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On the dynamics of the singularly perturbed Riccati differential equation with two different delays

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Abstract

In this paper, we consider the singularly perturbation of the Riccati difference equation with two different delays. At first, we study the local stability of the fixed points and its corresponding characteristic equation of the linearized system. At second, we show that there is Hopf bifurcation with restricted condition for occurrence. Then we get out the discretized system by applying the method of steps. Local stability and bifurcation analysis of the discretized system. We compare the results with the results of the Riccati differential equation with two different delays. Finally, numerical simulations including bifurcation diagram, Lyapunov exponent and phase portraits are carried out to confirm the analytical findings .

Keywords: Singular perturbation, Differential delay, Fixed points, Chaos, Bifurcation.

1. INTRODUCTION

The ordinary differential equation involving at least one delay term and the highest derivative is multiplied by a small parameter namely singularly perturbed delay differential equation [1, 2, 3, 4, 10, 8, 6]. In recent decades, the analysis and design

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theories for singularly perturbed systems with time-delay have been of considerable concern. For example, Fridman has considered the effect of small delay on stability of the singularly perturbed systems [7, 12, 15]. Two approaches to the asymptotic analysis and solution of this problem are proposed. In the first approach, an asymptotic solution of the singularly-perturbed system of functional-differential equations of Riccati type, associated with the original problem by the sufficient conditions of the existence of its solution, is constructed[13, 14]. Based on this asymptotic solution, conditions for the existence of a solution of the original problem, independent of the small parameter of singular perturbations, are derived [5, 16].
In this paper, we will discuss the dynamic behavior of the singularly perturbed Riccati differential equation with two different delays given in the form

$$\begin{aligned} \epsilon\frac{dx}{dt} &= -x(t) + 1 - \rho x(t-1)x(t-2), && t \in [0,T], \\ x(t) &= x_0, && t \leq 0. \end{aligned} \tag{1}$$

1.1. Stability and bifurcation

Consider the problem (1). Solving the equation ([11])

$$-x + 1 - \rho x^2 = 0,$$

we obtain the two fixed points,

$$(x_{1,2})^* = (\frac{-1}{2\rho})(1 \pm \sqrt{1+4\rho}).$$

At the neighborhood of $(x_1)^*$ the linearized equation is

$$\epsilon\frac{dy}{dt} = -y(t) + \frac{1}{2}(1+\sqrt{1+4\rho})y(t-1) + \frac{1}{2}(1+\sqrt{1+4\rho})y(t-2) \tag{2}$$

where

$$y(t) = y(t) - ((\frac{-1}{2\rho})(1+\sqrt{1+4\rho})).$$

Then the characteristic equation is

$$\lambda + \frac{1}{\epsilon} - \frac{1}{2\epsilon}(1+\sqrt{1+4\rho})e^{-\lambda} - \frac{1}{2\epsilon}(1+\sqrt{1+4\rho})e^{-2\lambda} = 0. \tag{3}$$

Theorem 1. *When the parameter ρ passes through the critical value $\rho = \rho_* = \frac{1}{4}[(\frac{1+\epsilon^2\omega_0^2-(\cos(\omega_0)+\epsilon\omega_0\sin(\omega_0))}{\cos(\omega_0)+\epsilon\omega_0\sin(\omega_0)})^2 - 1]$, $\omega_0 = \tan(2\omega_0)(1 - s\cos(\omega_0)) + s\sin(\omega_0)$, there is Hopf bifurcation from the equilibrium $(x_1)^* = (\frac{-1}{2\rho})(1+\sqrt{1+4\rho})$ to a periodic orbit.*

