Stability of a Quadratic Functional Equation

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Abstract

This paper deals with the Ulam-Hyers stability of a quadratic functional equation

\[ q\left(\frac{x - y + z}{2}\right) = \frac{1}{2} (q(x - z) + q(x - y)) - \frac{1}{4} q(z - y) \]

using direct and fixed point methods in fuzzy normed space.

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1. INTRODUCTION

One of the most interesting questions in the theory of functional analysis concerning the Ulam stability problem of functional equations is as follows: when is it true that a mapping satisfying a functional equation approximately must be close to an exact solution of the given functional equation?

The functional equation
\[ f(x + y) + f(x - y) = 2f(x) + 2f(y) \] (1.1)

is said to be quadratic functional equation because the quadratic function \( f(x) = ax^2 \) is a solution of the functional equation (1.1).

This paper established the Ulam-Hyers stability of a quadratic functional equation
\[ q \left( x - \frac{y + z}{2} \right) = \frac{1}{2} (q(x - z) + q(x - y)) - \frac{1}{4} q(z - y) \] (1.2)

using the direct and fixed point methods in fuzzy normed space.

2. PRELIMINARIES

A.K. Katsaras [17] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [11, 19, 35]. In particular, T. Bag and S.K. Samanta [6], following S.C. Cheng and J.N. Mordeson [8], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [18]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [7].

We use the definition of fuzzy normed spaces given in [6] and [22, 23, 24, 25].

**Definition 2.1.** Let \( X \) be a real linear space. A function \( N : X \times \mathbb{R} \rightarrow [0, 1] \)(the so-called fuzzy subset) is said to be a fuzzy norm on \( X \) if for all \( x, y \in X \) and all \( s, t \in \mathbb{R} \),

(F1) \( N(x, c) = 0 \) for \( c \leq 0 \);
(F2) \( x = 0 \) if and only if \( N(x, c) = 1 \) for all \( c > 0 \);
(F3) \( N(cx, t) = N \left( x, \frac{t}{|c|} \right) \) if \( c \neq 0 \);
(F4) \( N(x + y, s + t) \geq \min \{ N(x, s), N(y, t) \} \);
(F5) $N(x, \cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t \to \infty} N(x, t) = 1$;
(F6) for $x \neq 0$, $N(x, \cdot)$ is (upper semi) continuous on $\mathbb{R}$.

The pair $(X, N)$ is called a fuzzy normed linear space. One may regard $N(X, t)$ as the truth-value of the statement the norm of $x$ is less than or equal to the real number $t$.

Example 2.2. Let $(X, \| \cdot \|)$ be a normed linear space. Then

$$N(x, t) = \begin{cases} \frac{t}{t + \|x\|}, & t > 0, \ x \in X, \\ 0, & t \leq 0, \ x \in X \end{cases}$$

is a fuzzy norm on $X$.

Definition 2.3. Let $(X, N)$ be a fuzzy normed linear space. Let $x_n$ be a sequence in $X$. Then $x_n$ is said to be convergent if there exists $x \in X$ such that $\lim_{n \to \infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, $x$ is called the limit of the sequence $x_n$ and we denote it by $N \lim_{n \to \infty} x_n = x$.

Definition 2.4. A sequence $x_n$ in $X$ is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists $n_0$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.

Definition 2.5. Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

Definition 2.6. A mapping $f : X \to Y$ between fuzzy normed spaces $X$ and $Y$ is continuous at a point $x_0$ if for each sequence $\{x_n\}$ covering to $x_0$ in $X$, the sequence $f\{x_n\}$ converges to $f(x_0)$ . If $f$ is continuous at each point of $x_0 \in X$ then $f$ is said to be continuous on $X$.

The stability of various functional equations in fuzzy normed spaces were investigated in [3, 4, 15, 21, 22, 23, 24, 25, 29, 32].

