Initial Coefficient Estimates for Certain Subclasses of m-fold Symmetric bi-univalent Functions

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Abstract

This paper provides the two new subclasses of the function class $S_{\Sigma}^m(\alpha, \tau, \lambda)$ and $S_\Sigma^m(\beta, \tau, \lambda)$ of analytic and bi-univalent functions defined in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. Besides, find estimates on the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in these new subclasses. Many interesting new and already existing corollaries are also presented.

Key words and phrases: m-Fold symmetry, bi-univalent functions, coefficient estimates

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1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

which are univalent in $\mathbb{U}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ subclass class of function of $f \in \mathcal{A}$ consisting of the form (1.1) which are also univalent in $\mathbb{U}$. 
The Koebe one-quarter theorem [8] ensures that the image of $U$ under every univalent function $f \in S$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function $f$ has an inverse $f^{-1}$ satisfying $f^{-1}(f(z)) = z, (z \in U)$ and

$$f\left(f^{-1}(w)\right) = w, \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right)$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots$$  (1.2)

A function $f \in A$ is said to be bi-univalent in $U$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $U$. Let $\Sigma$ denote the class of bi-univalent functions in $U$ given by (1.1). Lewin [12] investigated the class $\Sigma$ of bi-univalent functions and showed that $|a_2| < 1.51$ for the functions belonging to $\Sigma$. Subsequently, Brannan and Clunie [5] conjectured that $|a_2| \leq \sqrt{2}$. An analytic function $f$ is subordinate to an analytic function $g$, written $f(z) \prec g(z)$, provided there is a schwarz function $w$ defined on $U$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. Ma and Minda [13], unified various subclasses of starlike and convex functions for which either of the quantity $zf^\prime(z)f^\prime(z)$ or $1 + zf^{\prime\prime}(z)/f^\prime(z)$ is subordinate to a more general superordinate function.

In recent years, the study of bi-univalent functions has gained momentum mainly due to the work of Srivastava et al. [15], which has apparently revived the subject. Motivated by their work [15], many researchers (see, for example, [1, 2, 5, 9, 10, 11, 12]); see also the various closely-related papers on the subject, which are cited in some of these works) have recently investigated several interesting subclasses of the bi-univalent function class $\Sigma$ and found non-sharp estimates on the first two Taylor-Maclaurin coefficients of functions belonging to these subclasses.

Let $m \in \mathbb{N} = 1, 2, 3, \ldots$. A domain $D$ is said to be $m$-fold symmetric if a relation of $D$ about the origin through an angle $\frac{2\pi}{m}$ carries $D$ on itself. It follows that, a function $f(z)$ analytic in $U$ is said to be $m$-fold symmetric ($m \in \mathbb{N}$) if

$$f(e^{\frac{2\pi i}{m}}z) = e^{\frac{2\pi i}{m}}f(z)$$

. In particular, every $f(z)$ is 1-fold symmetric and odd $f(z)$ is 2-fold symmetric. We denote by $S_m$ the class of $m$-fold symmetric univalent functions in $U$ if it has the following normalized form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1}z^{mk+1}, \quad (z \in U, m \in \mathbb{N})$$  (1.3)

Analogous to the concept of $m$-fold symmetric univalent functions, we here introduced the concept of $m$-fold symmetric bi-univalent functions. Each function $f \in \Sigma$ generates
an $m$-fold symmetric bi-univalent function for each integer $m \in \mathbb{N}$. The normalized form of $f$ is given as in (1.3) and the series expansion for $f^{-1}$ is given as follows

$$g(w) = f^{-1}(w) = w - a_{m+1}w^{m+1} + [(m + 1)a_{m+1}^2 - a_{2m+1}]w^{2m+1} - \frac{1}{2}(m + 1)(3m + 3)a_{m+1}^3 - (3m + 2)a_{m+1}a_{2m+1} + a_{3m+1}w^{3m+1} + \cdots .$$

(1.4)

where $f^{-1} = g$. We denote by $\Sigma_m$ the class of $m$-fold symmetric bi-univalent functions in $U$. For $m=1$, the formula (1.4) coincides with the formula (1.2) of the class $\Sigma$.