Proof. Let $\lambda = i\omega_0, \omega_0 \in R^+$ be a for (3) for $\rho = \rho_*$, then we obtain

$$i\epsilon\omega_0 + 1 - \frac{1}{2}(1 + \sqrt{1+4\rho_*})e^{-i\omega_0} - \frac{1}{2}(1 + \sqrt{1+4\rho_*})e^{-2i\omega_0} = 0,$$

$$then,\ 1 - \frac{1}{2}(1 + \sqrt{1+4\rho_*})\cos(\omega_0) - \frac{1}{2}(1 + \sqrt{1+4\rho_*})\cos(2\omega_0) = 0,$$

$$\epsilon\omega_0 - \frac{1}{2}(1 + \sqrt{1+4\rho_*})\sin(\omega_0) - \frac{1}{2}(1 + \sqrt{1+4\rho_*})\sin(2\omega_0) = 0,$$

$$let\ \ \frac{1}{2}(1 + \sqrt{1+4\rho_*}) = s,$$

$$then, \quad 1 - s\cos(\omega_0) - s\cos(2\omega_0) = 0, \tag{4}$$

$$\epsilon\omega_0 - s\sin(\omega_0) - s\sin(2\omega_0) = 0. \tag{5}$$

By solving equation (4) and equation (5) We get,

$$s = \frac{1 + \epsilon^2\omega_0^2}{2(\cos(\omega_0) + \epsilon\omega_0\sin(\omega_0))}.$$

$$Then, \quad \rho_* = \frac{1}{4}[(\frac{1 + \epsilon^2\omega_0^2 - (\cos(\omega_0) + \epsilon\omega_0\sin(\omega_0)}{\cos(\omega_0) + \epsilon\omega_0\sin(\omega_0)})^2 - 1],$$

to get ω_0,

$$\frac{\epsilon\omega_0 - s\sin(\omega_0)}{1 - s\cos(\omega_0)} = \frac{\sin(2\omega_0)}{\cos(2\omega_0)},$$

$$\omega_0 = \frac{1}{\epsilon}[\tan(2\omega_0)(1 - s\cos(\omega_0)) + s\sin(\omega_0)].$$

Now, we are left with the condition $\frac{d(Re(\lambda))}{d\rho}\ |_\rho \neq\ 0$. To show that this condition is satisfied,
let $\lambda = k(\rho) + i\omega(\rho)$ and using equation (3),
we get,

$$\epsilon(k + i\omega) + 1 - \frac{1}{2}(1 + \sqrt{1+4\rho})e^{-k-i\omega} - \frac{1}{2}(1 + \sqrt{1+4\rho})e^{-2(k+i\omega)} = 0,$$

then, we have,

$$\epsilon k+1-\frac{1}{2}(1+\sqrt{1+4\rho})e^{-k}\cos(\omega)-\frac{1}{2}(1+\sqrt{1+4\rho})e^{-2k}\cos(2\omega)=0, \qquad (6)$$

$$\epsilon\omega+\frac{1}{2}(1+\sqrt{1+4\rho})e^{-k}\sin(\omega)+\frac{1}{2}(1+\sqrt{1+4\rho})e^{-2k}\sin(2\omega)=0. \qquad (7)$$

differentiate equation(6) and equation (7) with respect to ρ, we get,

$$\begin{aligned}
&\epsilon\frac{dk}{d\rho}+\frac{1}{2}e^{-k}\cos(\omega)\frac{dk}{d\rho}+\frac{1}{2}e^{-k}\sin(\omega)\frac{d\omega}{d\rho}\\
&\quad-\frac{1}{2}e^{-k}\cos(\omega)\frac{4}{2\sqrt{1+4\rho}}+\frac{1}{2}(\sqrt{1+4\rho})e^{-k}\cos(\omega)\frac{dk}{d\rho}\\
&\quad+\frac{1}{2}(\sqrt{1+4\rho})e^{-k}\sin(\omega)\frac{d\omega}{d\rho}+e^{-2k}\cos(2\omega)\frac{dk}{d\rho}\\
&\quad+e^{-2k}\sin(2\omega)\frac{d\omega}{d\rho}+(\sqrt{1+4\rho})e^{-2k}\cos(2\omega)\frac{dk}{d\rho}\\
&\quad+(\sqrt{1+4\rho})e^{-2k}\sin(2\omega)\frac{d\omega}{d\rho}-\frac{1}{2}e^{-2k}\frac{4}{2\sqrt{1+4\rho}}\cos(2\omega)=0,
\end{aligned}$$