Hereafter throughout this paper, assume that $X, (Z, N')$ and $(Y, N')$ are linear space, fuzzy normed space and fuzzy Banach space, respectively. We use the following abbreviation for a given function $f : X \to Y$ by

$$D_q(x, y, z) = q \left( x - y + \frac{z}{2} \right) - \frac{1}{2} (q(x - z) + q(x - y)) + \frac{1}{4} q(z - y)$$

for all $x, y, z \in X$. 
3. FUZZY STABILITY RESULTS: DIRECT METHOD

Now, we investigate the generalized Ulam-Hyers stability of the functional equation (1.2) in fuzzy normed space using direct method.

**Theorem 3.1.** Let \( \varsigma \in \{-1, 0, 1\} \) be fixed and let \( \vartheta : X^3 \to Z \) be a mapping with \( 0 < \left( \frac{d}{4} \right)^\varsigma < 1 \)

\[
N(\vartheta(2^\varsigma x, 2^\varsigma y, 2^\varsigma z), r) \geq N(\vartheta(x, y, z), r) \tag{3.1}
\]

for all \( x, y, z \in X \) and all \( d > 0 \) and

\[
\lim_{n \to \infty} N'(\vartheta(2^{c_n} x, 2^{c_n} y, 2^{c_n} z), 4^{c_n} r) = 1 \tag{3.2}
\]

for all \( x, y, z \in X \) and all \( r > 0 \). Suppose that a mapping \( q : X \to Y \) satisfies the inequality

\[
N(D_q(x, y, z), r) \geq N'(\vartheta(x, y, z), r) \tag{3.3}
\]

for all \( x, y, z \in X \) and all \( r > 0 \). Then the limit

\[
Q(z) = N - \lim_{n \to \infty} \frac{q(2^{nc} z)}{4^{nc}} \tag{3.4}
\]

exists for all \( z \in X \) and all \( r > 0 \) and the mapping \( Q : X \to Y \) is a unique quadratic mapping satisfying (1.2) and

\[
N(q(z) - Q(z), r) \geq N'(\vartheta(0, -z, z), r|4 - d|) \tag{3.5}
\]

for all \( z \in X \) and all \( r > 0 \).

**Proof.** First assume \( \varsigma = 1 \). Replacing \((x, y, z)\) by \((0, -z, z)\) in (3.3), we get

\[
N(q(2z) - 4q(z), r) \geq N'(\vartheta(0, -z, z), r) \tag{3.6}
\]

for all \( z \in X \) and all \( r > 0 \). Replacing \( z \) by \( 2^n z \) in (3.6), we obtain

\[
N\left(\frac{q(2^{n+1} z)}{2^2} - q(2^n z), \frac{r}{2^2}\right) \geq N'(\vartheta(0, -2^n z, 2^n z), r) \tag{3.7}
\]

for all \( z \in X \) and all \( r > 0 \). Using (3.1), \((F3)\) in (3.7), we arrive