Some examples of $m$-fold symmetric bi-univalent functions are given as follows

$$\left( \frac{z^m}{1 - z^m} \right)^{\frac{1}{m}}, \left[ \frac{1}{2} \log \left( \frac{1 + z^m}{1 - z^m} \right) \right]^{\frac{1}{m}} \text{ and } \left[ -\log (1 - z^m) \right]^{\frac{1}{m}}$$

with the corresponding inverse functions

$$\left( \frac{w^m}{1 + w^m} \right)^{\frac{1}{m}}, \left[ \frac{e^{2w^m} - 1}{e^{2w^m} + 1} \right]^{\frac{1}{m}} \text{ and } \left( \frac{e^{w^m} - 1}{e^{w^m}} \right)^{\frac{1}{m}}$$

respectively. Recently, many authors investigated bounds for various subclasses of $m$-fold bi-univalent functions (see [3, 15, 16, 17, 18, 19, 20]).

The aim of the present paper is to introduces the certain subclasses $\mathcal{S}_{\Sigma_m}(\alpha, \tau, \lambda)$ and $\mathcal{S}_{\Sigma_m}(\beta, \tau, \lambda)$. Derive the estimates on initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in these subclasses.

**1.1. The class $\mathcal{S}_{\Sigma_m}(\alpha, \tau, \lambda)$**

**Definition 1.1.** For $\tau \in \mathbb{C} \setminus \{0\}, 0 \leq \lambda \leq 1, 0 < \alpha \leq 1, m \in \mathbb{N}$, a function $f \in \Sigma_m$ is said to be in class $\mathcal{S}_{\Sigma_m}(\alpha, \tau, \lambda)$ if the following conditions are satisfied

$$\left| \arg \left[ 1 + \frac{1}{\tau} \left( \frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda) f(z) + \lambda zf'(z)} - 1 \right) \right] \right| < \frac{\alpha \pi}{2}$$

(1.5)

and

$$\left| \arg \left[ 1 + \frac{1}{\tau} \left( \frac{zg'(z) + \lambda z^2 g''(z)}{(1 - \lambda) g(z) + \lambda zg'(z)} - 1 \right) \right] \right| < \frac{\alpha \pi}{2}$$

(1.6)

where function $g = f^{-1}$.

**Remark 1.2.** On specializing the parameter $\tau, \lambda, m$ one can state the various new as well as known subclasses of analytic bi-univalent functions studied earlier in the literature.

(i) For $m = 1$, we obtain new class of bi-univalent function.

$$\mathcal{S}_{\Sigma_m}(\alpha, \tau, \lambda) = \mathcal{S}_\Sigma(\alpha, \tau, \lambda).$$
(ii) For $\lambda = 0$, we obtain new class which consists $m$-fold symmetric bi starlike function.
\[ S_{\Sigma_m} (\alpha, \tau, \lambda) = S_{\Sigma_m}^* (\alpha, \tau). \]

(iii) For $\lambda = 1$, we obtain new class which consists $m$-fold symmetric convex bi univalent function.
\[ S_{\Sigma_m} (\alpha, \tau, \lambda) = C_{\Sigma_m} (\alpha, \tau). \]

(iv) For $\lambda = 0, \tau = 1$, we obtain class which consists $m$-fold symmetric bi-univalent function by S. Altinkaya, S. Yalcin [3].
\[ S_{\Sigma_m} (\alpha, \tau, \lambda) = \delta_{\Sigma_m}^\alpha. \]

(v) For $\lambda = 0, m = 1, \tau = 1$, we obtain class of bi-univalent function introduced by Brannan and Taha [7].
\[ S_{\Sigma_m} (\alpha, \tau, \lambda) = \delta_{\Sigma}^\alpha (\alpha). \]

(vi) For $\lambda = 1, \tau = 1$, we obtain class which consists $m$-fold symmetric convex bi univalent function by A. K. Wanas and A. H. Majeed [20].
\[ S_{\Sigma_m} (\alpha, \tau, \lambda) = E_{\Sigma_m} (0, 1, 1, \alpha). \]