$$\begin{aligned}
&=\frac{dk}{d\rho}(\epsilon+\frac{1}{2}e^{-k}\cos(\omega)+\frac{1}{2}(\sqrt{1+4\rho})e^{-k}\cos(\omega)+e^{-2k}\cos(2\omega)+(\sqrt{1+4\rho})e^{-2k}\cos(2\omega))\\
&+\frac{d\omega}{d\rho}(\frac{1}{2}e^{-k}\sin(\omega)+\frac{1}{2}(\sqrt{1+4\rho})e^{-k}\sin(\omega)+e^{-2k}\sin(2\omega)+(\sqrt{1+4\rho})e^{-2k}\sin(2\omega))\\
&-\frac{e^{-k}\cos(\omega)}{\sqrt{1+4\rho}}-\frac{e^{-2k}\cos(2\omega)}{\sqrt{1+4\rho}}=0.
\end{aligned} \qquad (8)$$

$$\begin{aligned}
&\epsilon\frac{d\omega}{d\rho}-\frac{1}{2}e^{-k}\sin(\omega)\frac{dk}{d\rho}+\frac{1}{2}e^{-k}\cos(\omega)\frac{d\omega}{d\rho}\\
&\quad+\frac{1}{2}e^{-k}\sin(\omega)\frac{4}{2\sqrt{1+4\rho}}-\frac{1}{2}(\sqrt{1+4\rho})e^{-k}\sin(\omega)\frac{dk}{d\rho}\\
&\quad+\frac{1}{2}(\sqrt{1+4\rho})e^{-k}\cos(\omega)\frac{d\omega}{d\rho}-e^{-2k}\sin(2\omega)\frac{dk}{d\rho}\\
&\quad+e^{-2k}\cos(2\omega)\frac{d\omega}{d\rho}-(\sqrt{1+4\rho})e^{-2k}\sin(2\omega)\frac{dk}{d\rho}\\
&\quad+(\sqrt{1+4\rho})e^{-2k}\cos(2\omega)\frac{d\omega}{d\rho}+\frac{1}{2}e^{-2k}\frac{4}{2\sqrt{1+4\rho}}\sin(2\omega)=0,
\end{aligned}$$

$$
\begin{aligned}
&= \frac{dk}{d\rho}(-\frac{1}{2}e^{-k}\sin(\omega) - \frac{1}{2}(\sqrt{1+4\rho})e^{-k}\sin(\omega) \\
&- e^{-2k}\sin(2\omega) - (\sqrt{1+4\rho})e^{-2k}\cos(2\omega)) \\
&+ \frac{d\omega}{d\rho}(\epsilon + \frac{1}{2}e^{-k}\cos(\omega) + \frac{1}{2}(\sqrt{1+4\rho})e^{-k}\cos(\omega) \\
&+ e^{-2k}\cos(2\omega) + (\sqrt{1+4\rho})e^{-2k}\cos(2\omega)) \\
&+ \frac{e^{-k}\sin(\omega)}{\sqrt{1+4\rho}} + \frac{e^{-2k}\sin(2\omega)}{\sqrt{1+4\rho}} = 0.
\end{aligned} \tag{9}
$$

Solving equation (8) and equation (9) for $\frac{dk}{d\rho}$, we obtain

$$
\frac{d(Re(\lambda))}{d\rho}\;|_{\rho=\rho_*} = \frac{dk}{d\rho}\;|_{k=0,\omega=\omega_0,\rho=\rho_*} \neq 0
$$

$\square$

1.2. The discretized system

By using the method of steps, we can get out the discrete-time version of The system (1) by the following steps, the system can be written as ([12])

$$
\begin{aligned}
\epsilon\frac{dx}{dt} &= -x(t) + 1 - \rho x(t-1)y(t-1), \\
y(t) &= x(t-1), \\
x(t) &= x_0, t \leq 0.
\end{aligned} \tag{10}
$$

