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On Some Results in L^2 (m) Space

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Abstract

This article is an extension of the classical theory of measures and integrals. The concept of fuzzy m-integral of complex valued B-functions introduced in [6], using Butnariu approach, is reviewed. We initiate a new outlook for studying various properties of the spaces L^1(m) and L^2(m) of complex valued B-functions.

Keywords σ–algebra of fuzzy sets, fuzzy measure, Complex-valued B-function, integrals of a complex B-function, L^1(m) space, L^2(m) space

1. Introduction

Following the introduction of fuzzy set theory by Zadeh [1965], fuzzy measures and fuzzy integrals were introduced by Sugeno. The notion of an 'additive fuzzy measure' was introduced in [1] to deal with the concept of additive fuzzy integral for extended real valued functions [2]. In this paper, we introduce the spaces L^1 (m) and L^2 (m) of complex valued B-functions and study some of their properties.

2. Fuzzy Preliminaries

The following fuzzy preliminaries are taken from [1-4] which gives a brief outline of Butnariu fuzzy measure.

Let X be a nonempty set, ***C*** be a σ–algebra of fuzzy sets in X and F be the class of all real valued B-functions on the Borel space (X, ***C***) and let **R*** denote **R** $\cup$ {±∞} and let $R_+^* = [\,0, +\infty]$.

Definition 2.1 A set function m: ***C*** → R_+^* is called a fuzzy measure if it satisfies the following properties:

1. m(Φ) = 0. (vanishing at Φ)
2. If $(A_n)_{n\in N}$ is a sequence in ***C***, then m ($\oplus_{n\in N} A_n) = \sum_1^\infty m(A_n)$ (σ-additivity)

Then, (X, ***C***, m) is called a measure space.

Definition 2.2 A measure space is a triple *M* = (X, ***C***, m) where *B* = (X, ***C***) is a *Borel space* and m is a *fuzzy measure* on ***C***.

Definition 2.3 Let f: X $\rightarrow$ R^*_+ = (0, +∞] be a function. We say that f is a *Borel function* w.r.t *B* or *B-function* iff there exists a non-negative increasing sequence of simple B-functions which converges to f.

Definition 2.4 [4] Let f: X $\rightarrow$ R^*_+ = (0, +∞] be a B-function and A be an element of a σ-algebra ***C.*** If $(s_n)_{n\in N}$ is a sequence in B^+ (f, A) so that $s_{n+1} \geq_A s_n$ for all n ∈N and $\lim_{n\in N} s_n$ (x).A(x) = f(x). A(x) (for all x ∈X) holds, then we call it m-integral of f over A and denote it by $\int_A f\, dm = \lim_{n\in N} \int_A s_n\, dm$.
If $\int_A f$ dm is finite, then f is said to be m-*integrable* over A.
In this section we extend the concept of B-functions and m-integrals for complex-valued functions also.

Definition 2.5 [6] If (X,***C*****,** m) is a measure space and f be a complex-valued function defined on X so that f = Re f + i Im f. Then f is said to be a complex-valued Borel function or simply complex B-function on the Borel space (X, ***C***) if both Re f and Im f are real-valued B-functions.

Definition 2.6 [6] If m is a fuzzy measure on ***C***, f is a complex B-function on X and A ∈ ***C***. Then, f is said to be fuzzy integrable w.r.t m (m-integrable) over A iff f is a complex B-function and $\int_A |f|\, dm < +\infty$ and we define $\int_A f\, dm = \int_A \mathrm{Re}\, f\, dm + i \int_A \mathrm{Im}\, f\, dm$, provided $\int_A \mathrm{Re}\, f\, dm$ and $\int_A \mathrm{Im}\, f\, dm$ are both finite.When A = X, we write $\int f\, dm$ instead of $\int_X f\, dm$.

Proposition 2.7 [6] f is integrable if and only if | f | is integrable.

Proposition 2.8[6] Let f be a complex B-function. If f is m-integrable on A then c f is also m-integrable on A and $\int_A c f\, dm = c \int_A f\, dm$.

Proposition 2.9 [6] If f is m-integrable on A and B ⊂ A, then f is m-integrable on B also.
Follows from definition

3. The Spaces $L^1(m)$ & $L^2(m)$

In this section, we introduce the spaces $L^1(m)$ & $L^2(m)$

Definition 3.1 $L^1(m)$ is defined as the collection of all complex B-functions f on X for which $\int |f|\, dm < +\infty$. If f is a complex B-function then | f | is also a B-function and hence the above integral is well defined.The members of $L^1(m)$ are called m-integrable functions.