\[
N\left(\frac{q(2^{n+1} z)}{2^2} - q(2^n z), \frac{r}{4}\right) \geq N'(\vartheta(0, -z, z), \frac{r}{d^n}) \tag{3.8}
\]
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for all \( z \in X \) and all \( r > 0 \). It is easy to verify from (3.8), that
\[
N \left( \frac{q(n+1)^2}{2^{2(n+1)}} - \frac{q(n)^2}{2^{2n}} - \frac{r}{2^2} \right) \geq N' \left( \vartheta(0, -z, z) \right)
\]
(3.9)
holds for all \( z \in X \) and all \( r > 0 \). Replacing \( r \) by \( d^n r \) in (3.9), we get
\[
N \left( \frac{q(n+1)^2}{2^{2(n+1)}} - \frac{q(n)^2}{2^{2n}} - \frac{d^n r}{2^{2(n+1)}} \right) \geq N' \left( \vartheta(0, -z, z), r \right)
\]
for all \( z \in X \) and all \( r > 0 \). It is easy to see that
\[
\frac{q(n)^2}{2^{2n}} - q(z) = \sum_{i=0}^{n-1} \left[ \frac{q(2i+1)^2}{2^{2(i+1)}} - \frac{q(2i)^2}{2^{2i}} \right]
\]
(3.11)
for all \( z \in X \). From equations (3.10) and (3.11), we have
\[
N \left( \frac{q(n)^2}{2^{2n}} - q(z) \right) \geq \min \left\{ \frac{q(2i+1)^2}{2^{2(i+1)}} - \frac{q(2i)^2}{2^{2i}} : i \in \mathbb{N} \right\}
\]
\[
\geq \min \left\{ N' \left( \vartheta(0, -z, z), r \right) \right\}
\]
(3.12)
for all \( z \in X \) and all \( r > 0 \). Replacing \( z \) by \( 2^m z \) in (3.12) and using (3.1), (F3), we obtain
\[
N \left( \frac{q(2n+m)^2}{2^{2(n+m)}} - \frac{q(2m)^2}{2^{2m}} \right) \geq N' \left( \vartheta(0, -z, z), \frac{r}{d^m} \right)
\]
(3.13)
for all \( z \in X \) and all \( r > 0 \) and all \( m, n \geq 0 \). Replacing \( r \) by \( d^m r \) in (3.13), we get
\[
N \left( \frac{q(2n+m)^2}{2^{2(n+m)}} - \frac{q(2m)^2}{2^{2m}} \right) \geq N' \left( \vartheta(0, -z, z), \frac{d^m r}{2^{2i}} \right)
\]
(3.14)
for all \( z \in X \) and all \( r > 0 \) and all \( m, n \geq 0 \). Using (F3) in (3.14), we obtain
\[
N \left( \frac{q(2n+m)^2}{2^{2(n+m)}} - \frac{q(2m)^2}{2^{2m}} \right) \geq N' \left( \vartheta(0, -z, z), \sum_{i=m}^{m+n-1} \frac{r}{2^{2i}} \right)
\]
(3.15)
for all \( z \in X \) and all \( r > 0 \) and all \( m, n \geq 0 \). Since \( 0 < d < 2^2 \) and \( \sum_{i=0}^{n} \left( \frac{d}{2^2} \right)^i < \infty \),
the cauchy criterion for convergence and (F5) implies that \( \left\{ \frac{q(2^n z)}{2^{2n}} \right\} \) is a Cauchy
sequence in \((Y, N)\). Since \((Y, N)\) is a fuzzy Banach space, this sequence converges to some point \(Q(z) \in Y\). So one can define the mapping \(Q : X \rightarrow Y\) by

\[
Q(z) = N - \lim_{n \to \infty} \frac{q(2^n z)}{2^n}
\]

for all \(z \in X\). Letting \(m = 0\) in (3.15), we get

\[
N \left( \frac{q(2^n z)}{2^n} - q(z), r \right) \geq N' \left( \vartheta(0, -z, z), \frac{r}{\sum_{i=0}^{n-1} d_i} \right)
\]

for all \(z \in X\) and all \(r > 0\). Letting \(n \to \infty\) in (3.16) and using (F6), we arrive

\[
N \left( q(z) - Q(z), r \right) \geq N' \left( \vartheta(0, -z, z), r(2^a - d) \right)
\]

for all \(z \in X\) and all \(r > 0\). To prove \(Q\) satisfies the functional equation (1.2), replacing \((x, y, z)\) by \((2^n x, 2^n y, 2^n z)\) in (3.3), respectively, we obtain

\[
N \left( \frac{1}{2^n} D_q(2^n x, 2^n y, 2^n z), r \right) \geq N' \left( \vartheta(2^n x, 2^n y, 2^n z), 2^{2n} r \right)
\]

for all \(r > 0\) and all \(x, y, z \in X\). Now,

\[
N \left( \vartheta(x - \frac{y + z}{2}) - \frac{1}{2} (Q(x - z) + Q(x - y)) + \frac{1}{4} Q(z - y), r \right)
\]