(vii) For $\lambda = 1, m = 1, \tau = 1$, we obtain class which consists convex bi univalent function introduced by Brannan and Taha [7].
\[ S_{\Sigma_m} (\alpha, \tau, \lambda) = \delta_{\Sigma_1} (\alpha). \]

1.2. The class $S_{\Sigma_m} (\beta, \tau, \lambda)$

**Definition 1.3.** For $\tau \in \mathbb{C} \setminus \{0\}, 0 \leq \lambda \leq 1, 0 < \beta \leq 1, m \in \mathbb{N}$, a function $f \in \Sigma_m$ is said to be in class $S_{\Sigma_m} (\beta, \tau, \lambda)$ if the following conditions are satisfied
\[
\mathcal{R} \left[ 1 + \frac{1}{\tau} \left( \frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda) f(z) + \lambda zf'(z)} - 1 \right) \right] > \beta \tag{1.7}
\]
and
\[
\mathcal{R} \left[ 1 + \frac{1}{\tau} \left( \frac{zg'(w) + \lambda z^2 g''(w)}{(1 - \lambda) g(w) + \lambda zg'(w)} - 1 \right) \right] > \beta \tag{1.8}
\]
where function $g = f^{-1}$.

**Remark 1.4.** On specializing the parameter $\tau, \lambda, m$ one can state the various new as well as known subclasses of analytic bi-univalent functions studied earlier in the literature.

(i) For $m = 1$, we obtain new class of bi-univalent function.
\[ S_{\Sigma_m} (\beta, \tau, \lambda) = S_{\Sigma} (\beta, \tau, \lambda). \]
(ii) For \( \lambda = 0 \), we obtain new class which consists \( m \)-fold symmetric bi-starlike function.

\[
S_{\Sigma_m} (\beta, \tau, \lambda) = S_{\Sigma_m}^s (\beta, \tau).
\]

(iii) For \( \lambda = 1 \), we obtain new class which consists \( m \)-fold symmetric convex bi-univalent function.

\[
S_{\Sigma_m} (\beta, \tau, \lambda) = C_{\Sigma_m} (\beta, \tau).
\]

(iv) For \( \lambda = 0, \tau = 1 \), we obtain class which consists \( m \)-fold symmetric bi-univalent function by S. Altinkaya, S. Yalcin [3].

\[
S_{\Sigma_m} (\beta, \tau, \lambda) = N_0^{\Sigma_m} (\beta, 1).
\]

(v) For \( \lambda = 0, m = 1, \tau = 1 \), we obtain class of bi-univalent function introduced by Brannan and Taha [7].

\[
S_{\Sigma_m} (\beta, \tau, \lambda) = \delta_{\Sigma}^* (\beta).
\]

(vi) For \( \lambda = 1, \tau = 1 \), we obtain class which consists \( m \)-fold symmetric convex bi-univalent function by A. K. Wanas and A. H. Majeed [20].

\[
S_{\Sigma_m} (\beta, \tau, \lambda) = E_{\Sigma}^* (0, 1, 1, \beta).
\]

(vii) For \( \lambda = 1, m = 1, \tau = 1 \), we obtain class which consists convex bi-univalent function introduced by Brannan and Taha [7].

\[
S_{\Sigma_m} (\beta, \tau, \lambda) = \delta_{\Sigma_1} (\beta).
\]

In order to prove our main results, we required the following lemma.

**Lemma 1.5.** (see [8]) If \( P(z) = 1 + p_1 z + p_2 z^2 + p_2 z^2 + \cdots \) is an analytic function in \( \mathbb{U} \) with positive real part, then

\[
|p_n| \leq 2 \quad (n \in \mathbb{N} = 1, 2, 3, \cdots)
\]