The discretized model of the system (1) is obtained via the method of steps as follows let $t \in (0, 1]$, then,

$$
\begin{aligned}
y_1(t) &= x_0, \\
x_1(t) &= x_0 e^{\frac{-t}{\epsilon}} + \frac{1}{\epsilon}\int_0^t e^{\frac{s-t}{\epsilon}}(1 - \rho x(s-1)y(s-1))ds, \\
x_1(t) &= x_0 e^{\frac{-t}{\epsilon}} + (1 - \rho x_0 y_1)(1 - e^{\frac{-t}{\epsilon}}).
\end{aligned}
$$

let $t \longrightarrow 1$, then,

$$
\begin{aligned}
y_1(1) &= x_0, \\
x_1(1) &= x_0 e^{\frac{-1}{\epsilon}} + (1 - \rho x_0 y_1)(1 - e^{\frac{-1}{\epsilon}}).
\end{aligned}
$$

For $t \in (1,2]$, take $x(t) = x_1 = x_1(1)$, $y_1(t) = y_1(1) = y_1$, when $t \leq 1$,

$$then, \quad y_2(t) = x_1(t),$$

$$x_2(t) = x_1 e^{\frac{-(t-1)}{\epsilon}} + \frac{1}{\epsilon}\int_1^t e^{\frac{s-t}{\epsilon}}(1-\rho x_1 y_1)ds,$$
$$= x_1 e^{\frac{-(t-1)}{\epsilon})} + (1-\rho x_1 y_1)(1-e^{\frac{-(t-1)}{\epsilon}}).$$

Let $t \longrightarrow 2$, then,

$$y_2(1) = x_1,$$
$$x_2(2) = x_1(1)e^{\frac{-1}{\epsilon}} + (1-\rho x_1(1)y_1(1))(1-e^{\frac{-1}{\epsilon}}).$$

For $t \in (2,3]$, take $x(t) = x_2 = x_2(2)$, $y_2(t) = y_2(2) = y_2$, when $t \leq 2$,

$$then, \quad y_3(t) = x_2(t),$$

$$x_3(t) = x_1 e^{\frac{-(t-1)}{\epsilon}} + \frac{1}{\epsilon}\int_2^t e^{\frac{s-t}{\epsilon}}(1-\rho x_2 y_2)ds,$$

$$= x_2 e^{\frac{-(t-2)}{\epsilon}} + (1-\rho x_1 y_2)(1-e^{\frac{-(t-2)}{\epsilon}}).$$

Let $t \longrightarrow 3$, then,

$$y_3(3) = x_2,$$
$$x_3(3) = x_2 e^{\frac{-1}{\epsilon}} + (1-\rho x_2 y_2(1))(1-e^{\frac{-1}{\epsilon}}).$$

Repeating the operation, we get that the solution of the system (10) is given by

$$y_{n+1}(t) = x_n(t),$$

$$x_{n+1}(t) = x_n e^{\frac{-(t-n)}{\epsilon}} + (1-\rho x_n y_n)(1-e^{\frac{-(t-n)}{\epsilon}}).$$

Let $t \longrightarrow n+1$, then,

$$\begin{aligned} x_{n+1} = x_n e^{\frac{-1}{\epsilon}} + (1-\rho x_n y_n)(1-e^{\frac{-1}{\epsilon}}), \\ y_n = x_n. \end{aligned} \tag{11}$$

1.3. Local stability and bifurcation analysis of the discretized system

The system (11) has two fixed points $(x^*_{1,2}, y^*_{1,2}) = (\frac{-1\pm\sqrt{1+4\rho}}{2\rho}, \frac{-1\pm\sqrt{1+4\rho}}{2\rho})$.