Theorem 3.2 Suppose f and g $\in L^1(m)$ and a,b $\in$ **C** then a f + b g $\in L^1(m)$ and $\int (a f + b g)\, dm = a \int f\, dm + b \int g\, dm$.

Definition 3.3 Let (X, ***C***,m) be a measure space. We define the space $L^2(m)$ or simply L^2 as the collection of all complex B-functions f such that $\int | f |^2\, dm < +\infty$.

Definition 3.4 For f $\in L^2(m)$ we define fuzzy norm of f as $\| f \| = \{ \int_A | f |^2\, dm \}^{1/2}$
It follows that for any complex number c, $\| cf \| = | c | \, \| f \|$, f $\in L^2(m)$

Theorem 3.5 $L^2(m)$ forms a fuzzy linear space over the complex field.

Theorem 3.6 If f, g $\in L^2(m)$ then f g $\in L^1(m)$ and $| f g | \leq \| f \| \, \| g \|$.

Theorem 3.7 If f, g $\in L^2(m)$ then f + g $\in L^2(m)$ and $\| f + g \| \leq \| f \| + \| g \|$.

Definition 3.8 A fuzzy semi norm on a real or complex vector space L is a real valued function p with the following properties:

- $p(f) \geq 0$
- $p(cf) = c\, p(f)$, for any complex number c
- $p(f + g) \leq p(f) + p(g)$, f and g are arbitrary elements of L.

If p is a fuzzy semi norm with the additional property that p(f) = 0 implies f = 0, then p is called a fuzzy norm.
Now $\|.\|$ is a a semi norm on $L^2(m)$.
If $\|.\|$ is a fuzzy semi norm on a vector space,we have the notion of distance
$d(f, g) = \| f-g \|$. By the definition of semi norm, we have,
$d(f,g) \geq 0$
$d(f,g) = 0$ if f = g
$d(f,g) = d(f,g)$
$d(f,g) \leq d(f,h) + d(h,g)$.

Theorem 3.9 If f is m-integrable then $| \int f\, dm | \leq \int | f |\, dm$.

Proof:-
If $\int f\, dm = r e^{i\theta}$, $r \geq 0$, then $\int e^{-i\theta} f\, dm = r = | \int f\, dm |$.
But if $f(\omega) = \rho(\omega) e^{i\varphi(\omega)}$ (taking $\rho \geq 0$) then,

$$\int e^{-i\theta} f\, dm = \int e^{i(\varphi-\theta)}\, dm = \int \rho \cos(\varphi-\theta)\, dm \leq \int \rho\, dm = \int | f |\, dm.$$

Theorem 3.10 Suppose f_n: X → R*, (n ∈ N) is a sequence of complex B-functions defined on X such that $\Sigma_{n\in N} \int | f_n | dm < +\infty$ …(1) Then the series $f(x) = \Sigma_{n\in N} f_n(x)$ …(2) converges to $f \in L^1(m)$ and
$\int_X f\, dm = \lim_{n\in N} \int_X f_n\, dm$ …(3).

If ||. || is a norm then d is actually a metric.

REFERENCES

1. D. Butnariu, Additive Fuzzy Measures and Integrals I, Journal of Mathematical analysis and applications 93,(1983) 436-452
2. D. Butnariu, Additive Fuzzy Measures and Integrals II, Journal of Mathematical analysis and applications
3. D. Butnariu, Additive Fuzzy Measures and Integrals III,Journal of Mathematical analysis and applications 125,,(1983) 288-303
4. D. Butnariu, Fuzzy Measurability and Integrability, Journal of Mathematical analysis and applications117,(1986) 385-410 (1986)
5. A. George and P.V Veeramani, On some results in fuzzy metric spaces. Fuzzy sets and systems 64 (1994), 395-399.
6. Mathews M. George, Fuzzy Integrals of Complex-valued Functions, The journal of Fuzzy Mathematics, LA, USA (accepted for publication)
7. Robert B. Ash and Catherine A. Doleans-Dade, Probability & Measure Theory, Second edition, Elsevier N.D, India.
8. W.Rudin, Real and Complex analysis, McGraw-Hill, Singapore 1987
9. W.Rudin, Principles of Mathematical Analysis, Third edition, International Students Edition, Singapore 1984
10. Z. Wang, G.J.Klir, Fuzzy Measure Theory, Plenum, New York 1992