\[
\geq \min \left\{ N \left( \vartheta(x - \frac{y + z}{2}) - \frac{1}{2^{2n} q} \left( 2^n \left( x - \frac{y + z}{2} \right) \right), \frac{r}{5} \right), \right.
\]

\[
N \left( -\frac{1}{2} Q(x - z) + \frac{1}{2^{2n} q} \left( 2^n (x - z) \right), \frac{r}{5} \right), \right.
\]

\[
N \left( -\frac{1}{2} Q(x - y) + \frac{1}{2^{2n} q} \left( 2^n (x - y) \right), \frac{r}{5} \right), \right.
\]

\[
N \left( \frac{1}{4} Q(z - y) - \frac{1}{2^{2n} q} \left( 2^n (z - y) \right), \frac{r}{5} \right), \right.
\]

\[
N \left( \frac{1}{2^{2n} q} \left( 2^n \left( x - \frac{y + z}{2} \right) \right) - \frac{1}{2^{2n} q} \left( 2^n (x - z) \right), \frac{r}{5} \right), \right.
\]

\[
-\frac{1}{2^{2n} q} \left( 2^n (x - y) \right) + \frac{1}{2^{2n} q} \left( 2^n (z - y) \right), \frac{r}{5} \right) \right\}
\]

(3.18)

for all \(x, y, z \in X\) and all \(r > 0\). Using (3.17) and (F5) in (3.18), we arrive

\[
N \left( \vartheta(x - \frac{y + z}{2}) - \frac{1}{2} (Q(x - z) + Q(x - y)) + \frac{1}{4} Q(z - y), r \right)
\]

\[
\geq \min \left\{ 1, 1, 1, 1, N' \left( \vartheta(2^n x, 2^n y, 2^n z), 2^{2n} r \right) \right\}
\]

\[
\geq N' \left( \vartheta(2^n x, 2^n y, 2^n z), 2^{2n} r \right)
\]

(3.19)
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for all \(x, y, z \in X\) and all \(r > 0\). Letting \(n \to \infty\) in (3.19) and using (3.2), we see that
\[
N \left( Q \left( x - \frac{y + z}{2} \right) - \frac{1}{2} (Q(x - z) + Q(x - y)) + \frac{1}{4} Q(z - y), r \right) = 1 \tag{3.20}
\]
for all \(x, y, z \in X\) and all \(r > 0\). Using \((F2)\) in the above inequality gives
\[
Q \left( x - \frac{y + z}{2} \right) = \frac{1}{2} (Q(x - z) + Q(x - y)) - \frac{1}{4} Q(z - y)
\]
for all \(x, y, z \in X\). Hence, \(Q\) satisfies the quadratic functional equation (1.2). In order to prove \(Q\) is unique, let \(Q'\) be another quadratic functional equation satisfying (1.2) and (3.5). Hence,
\[
N (Q(z) - Q'(z), r) = N \left( \frac{Q(2^n z)}{2^{2n}} - \frac{Q'(2^n z)}{2^{2n}}, r \right)
\geq \min \left\{ N \left( \frac{Q(2^n z)}{2^{2n}} - \frac{q(2^n z)}{2^{2n}}, \frac{r}{2} \right), N \left( \frac{q(2^n z)}{2^{2n}} - \frac{Q'(2^n z)}{2^{2n}}, \frac{r}{2} \right) \right\}
\geq N' \left( \theta(0, -2^n z, 2^n z), r 2^{2n}(2^2 - d) \right)
\geq N' \left( \theta(0, -z, z), r 2^{2n}(2^2 - d) \right)
\]
for all \(z \in X\) and all \(r > 0\). Since
\[
\lim_{n \to \infty} \frac{r 2^{2n}(2^2 - d)}{2d^n} = \infty,
\]
we obtain
\[
\lim_{n \to \infty} N' \left( \theta(0, -z, z), \frac{r 2^{2n}(2^2 - d)}{2d^n} \right) = 1.
\]
Thus
\[
N (Q(z) - Q'(z), r) = 1
\]
for all \(z \in X\) and all \(r > 0\), hence \(Q(z) = Q'(z)\). Therefore \(Q(z)\) is unique.