Next, we calculate the Jacobian matrix at the first fixed point (x^*_1, y^*_1)

$$J(x^*, y^*) = \begin{pmatrix} e^{\frac{-1}{\epsilon}} - \rho y^*(1 - e^{\frac{-1}{\epsilon}}) & -\rho x^*(1 - e^{\frac{-1}{\epsilon}}) \\ 1 & 0 \end{pmatrix}.$$

Let us rename $-\rho x^*(1 - e^{\frac{-1}{\epsilon}}) = z$, and $e^{\frac{-1}{\epsilon}} - \rho y^*(1 - e^{\frac{-1}{\epsilon}}) = m.$

The characteristic equation

$$\lambda^2 - m\lambda - z = 0,$$

has two roots

$$\lambda_{1,2} = \frac{m \pm \sqrt{m^2 + 4z}}{2}.$$

Lemma 2. *[9] Let $F(\lambda) = \lambda^2 + P\lambda + Q$. Suppose that $F(1) > 0$, and $F(\lambda) = 0$ has two roots λ_1 and λ_2. Then*

1. *$F(-1) > 0$ and $Q < 1$ if and only if $|\lambda_1| < 1$ and $|\lambda_2| < 1$;*
2. *$F(-1) < 0$ if and only if $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$);*
3. *$F(-1) > 0$ and $Q > 1$ if and only if $|\lambda_1| > 1$ and $|\lambda_2| > 1$;*
4. *$F(-1) = 0$ and $P \neq 0, 2$ if and only if $\lambda_1 = -1$ and $|\lambda_2| \neq 1$;*
5. *$P^2 - 4Q < 0$ and $Q = 1$ if and only if λ_1 and λ_2 are complex and $|\lambda_{1,2}| = 1$.*

By applying (2), we get

$$F(\lambda) = \lambda^2 - m\lambda - z = \lambda^2 + P\lambda + Q = 0. \tag{12}$$

Then

$$P = -m,$$
$$Q = -z.$$

We have

$$F(1) = 1 - m - z > 0, \\ we\ \ should\ \ have\ \ 1 > m + z. \tag{13}$$

By applying condition 1 of lemma (2)

$$Where, \quad F(-1) = 1 + m - z > 0,$$

$$1 + m > z, \tag{14}$$

$$Q < 1 \Rrightarrow -z < 1,\ z > -1.$$

$$Where,\ -\rho x^*(1 - e^{\frac{-1}{\epsilon}}) = z,$$

substitute by the value of x^*, we get,

$$-\rho[\frac{-1 + \sqrt{1 + 4\rho}}{2\rho}](1 - e^{\frac{-1}{\epsilon}})$$

$$= (\frac{1 - \sqrt{1 + 4\rho}}{2})(1 - e^{\frac{-1}{\epsilon}}) > -1. \tag{15}$$

If (14) and (15) satisfied, then (x_1^*, y_1^*) is stable.

The seconed fixed point is the same way of first fixed point.

2. NUMERICAL SIMULATIONS

We confirm all the previous analytical findings with the help of numerical simulations performed via Matlab. In all numerical simulations the initial condition is taken as $(x_0, y_0) = (0.4, 0.4)$ and the Bif. parameter is taken as ρ.
We have the following examples

