$ and so $x_\lambda \leq \bigvee \eta_{sp}(x_\lambda)$. ■

Theorem 3.9. Let (X, τ) be an L-topological space. Then

(1) (X, τ) is SP-T_0 if and only if for any two distinct molecules x_λ, y_μ, we have $\eta_{sp}(x_\lambda) \neq \eta_{sp}(y_\mu)$;

(2) (X, τ) is SP-T_0 if and only if for any two distinct molecules x_λ, y_μ, we have $x_\lambda \not\leq cl_{sp}(y_\mu)$ or $y_\mu \not\leq cl_{sp}(x_\lambda)$.

Proof. (1) Suppose that (X, τ) is not SP-T_0, then there exist $x_\lambda, y_\mu \in M(L^X)$, $x_\lambda \neq y_\mu$ such that for any $P \in \eta_{sp}(x_\lambda)$, $y_\mu \not\leq P$, so $P \in \eta_{sp}(y_\mu)$, on the other hand for any $Q \in \eta_{sp}(y_\mu)$, $x_\lambda \not\leq Q$, so $Q \in \eta_{sp}(x_\lambda)$, i.e., $\eta_{sp}(x_\lambda) = \eta_{sp}(y_\mu)$. Conversely, Suppose that (X, τ) is SP-T_0, then for any $x_\lambda \neq y_\mu$ in $M(L^X)$, there exists $P \in \eta_{sp}(x_\lambda)$ such that $y_\mu \leq P$, so $P \notin \eta_{sp}(y_\mu)$ or $Q \in \eta_{sp}(y_\mu)$ such that $x_\lambda \leq Q$, so $Q \notin \eta_{sp}(x_\lambda)$. Hence $\eta_{sp}(x_\lambda) \neq \eta_{sp}(y_\mu)$.

(2) Suppose that (X, τ) is not SP-T_0, then by (1), we know that there exist $x_\lambda, y_\mu \in M(L^X)$, $x_\lambda \neq y_\mu$ such that $\eta_{sp}(x_\lambda) = \eta_{sp}(y_\mu)$, hence $x_\lambda \leq cl_{sp}(y_\mu)$ and $y_\mu \leq cl_{sp}(x_\lambda)$, because if $x_\lambda \not\leq cl_{sp}(y_\mu)$ or $y_\mu \not\leq cl_{sp}(x_\lambda)$, then $cl_{sp}(y_\mu) \in \eta_{sp}(x_\lambda) = \eta_{sp}(y_\mu)$, or $cl_{sp}(x_\lambda) \in \eta_{sp}(y_\mu) = \eta_{sp}(x_\lambda)$, those are contradictions.

Conversely, suppose that (X, τ) is SP-T_0. Then for any $x_\lambda \neq y_\mu$ in $M(L^X)$, there exists $P \in \eta_{sp}(x_\lambda)$ such that $y_\mu \leq P$, so $cl_{sp}(y_\mu) \leq P$. Hence $x_\lambda \not\leq cl_{sp}(y_\mu)$. ■

Similarly we can prove following theorem.

Theorem 3.10. Let (X, τ) be an L-topological space. Then

(1) (X, τ) is sub-SP-T_0 if and only if for any two distinct crisp points $x, y \in X$, there exists such that $\eta_{sp}(x_\lambda) \neq \eta_{sp}(y_\lambda)$;

(2) (X, τ) is sub-SP-T_0 if and only if for any two distinct crisp points x, y, there exists $\lambda \in M(L)$ such that $x_\lambda \not\leq cl_{sp}(y_\lambda)$ or $y_\lambda \not\leq cl_{sp}(x_\lambda)$.

The following Theorem is obvious:

Theorem 3.11. SP-T_{-1}, SP-T_0 and sub-SP-T_0 separability are hereditary.

Theorem 3.12. Let $f : (X, \tau) \to (Y, \mu)$ be a bijection and a SP-irresolute closed mapping. If (X, τ) is SP-$T_i (i = -1, 0)$, then (Y, μ) is SP-$T_i (i = -1, 0)$ too.

Proof. We only prove $i = 0$.

Let (X, τ) be a SP-T_0, for any $x_\lambda \neq y_\mu \in M(L^Y)$, then $f_L^{\leftarrow}(x_\lambda), f_L^{\leftarrow}(y_\mu) \in M(L^X)$ and $f_L^{\leftarrow}(x_\lambda) \neq f_L^{\leftarrow}(y_\mu)$. Since$(X, \tau)$ is a SP-T_0, there exists a $P \in \eta_{sp}(f_L^{\leftarrow}(x_\lambda))$

such that $f_L^{\leftarrow}(y_\mu) \leq P$ or there exists a $Q \in \eta_{sp}(f_L^{\leftarrow}(y_\mu))$ such that $f_L^{\leftarrow}(x_\lambda) \leq Q$. So that $x_\lambda \not\leq f_L^{\rightarrow}(P)$ and $y_\mu \leq f_L^{\rightarrow}(P)$ or $y_\mu \not\leq f_L^{\rightarrow}(P)$ and $x_\lambda \leq f_L^{\rightarrow}(P)$. This implies that there exists a $f_L^{\rightarrow}(P) \in \eta_{sp}^{-}(x_\lambda)$ such that $y_\mu \leq f_L^{\rightarrow}(P)$ or there exists a $f_L^{\rightarrow}(Q) \in \eta_{sp}^{-}(y_\mu)$ such that $x_\lambda \leq f_L^{\rightarrow}(Q)$ from $f_L^{\rightarrow}$ is SP-irresolute closed mapping. ■

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