2. **COEFFICIENT ESTIMATES**

**Theorem 2.1.** If \( f \in S_{\Sigma_m} (\alpha, \tau, \lambda) \quad (\tau \in \mathbb{C} \setminus \{0\}, 0 \leq \lambda \leq 1, 0 < \alpha \leq 1, m \in \mathbb{N}) \), then

\[
|a_{m+1}| \leq \frac{2\alpha |\tau|}{\sqrt{2m\alpha \tau \left[ (m + 1) (1 + 2\lambda m) - (1 + \lambda m)^2 \right] + m^2 (1 - \alpha) (1 + \lambda m)^2}}
\]

and

\[
|a_{2m+1}| \leq \frac{\alpha \tau}{m (1 + 2\lambda m)} + \frac{2\alpha^2 \tau^2 (m + 1)}{m^2 (1 + \lambda m)^2}.
\]
Proof. Let \( f \in S_{\Sigma_m} (\tau, \lambda, \alpha) \). Then

\[
1 + \frac{1}{\tau} \left( \frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda) f(z) + \lambda z f'(z)} - 1 \right) = [p(z)]^\alpha
\]  

(2.11)

and

\[
1 + \frac{1}{\tau} \left( \frac{zg'(z) + \lambda z^2 g''(z)}{(1 - \lambda) g(z) + \lambda z g'(z)} - 1 \right) = [q(w)]^\alpha
\]  

(2.12)

where \( p(z) \) and \( q(z) \) are in familiar Caratheodory class \( \mathcal{P} \) and following series expansions:

\[
p(z) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \cdots
\]  

(2.13)

and

\[
q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \cdots
\]  

(2.14)

Now, equating the coefficients of (2.11) and (2.12), we get

\[
m \tau (1 + m \lambda) a_{m+1} = \alpha p_m
\]  

(2.15)

\[
m \left[ 2 (1 + 2m \lambda) a_{2m+1} - (1 + m \lambda)^2 a_{m+1}^2 \right] = \alpha p_{2m} + \frac{\alpha (\alpha - 1)}{2} p_m^2
\]  

(2.16)

and

\[
- \frac{m}{\tau} (1 + m \lambda) a_{m+1} = \alpha q_m
\]  

(2.17)

\[
m \left[ \{2 (m + 1) (1 + 2m \lambda) - (1 + m \lambda)^2 \} a_{m+1}^2 - 2 (1 + 2m \lambda) a_{2m+1} \right] = \alpha q_{2m} + \frac{\alpha (\alpha - 1)}{2} q_m^2
\]  

(2.18)

Now considering (2.15) and (2.17), we get

\[
p_m = -q_m
\]  

(2.19)

and

\[
\frac{2m^2}{\tau^2} (1 + m \lambda)^2 a_{m+1}^2 = \alpha^2 (p_m^2 + q_m^2)
\]  

(2.20)

Now from (2.16), (2.18) and (2.20) we get

\[
a_{m+1}^2 = \frac{\alpha^2 \tau^2 (p_{2m} + q_{2m})}{2m \tau \alpha \{ (m + 1) (1 + 2m \lambda) - (1 + m \lambda)^2 \} + m^2 (1 - \alpha) (1 + m \lambda)^2}
\]  

(2.21)

Now, taking absolute value of (2.21) and applying lemma 1.1 for the coefficients \( p_{2m} \) and \( q_{2m} \), we obtain

\[
|a_{m+1}| \leq \frac{2\alpha |\tau|}{\sqrt{[2m \tau \alpha \{ (m + 1) (1 + 2m \lambda) - (1 + m \lambda)^2 \} + m^2 (1 - \alpha) (1 + m \lambda)^2]}}
\]  

(2.22)
This gives the desired estimate for $|a_{m+1}|$ as asserted in (2.9). In order to find the bound on $|a_{2m+1}|$, by subtracting (2.18) from (2.16), we get

$$\frac{m}{\tau} \left[ 4 (1 + 2m\lambda) a_{2m+1} - 2 (m + 1) (1 + 2m\lambda) a_{m+1}^2 \right] = \alpha (p_{2m} - q_{2m}) + \frac{\alpha (\alpha - 1)}{2} (p_m^2 - q_m^2)$$

(2.23)

It follows from (2.19), (2.20) and (2.23)

$$a_{2m+1} = \frac{\alpha \tau (p_{2m} - q_{2m})}{4m (1 + 2m\lambda)} + \frac{\alpha^2 \tau^2 (m + 1) (p_m^2 + q_m^2)}{4m^2 (1 + m\lambda)^2}$$

(2.24)

Taking the absolute value of (2.24) and applying Lemma 1.1 once again for the coefficients $p_{2m}$ and $q_{2m}$, we obtain

$$|a_{2m+1}| \leq \frac{\alpha |\tau|}{m (1 + 2m\lambda)} + \frac{2\alpha^2 \tau^2 (m + 1)}{m^2 (1 + m\lambda)^2}$$

(2.25)

Which completes the proof of Theorem 2.1.