For \(\varsigma = -1\), we can prove the result by a similar method. This completes the proof of the theorem.

From Theorem 3.1, we obtain the following corollaries concerning the Ulam-Hyers stability for the functional equation (1.2).

**Corollary 3.2.** Suppose that a mapping \(q : X \to Y\) satisfies the inequality
\[
N (D_q(x, y, z), r)
\geq \begin{cases} 
N' (\epsilon, r), \\
N' (\epsilon \{ ||x||^s + ||y||^s + ||z||^s \}, r), & s \neq 2; \\
N' (\epsilon \{ ||x||^s ||y||^s ||z||^s + (||x||^{3s} + ||y||^{3s} + ||z||^{3s}) \}, r), & s \neq \frac{2}{3};
\end{cases} \tag{3.21}
\]
for all $x, y, z \in X$ and all $r > 0$, where $\epsilon, s$ are constants. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$N(q(z) - Q(z), r) \geq \begin{cases} N'(\epsilon, 3r), \\ N'(2\epsilon ||z||^s, r|2^s - 2^s|), \\ N'(2\epsilon ||z||^{3s}, r|2^s - 2^{3s}|) \end{cases}$$

(3.22)

for all $z \in X$ and all $r > 0$.

4. FUZZY STABILITY RESULTS: FIXED POINT METHOD

In this section, the authors present the generalized Ulam-Hyers stability of the functional equation (1.2) in fuzzy normed space using fixed point method. Now we will recall the fundamental results in fixed point theory.

**Theorem 4.1.** (Banach’s contraction principle) Let $(X, d)$ be a complete metric space and consider a mapping $T : X \to X$ which is strictly contractive mapping, that is

(A1) $d(Tx, Ty) \leq Ld(x, y)$ for some (Lipschitz constant) $L < 1$. Then,

(i) The mapping $T$ has one and only fixed point $x^* = T(x^*)$;

(ii) The fixed point for each given element $x^*$ is globally attractive, that is

(A2) $\lim_{n \to \infty} T^n x = x^*$, for any starting point $x \in X$;

(iii) One has the following estimation inequalities:

(A3) $d(T^n x, x^*) \leq \frac{1}{1-L} d(T^n x, T^{n+1} x), \forall \ n \geq 0, \forall \ x \in X$;

(A4) $d(x, x^*) \leq \frac{1}{1-L} d(x, x^*), \forall \ x \in X$.

**Theorem 4.2.** [20](The alternative of fixed point) Suppose that for a complete generalized metric space $(X, d)$ and a strictly contractive mapping $T : X \to X$ with Lipschitz constant $L$. Then, for each given element $x \in X$, either

(B1) $d(T^n x, T^{n+1} x) = \infty$ $\forall \ n \geq 0$,

or

(B2) there exists a natural number $n_0$ such that:

(i) $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$ ;

(ii) The sequence $(T^n x)$ is convergent to a fixed point $y^*$ of $T$

(iii) $y^*$ is the unique fixed point of $T$ in the set $Y = \{y \in X : d(T^{n_0} x, y) < \infty\}$;

(iv) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in Y$. 