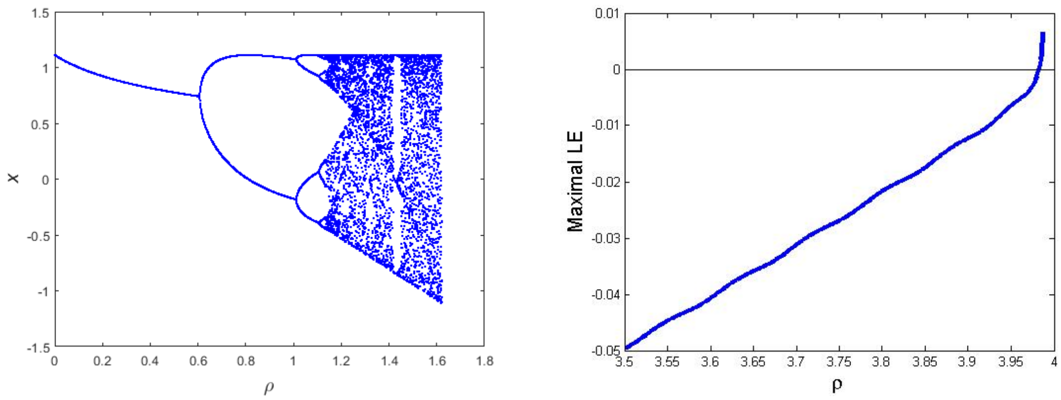
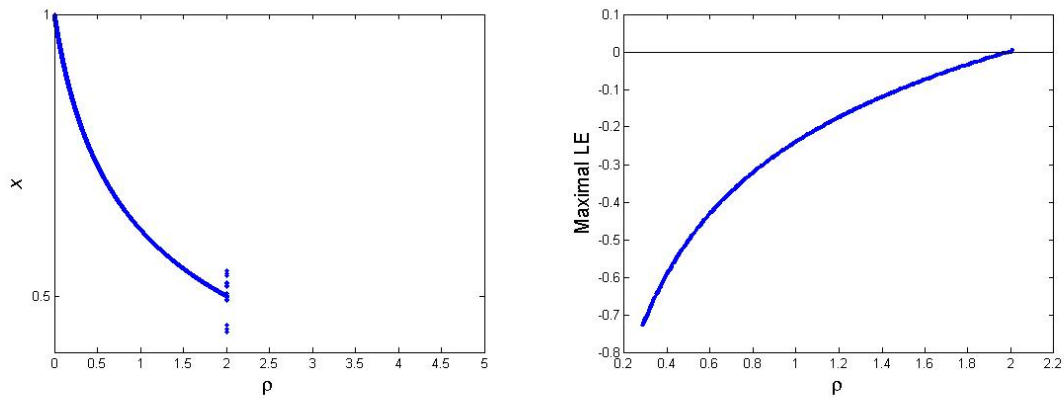
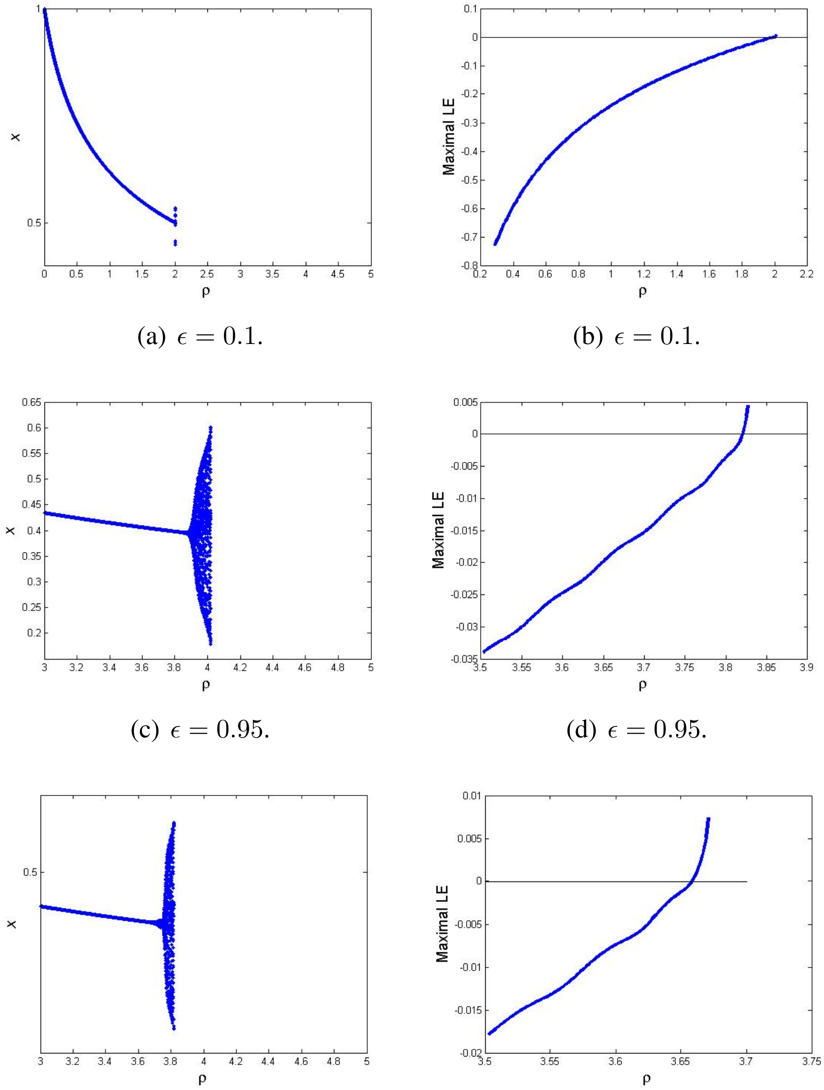