For $m = 1$, in Theorem 2.1, we have the following Corollary.

**Corollary 2.2.** Let $f$ given by 1.3 is in the class $\mathcal{S}_\Sigma (\alpha, \tau, \lambda)$, then

$$|a_2| \leq \frac{2\alpha |\tau|}{\sqrt{2\alpha \tau [2 (1 + 2\lambda) - (1 + \lambda)^2] + (1 - \alpha) (1 + \lambda)^2}}$$

and

$$|a_3| \leq \frac{\alpha \tau (1 + 2\lambda) + 4\alpha^2 \tau^2}{(1 + \lambda)^2}.$$  

For $\lambda = 0$, in Theorem 2.1, we have the following Corollary.

**Corollary 2.3.** Let $f$ given by 1.3 is in the class $\mathcal{S}_{\Sigma m}^c (\alpha, \tau)$, then

$$|a_{m+1}| \leq \frac{2\alpha |\tau|}{m \sqrt{1 + \alpha (2\tau - 1)}}$$

and

$$|a_{2m+1}| \leq \frac{\alpha \tau}{m} + \frac{2\alpha^2 \tau^2 (m + 1)}{m^2}.$$  

For $\lambda = 1$, in Theorem 2.1, we have the following Corollary.

**Corollary 2.4.** Let $f$ given by 1.3 is in the class $\mathcal{C}_\Sigma (\alpha, \tau)$, then

$$|a_{m+1}| \leq \frac{2\alpha |\tau|}{m \sqrt{2\alpha \tau (m + 1) + (1 - \alpha) (1 + m)^2}}$$

and

$$|a_{2m+1}| \leq \frac{\alpha \tau}{m (1 + 2m)} + \frac{2\alpha^2 \tau^2}{m^2 (1 + m)}.$$
For \( \lambda = 0, \tau = 1 \), in Theorem 2.1, we have the following Corollary.

**Corollary 2.5.** Let \( f \) given by 1.3 is in the class \( \delta_{\Sigma, m}^{\alpha} \), then

\[
|a_{m+1}| \leq \frac{2\alpha}{m\sqrt{1 + \alpha}}
\]

and

\[
|a_{2m+1}| \leq \frac{\alpha}{m} + \frac{2\alpha^2 (m + 1)}{m^2}.
\]

For \( \lambda = 0, m = 1, \tau = 1 \), in Theorem 2.1, we have the following Corollary.

**Corollary 2.6.** Let \( f \) given by 1.3 is in the class \( \delta^{\alpha}_\Sigma (\alpha) \), then

\[
|a_{2}\rangle \leq \frac{2\alpha}{\sqrt{1 + \alpha}}
\]

and

\[
|a_{3}| \leq \alpha + 4\alpha^2 = \alpha (1 + 4\alpha).
\]

For \( \lambda = 1, \tau = 1 \), in Theorem 2.1, we have the following Corollary.

**Corollary 2.7.** Let \( f \) given by 1.3 is in the class \( E_{\Sigma, m} (0, 1, 1, \alpha) \), then

\[
|a_{m+1}| \leq \frac{2\alpha}{m\sqrt{2\alpha (m + 1) + (1 - \alpha)(1 + m)^2}}
\]

and

\[
|a_{2m+1}| \leq \frac{\alpha}{m(1 + 2m)} + \frac{2\alpha^2}{m^2(1 + m)}.
\]

For \( \lambda = 1, m = 1, \tau = 1 \), in Theorem 2.1, we have the following Corollary.