In order to prove the stability results we define the following: 
\[ \delta_i = \begin{cases} 2 & \text{if } i = 1, \\ \frac{1}{2} & \text{if } i = 0 \end{cases} \]
and \( \Omega \) is the set such that
\[ \Omega = \{ g : g(z) = 0 \}. \]

**Theorem 4.3.** Let \( q : X \to Y \) be a mapping for which there exist a mapping \( \vartheta : X^3 \to Z \) with the condition
\[ \lim_{n \to \infty} N'(\vartheta(\mu_i^n x, \mu_i^n y, \mu_i^n z), \mu_i^{2n} r) = 1 \] (4.1)
for all \( x, y, z \in X, r > 0 \) and satisfying the functional inequality
\[ N(D_q(x, y, z), r) \geq N'(\vartheta(x, y, z), r) \] (4.2)
for all \( x, y, z \in X, r > 0 \). If there exists \( L = L(i) > 0 \) such that the function
\[ z \to \gamma(z) = \vartheta\left(0, -\frac{z}{2}, \frac{z}{2}\right), \]
has the property
\[ N'(\frac{L \gamma(\mu_i z)}{\mu_i^i}, r) = N'(\gamma(z), r), \forall z \in X, r > 0. \] (4.3)
Then there exists unique quadratic mapping \( Q : X \to Y \) satisfying the functional equation (1.2) and
\[ N(q(z) - Q(z), r) \geq N'(\frac{L^{1-i}}{1-L} \gamma(z), r) \forall z \in X, r > 0. \] (4.4)

**Proof.** Let \( d \) be a general metric on \( \Omega \), such that
\[ d(g, h) = \inf \{ K \in (0, \infty) | N(g(z) - h(z), r) \geq N'(\gamma(z), r), z \in X, r > 0 \}. \]
It is easy to see that \( (\Omega, d) \) is complete. Define \( T : \Omega \to \Omega \) by \( Tg(z) = \frac{1}{\delta_i^2} g(\delta_i z) \), for all \( z \in X \). For \( g, h \in \Omega \), we have \( d(g, h) \leq K \)
\[ \Rightarrow \quad N(g(z) - h(z), r) \geq N'(K \gamma(z), r), \forall z \in X, r > 0 \]
\[ \Rightarrow \quad N\left(\frac{g(\delta_i z)}{\delta_i^i} - \frac{h(\delta_i z)}{\delta_i^i}, r\right) \geq N'(K \gamma(\delta_i z), \delta_i^2 r), \forall z \in X, r > 0 \]
\[ \Rightarrow \quad N(Tg(z) - Th(z), r) \geq N'(KL\gamma(z), r), \forall z \in X, r > 0 \]
\[ \Rightarrow \quad d(Tg(z), Th(z)) \leq KL, \forall z \in X \]
\[ \Rightarrow \quad d(Tg, Th) \leq Ld(g, h) \] (4.5)
for all \( g, h \in \Omega \). Therefore \( T \) is strictly contractive mapping on \( \Omega \) with Lipschitz constant \( L \). Replacing \((x, y, z)\) by \((0, -z, z)\) in (4.2), we get

\[
N\left(\frac{q(2z)}{2^2} - q(z), r\right) \geq N'(\vartheta(0, -z, z), 2^2r) \tag{4.6}
\]

for all \( z \in X, r > 0 \). Using (F3) in (4.6), we arrive

\[
N\left(\frac{q(2z)}{2^2} - q(z), r\right) \geq N'(\vartheta(0, -z, z), 2^2r) \tag{4.7}
\]

for all \( z \in X, r > 0 \) with the help of (4.3) when \( i = 0 \), it follows from (4.7), we get

\[
\Rightarrow \quad N\left(\frac{q(2z)}{2^2} - q(z), r\right) \geq N'(L\gamma(z), r) \tag{4.8}
\]

Replacing \( z \) by \( \frac{z}{2} \) in (4.6), we obtain

\[
N\left(q(z) - 2^2q\left(\frac{z}{2}\right), r\right) \geq N'(\vartheta\left(0, -\frac{z}{2}, \frac{z}{2}\right), r) \tag{4.9}
\]

for all \( z \in X, r > 0 \) with the help of (4.3) when \( i = 1 \), it follows from (4.9), we get

\[
\Rightarrow \quad N\left(q(z) - 2^2q\left(\frac{z}{2}\right), r\right) \geq N'(\gamma(z), r) \tag{4.10}
\]