Figure 1: Bif.D. and the graph of Lyp.Ex. of system (11) as $\epsilon \longrightarrow 1$.



(a) (b)

Figure 2: Hopf Bif.D. and the graph of Lyp.Ex. of system (11) as $\epsilon \longrightarrow 0$.



(a) $\epsilon = 0.1$. (b) $\epsilon = 0.1$.

(c) $\epsilon = 0.95$. (d) $\epsilon = 0.95$.

(e) $\epsilon = 0.9$. (f) $\epsilon = 0.9$.

Figure 3: Hopf Bif.D. and the graph of Lyp.Ex. of system (11) for different values of ϵ

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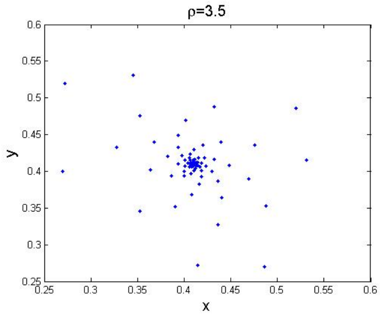
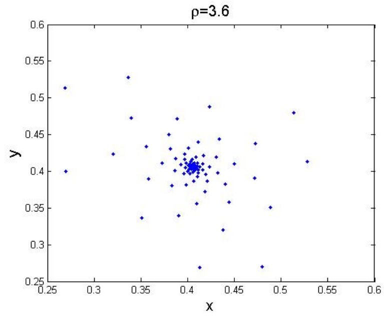
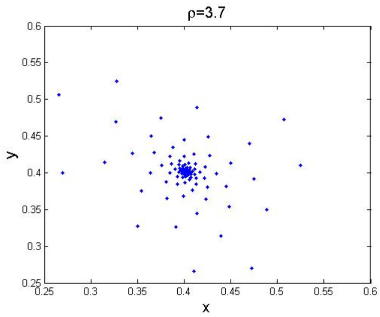


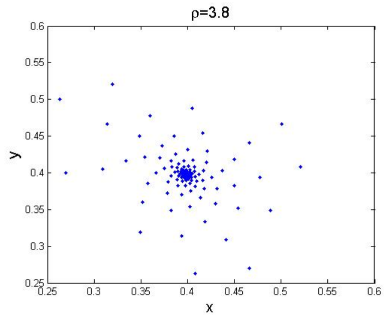
(a)



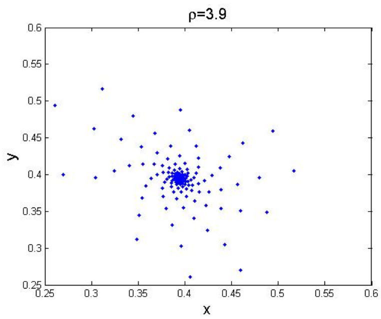
(b)



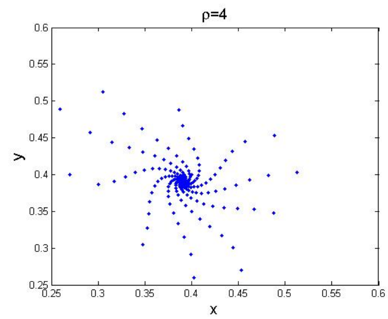
(c)