**Corollary 2.8.** Let \( f \) given by 1.3 is in the class \( \delta_{\Sigma, 1}^{\alpha} (\alpha) \), then

\[
|a_{2}| \leq \alpha
\]

and

\[
|a_{3}| \leq \frac{\alpha}{3} + \alpha^2.
\]

### 3. COEFFICIENT ESTIMATES

**Theorem 3.1.** If \( f \in S_{\Sigma, m} (\beta, \tau, \lambda) \) \( \tau \in \mathbb{C} \setminus \{0\}, 0 \leq \lambda \leq 1, 0 < \alpha \leq 1, m \in \mathbb{N} \), then

\[
|a_{m+1}| \leq \sqrt{\frac{2(1 - \beta)\tau}{m[(m + 1)(1 + 2m\lambda) - (1 + m\lambda)^2]}}
\]

\[
|a_{2m+1}| \leq \frac{|\tau|(1 - \beta)}{m(1 + 2m\lambda)} + \frac{2\tau^2 (m + 1)(1 - \beta)^2}{m^2(1 + m\lambda)^2}.
\]
Proof. Let \( f \in S_{\Sigma_m} (\tau, \lambda, \beta) \). Then

\[
1 + \frac{1}{\tau} \left( \frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda) f(z) + \lambda zf'(z)} - 1 \right) = \beta + (1 - \beta)p(z) \tag{3.28}
\]

and

\[
1 + \frac{1}{\tau} \left( \frac{zg'(z) + \lambda z^2 g''(z)}{(1 - \lambda) g(z) + \lambda zg'(z)} - 1 \right) = \beta + (1 - \beta)q(w) \tag{3.29}
\]

where \( p(z) \) and \( q(z) \) have the forms (2.13) and (2.14) respectively. Equating the coefficients of (3.28) and (3.29), we get

\[
\frac{m}{\tau} (1 + m\lambda) a_{m+1} = (1 - \beta) p_m \tag{3.30}
\]

\[
\frac{m}{\tau} \left[ 2 (1 + 2m\lambda) a_{2m+1} - (1 + m\lambda)^2 a_{m+1}^2 \right] = (1 - \beta) p_{2m} \tag{3.31}
\]

and

\[
- \frac{m}{\tau} (1 + m\lambda) a_{m+1} = (1 - \beta) q_m \tag{3.32}
\]

\[
\frac{m}{\tau} \left[ \left\{ 2 (m + 1) (1 + 2m\lambda) - (1 + m\lambda)^2 \right\} a_{m+1}^2 - 2 (1 + 2m\lambda) a_{m+1} \right] = (1 - \beta) q_{2m} \tag{3.33}
\]

Now considering (3.30) and (3.32), we get

\[
p_m = -q_m \tag{3.34}
\]

and

\[
\frac{2m^2}{\tau^2} (1 + m\lambda)^2 a_{m+1}^2 = (1 - \beta)^2 \left( p_m^2 + q_m^2 \right) \tag{3.35}
\]

Now from (3.31) and (3.33) we get

\[
a_{m+1}^2 = \frac{(1 - \beta) \tau (p_{2m}^2 + q_{2m}^2)}{2m \left[ (m + 1) (1 + 2m\lambda) - (1 + m\lambda)^2 \right]} \tag{3.36}
\]

Now, taking absolute value of (3.36) and applying lemma 1.1 for the coefficients \( p_{2m} \) and \( q_{2m} \), we obtain

\[
|a_{m+1}| \leq \sqrt{\frac{2 (1 - \beta) \tau}{m \left[ (m + 1) (1 + 2m\lambda) - (1 + m\lambda)^2 \right]}} \tag{3.37}
\]

This gives the desired estimate for \( |a_{m+1}| \) as asserted in (3.26). In order to find the bound on \( |a_{2m+1}| \), by subtracting (3.33) from (3.31), we get

\[
\frac{m}{\tau} \left[ 4 (1 + 2m\lambda) a_{2m+1} - 2 (m + 1) (1 + 2m\lambda) a_{m+1}^2 \right] = (1 - \beta) (p_{2m} - q_{2m}) \tag{3.38}
\]

It follows from (3.34), (3.35) and (3.38)

\[
a_{2m+1} = \frac{(1 - \beta) \tau (p_{2m} - q_{2m})}{4m (1 + 2m\lambda)} + \frac{(1 - \beta)^2 \tau^2 (m + 1) (p_m^2 + q_m^2)}{4m^2 (1 + m\lambda)^2} \tag{3.39}
\]
Taking the absolute value of (3.39) and applying Lemma 1.1 once again for the coefficients \( p_{2m} \) and \( q_{2m} \), we obtain

\[
|a_{2m+1}| \leq \frac{(1 - \beta)|\tau|}{m(1 + 2m\lambda)} + \frac{2\tau^2 (1 - \beta)^2 (m + 1)}{m^2 (1 + m\lambda)^2}
\]  \hspace{1cm} (3.40)

Which completes the proof of Theorem 3.1.