Then from (4.8) and (4.10), we can conclude

\[
d(q, Tq) \leq L^1 = L^{1-i}. \tag{4.11}
\]

Now from the fixed point alternative in both cases, it follows that there exists a fixed point \( Q \) of \( T \) in \( \Omega \) such that

\[
Q(z) = N - \lim_{k \to \infty} \frac{q(2^k z)}{2^{2k}}, \quad \forall z \in X, r > 0. \tag{4.12}
\]

Replacing \((x, y, z)\) by \((\delta_i x, \delta_i y, \delta_i z)\) in (4.2), we arrive

\[
N\left(\frac{1}{\delta_i^{2n}}D_q(\delta_i x, \delta_i y, \delta_i z), r\right) \geq N'(\vartheta(\delta_i x, \delta_i y, \delta_i z), \delta_i^{2n}r) \tag{4.13}
\]

for all \( r > 0 \) and all \( x, y, z \in X \)

By proceeding the same procedure as in the Theorem 3.1, we can prove the mapping, \( Q : X \to Y \) satisfies the functional equation (1.2).
Stability of a quadratic functional equation...

By fixed point alternative, since $Q$ is unique fixed point of $T$ in the set
\[ \Delta = \{ q \in \Omega | d(q, Q) < \infty \} , \]
therefore $Q$ is a unique function such that
\[ N(q(z) - Q(z), r) \geq N'(K\gamma(z), r) \]  \hspace{1cm} (4.13)
for all $z \in X, r > 0$ and $K > 0$. Again using the fixed point alternative, we obtain
\begin{align*}
&d(q, Q) \leq \frac{1}{1-L}d(q, Tq) \\
&\Rightarrow \quad d(q, Q) \leq L \left( \frac{1}{1-L} \right) \\
&\Rightarrow \quad N(q(z) - Q(z), r) \geq N'(\frac{L^{1-i}}{1-L}\gamma(z), r) \hspace{1cm} (4.14)
\end{align*}
for all $z \in X$ and $r > 0$. This completes the proof of the theorem. \( \square \)

From Theorem 4.3, we obtain the following corollary concerning the stability for the functional equation (1.2).

**Corollary 4.4.** Suppose that a mapping $q : X \rightarrow Y$ satisfies the inequality
\[ N(D_q(x, y, z), r) \]
\begin{align*}
&\geq \left\{ \begin{array}{ll}
N'(\epsilon, r), \\
N'(\epsilon \{ ||x||^s + ||y||^s + ||z||^s \}, r), & s \neq 2; \\
N'(\epsilon \{ ||x||^s ||y||^s ||z||^s + (||x||^{3s} + ||y||^{3s} + ||z||^{3s}) \}, r), & s \neq \frac{2}{3}; 
\end{array} \right. \hspace{1cm} (4.15)
\end{align*}
for all $x, y, z \in X$ and all $r > 0$, where $\epsilon, s$ are constants. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that
\[ N(q(z) - Q(z), r) \geq \left\{ \begin{array}{ll}
N'(\epsilon, 3r), \\
N'(2\epsilon||z||^s, |2^2 - 2^s||r), \\
N'(2\epsilon||z||^{3s}, |2^2 - 2^{3s}||r).
\end{array} \right. \hspace{1cm} (4.16)
\]
for all $z \in X$ and all $r > 0$.