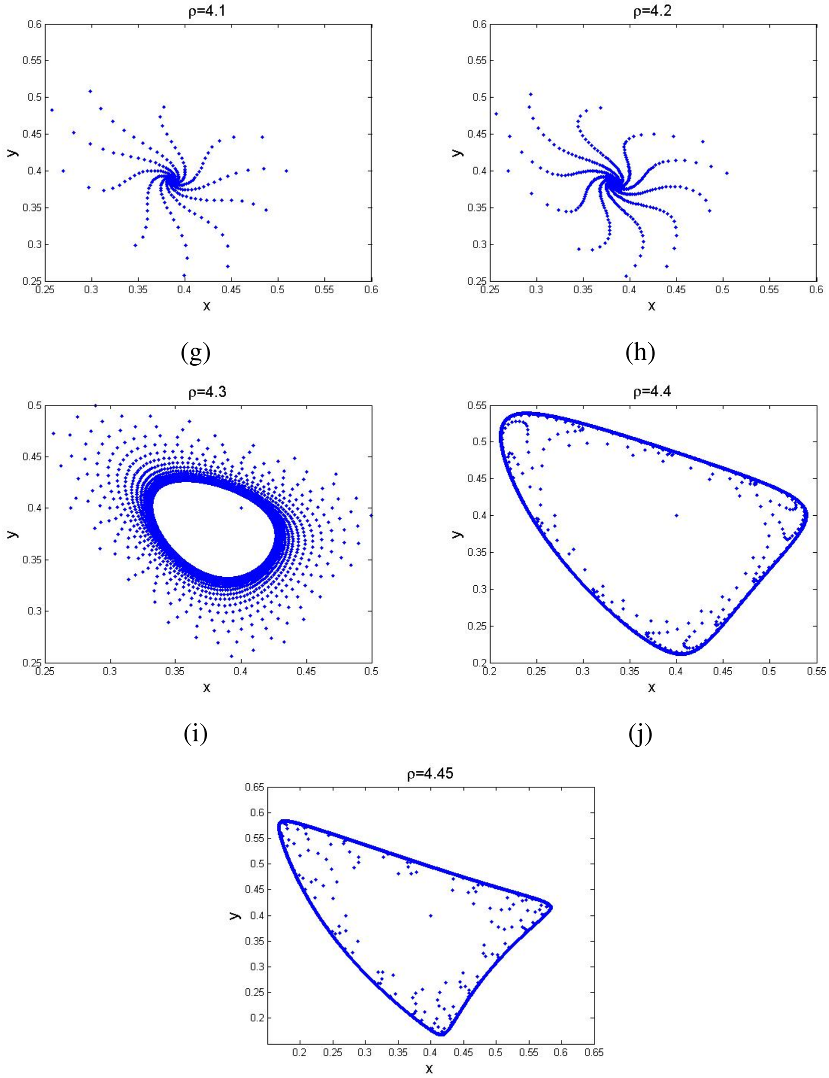
(d)



(e)



(f)



(g) (h) (i) (j) (k)

Figure 4: Phase portraits of system (11) for different ρ .

3. CONCLUSION

In this paper, we studied the dynamics of the S.P. Riccati difference equation with two different delays. First of all, we obtained F.P. and discussed their local stability by analyzing the corresponding characteristic equations of the linearized equations. secondly, we show that the equation exhibits Hopf Bif. and we have reached explicit conditions for its occurrence. Then, the method of steps is applied to obtain a discrete analogue of the considered system. We investigated local stability conditions of the F.P. of the discretized system. It is illustrated that the S.P. Riccati DDE with two delays behaves as the Riccati DDE with two delays. Finally, numerical simulations including Lyp. Ex., Bif.D. and ph.P.s carried out to confirm the theoretical analysis obtained and to illustrate more complex dynamics of the system.

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