For \( m = 1 \), in Theorem 3.1, we have the following Corollary.

**Corollary 3.2.** Let \( f \) given by 1.3 is in the class \( S_{\Sigma} (\beta, \tau, \lambda) \), then

\[
|a_2| \leq \sqrt{\frac{2\tau (1 - \beta)}{2(1 + 2\lambda) - (1 + \lambda)^2}}
\]

and

\[
|a_3| \leq \frac{|\tau| (1 - \beta)}{(1 + 2\lambda)} + \frac{4\tau^2 (1 - \beta)^2}{(1 + \lambda)^2}.
\]

For \( \lambda = 0 \), in Theorem 3.1, we have the following Corollary.

**Corollary 3.3.** Let \( f \) given by 1.3 is in the class \( S_{\Sigma m}^c (\beta, \tau) \), then

\[
|a_{m+1}| \leq \frac{1}{m} \sqrt{2\tau (1 - \beta)}
\]

and

\[
|a_{2m+1}| \leq \frac{|\tau| (1 - \beta)}{m} + \frac{2\tau^2 (m + 1) (1 - \beta)^2}{m^2}.
\]

For \( \lambda = 1 \), in Theorem 3.1, we have the following Corollary.

**Corollary 3.4.** Let \( f \) given by 1.3 is in the class \( C_{\Sigma m} (\beta, \tau) \), then

\[
|a_{m+1}| \leq \frac{1}{m} \sqrt{\frac{2\tau (1 - \beta)}{m + 1}}
\]

and

\[
|a_{2m+1}| \leq \frac{|\tau| (1 - \beta)}{m (1 + 2m)} + \frac{2\tau^2 (1 - \beta)^2}{m^2 (1 + m)}.
\]

For \( \lambda = 0, \tau = 1 \), in Theorem 3.1, we have the following Corollary.

**Corollary 3.5.** Let \( f \) given by 1.3 is in the class \( N_{\Sigma, m}^0 (\beta, 1) \), then

\[
|a_{m+1}| \leq \frac{1}{m} \sqrt{2 (1 - \beta)}
\]

and

\[
|a_{2m+1}| \leq \frac{(1 - \beta)}{m} + \frac{2(m + 1) (1 - \beta)^2}{m^2}.
\]
For $\lambda = 0, m = 1, \tau = 1$, in Theorem 3.1, we have the following Corollary.

**Corollary 3.6.** Let $f$ given by 1.3 is in the class $\delta^*_\Sigma (\beta)$, then

$$|a_2| \leq \sqrt{2(1 - \beta)}.$$

and

$$|a_3| \leq (1 - \beta) + 4(1 - \beta)^2.$$

For $\lambda = 1, \tau = 1$, in Theorem 3.1, we have the following Corollary.

**Corollary 3.7.** Let $f$ given by 1.3 is in the class $E^*_{\Sigma_m} (0, 1, 1, \beta)$, then

$$|a_{m+1}| \leq \frac{1}{m} \sqrt{\frac{2(1 - \beta)}{m+1}}$$

and

$$|a_{2m+1}| \leq \frac{(1 - \beta)}{m(1 + 2m)} + \frac{2(1 - \beta)^2}{m^2(1 + m)}.$$ 

For $\lambda = 1, m = 1, \tau = 1$, in Theorem 3.1, we have the following Corollary.

**Corollary 3.8.** Let $f$ given by 1.3 is in the class $\delta^*_\Sigma (\beta)$, then

$$|a_2| \leq \sqrt{1 - \beta}.$$

and

$$|a_3| \leq \frac{1 - \beta}{3} + (1 - \beta)^2.$$

REFERENCES