**Proof.** Setting
\[ \vartheta(x, y, z) = \left\{ \begin{array}{ll}
\epsilon, \\
\epsilon \{ ||x||^s + ||y||^s + ||z||^s \}, \\
\epsilon \{ ||x||^s ||y||^s ||z||^s + ||x||^{3s} + ||y||^{3s} + ||z||^{3s} \}.
\end{array} \right. \]
for all \(x, y, z \in X\). Then,
\[
N'(\partial(\delta^n_i x, \delta^n_i y, \delta^n_i z), \delta^{2n}_i r)
\]
\[
= \begin{cases}
N'(\frac{\epsilon}{\delta^n_i}, r), \\
N'(\frac{\epsilon}{\delta^{2n}_i} (||\delta^n_i x||^s + ||\delta^n_i y||^s + ||\delta^n_i z||^s), r), \\
N'(\frac{\epsilon}{\delta^{2n}_i} \{||\delta^n_i x||^s ||\delta^n_i y||^s + ||\delta^n_i z||^s + ||\mu^n_i x||^{3s} + ||\mu^n_i y||^{3s} + ||\mu^n_i z||^{3s}\}, r),
\end{cases}
\]
\[
= \begin{cases}
\to 1 \text{ as } n \to \infty, \\
\to 1 \text{ as } n \to \infty, \\
\to 1 \text{ as } n \to \infty.
\end{cases}
\]
Thus, (4.1) holds. But we have \(\gamma(z) = \partial \left(0, \frac{z}{2}, \frac{z}{2}\right)\) has the property
\[
N'\left(L \frac{1}{\delta^n_i} \gamma(\delta_i z), r\right) \geq N'(\gamma(z), r) \quad \forall z \in X, r > 0.
\]
Hence
\[
N'(\gamma(z), r) = N'(\partial \left(0, \frac{z}{2}, \frac{z}{2}\right), r) = \begin{cases}
N'(\epsilon, r), \\
N'(\epsilon 2^{1-s} ||z||^s, r), \\
N'(\epsilon 2^{1-3s} ||z||^{3s}, r).
\end{cases}
\]
Now,
\[
N'\left(\frac{1}{\delta^n_i} \gamma(\delta_i z), r\right) = \begin{cases}
N'(\frac{\epsilon}{\delta^n_i}, r), \\
N'(\frac{\epsilon}{\delta^{2n}_i} \frac{2}{2} ||\delta_i z||^s, r), \\
N'(\frac{\epsilon}{\delta^{2n}_i} \frac{2}{2} ||\delta_i z||^{3s}, r),
\end{cases}
\]
\[
= \begin{cases}
N'\left(\delta_i^{-2} \gamma(x), r\right), \\
N'\left(\delta_i^{-s} \gamma(z), r\right), \\
N'\left(\delta_i^{3s-2} \gamma(z), r\right).
\end{cases}
\]
for all \(z \in X\) and all \(r > 0\). Hence the inequality (4.3) holds either, \(L = 2^{s-2}\) for \(s < 2\)
if \(i = 0\) and \(L = 2^{2-s}\) for \(s > 0\) if \(i = 1\).

**Case 1:** \(L = 2^{s-2}\) for \(s < 2\) if \(i = 0\)
\[
N(\rho(z) - Q(z), r) \geq N'\left(\frac{2^{s-2}}{1 - 2^{s-2}}, \gamma(z), r\right) = N'\left(2\epsilon ||z||^s, \frac{r}{2^{2-2s}}, r\right).
\]

**Case 2:** \(L = 2^{2-s}\) for \(s > 2\) if \(i = 1\)
\[
N(\rho(z) - Q(z), r) \geq N'\left(\frac{1}{1 - 2^{2-s}}, \gamma(z), r\right) = N'\left(2\epsilon ||z||^s, \frac{r}{2^{2-2}}, r\right).
\]
Similarly, the inequality (4.3) holds either, \(L = 2^{-2}\) if \(i = 0\) and \(L = 2^{2}\) if \(i = 1\) for
condition (i) and also the inequality (4.3) holds either \(L = 2^{3s-2}\) for \(s < \frac{2}{3}\) if \(i = 0\) and
\(L = 2^{2-3s}\) for \(s > \frac{2}{3}\) if \(i = 1\) for condition (iii). Hence the proof is complete.
REFERENCES


