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Necessary and Sufficient Conditions for the Existence of a Weak Solution to the Maxwell-Stokes Type Equation

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Abstract

In this paper, we derive necessary and sufficient conditions for the existence of a weak solution to the Maxwell-Stokes type equation associated with slip-Navier boundary condition. Our equation is nonlinear and contains, so called, p-curlcurl system. Moreover, we give a result on the continuous dependence of the weak solution on the data.

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1. INTRODUCTION

In this paper, we give necessary and sufficient conditions for the existence of a weak solution to the Maxwell-Stokes type equation.

Amrouche and Seloula [3] considered the stationary Stokes equations:

$$-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \text{ and } \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega, \tag{1.1}$$

where $\boldsymbol{u}$ is the velocity vector field, π is the pressure, $\boldsymbol{f}$ is the external force and $\Omega \subset \mathbb{R}^3$ is a bounded possibly multi-connected domain with a boundary Γ. They imposed the following slip-Navier boundary conditions (cf. Amrouche and Rejaiba [2]).

$$\boldsymbol{u} \cdot \boldsymbol{n} = g \text{ and } \operatorname{curl} \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} \text{ on } \Gamma, \tag{1.2}$$

where $\boldsymbol{n}$ is the unit outward normal vector to Γ, g and $\boldsymbol{h}$ are given functions. They derived that the compatibility conditions are necessary and sufficient for the existence of a weak solution to (1.1) and the boundary conditions (1.2) plus some conditions associated with the multi-connected property of Ω.

We are interested in considering the following Maxwell-Stokes type system.

$$\operatorname{curl}[S_t(x, |\operatorname{curl}\boldsymbol{u}|^2)\operatorname{curl}\boldsymbol{u}] + \nabla\pi = \boldsymbol{f} \text{ and } \operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega, \tag{1.3}$$

where $S(x,t)$ is a Carathéodory function on $\Omega \times [0,\infty)$ satisfying some structure conditions. In the particular case where $S(x,t) = t^{p/2}$ $(1 < p < \infty)$, the first equation of (1.3) becomes, so called, p-curlcurl equation.

Such partial differential system involving the operator curl appear in many areas in mathematical physics. For example, for the Bohn-Infeld model in nonlinear electrodynamics, see Bohn and Infeld [9] and Yang [18]. For the several models in the theory of superconductivity, see Bates and Pan [8] and Chapman [11].

From a mathematical point of view, since the first equation of (1.3) involves the operator curl and is nonlinear, the system has special character that are quite different from the first equation of (1.1). The existence of solutions of such systems are sensitive from the nonlinearity. To show the existence of solutions of (1.1) with some boundary conditions, the Inf-sup theorem fulfills an important role. However, we can not apply this theorem to the system (1.3). To overcome it, we shall use a minimization problem that is developed in the author's previous paper Aramaki [6].

We impose the following boundary conditions.

$$\boldsymbol{u}\cdot\boldsymbol{n} = g \text{ and } S_t(x, |\operatorname{curl}\boldsymbol{u}|^2)\operatorname{curl}\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} \text{ on } \Gamma. \tag{1.4}$$

Moreover, since Ω may be multi-connected, if we assume that there exist cuts Σ_j $(j = 1,\dots,J)$ such that $\Omega^\circ = \Omega \setminus (\cup_{j=1}^J \Sigma_j)$ is simply connected, then we impose

$$\langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j} = 0 \text{ for } j = 1,\dots,J, \tag{1.5}$$

where $\langle\cdot,\cdot\rangle_{\Sigma_j}$ denotes the duality bracket of $W^{-1/p,p}(\Sigma_j)$ and $W^{1/p,p'}(\Sigma_j)$. In these situations, we give necessary and sufficient conditions for the existence of a weak solution to (1.3)-(1.5).

The paper is organized as follows. Section 2 consists of two subsections. In subsection 2.1, we give some preliminaries. In subsection 2.2, we give the main theorem (Theorem 2.6) that states necessary and sufficient conditions for the existence of the weak solution to the problem (1.3)-(1.5). Section 3 is devoted to the proof of the main theorem. In section 4, we consider the continuous dependence of the weak solution on the data. In Appendix, we convince the Green type equality.

2. PRELIMINARIES AND THE MAIN THEOREM

This section consists of two subsections. In subsection 2.1, we give some preliminaries that are necessary later. In subsection 2.2, we give the notion of a weak solution for the Maxwell-Stokes system (1.3)-(1.5) and state the main theorem (Theorem 2.6).

2.1. Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^3$ with a $C^{1,1}$ boundary Γ and let $1 < p < \infty$. We denote the conjugate exponent of p by p', i.e., $(1/p) + (1/p') = 1$. From now on we use $L^p(\Omega)$, $W^{m,p}(\Omega)$ and $W^{s,p}(\Gamma)$ for the standard L^p and Sobolev spaces of functions in Ω and Γ. For any Banach space B, we denote $B \times B \times B$ by boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vector and vector-valued functions, and we denote the standard Euclidean inner product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^3$ by $\boldsymbol{a} \cdot \boldsymbol{b}$. For the dual space $\boldsymbol{B}'$ of $\boldsymbol{B}$, we write $\langle \cdot, \cdot \rangle_{\boldsymbol{B}', \boldsymbol{B}}$ for the duality bracket.

We assume that a Carathéodory function $S(x,t)$ in $\Omega \times [0, \infty)$ satisfies the following structure conditions. For a.e. $x \in \Omega$, $S(x,t) \in C^2((0,\infty)) \cap C^0([0,\infty))$, and there exist positive constants $0 < \lambda \leq \Lambda < \infty$ such that for a.e. $x \in \Omega$,

$$S(x,0) = 0 \text{ and } \quad \lambda t^{(p-2)/2} \leq S_t(x,t) \leq \Lambda t^{(p-2)/2} \text{ for } t > 0, \tag{2.1a}$$

$$\lambda t^{(p-2)/2} \leq S_t(x,t) + 2t S_{tt}(x,t) \leq \Lambda t^{(p-2)/2} \text{ for } t > 0, \tag{2.1b}$$

$$\text{If } 1 < p < 2, S_{tt}(x,t) < 0, \text{ and if } p \geq 2, S_{tt}(x,t) \geq 0 \text{ for } t > 0, \tag{2.1c}$$

where $S_t = \partial S / \partial t$ and $S_{tt} = \partial^2 S / \partial t^2$. We note that from (2.1a), we have

$$\frac{2}{p} \lambda t^{p/2} \leq S(x,t) \leq \frac{2}{p} \Lambda t^{p/2} \text{ for } t \geq 0. \tag{2.2}$$

For a.e. $x \in \Omega$, it follows from (2.1b) that the function $G(t) = S(x, t^2)$ is a strictly convex function with respect to $t \in [0, \infty)$. Indeed, $G'(t) = 2t S_t(x, t^2)$ and so

$$G''(t) = 2(S_t(x,t^2) + 2t^2 S_{tt}(x,t^2)) \geq 2\lambda t^{p-2} > 0 \text{ for } t > 0.$$

Example 2.1. *If* $S(x,t) = \nu(x) t^{p/2}$, *where* ν *is a measurable function in* Ω *and satisfies* $0 < \nu_* \leq \nu(x) \leq \nu^* < \infty$ *for a.e.* $x \in \Omega$ *for some constants* ν_* *and* ν^*, *then it follows from elementary calculations that* (2.1a)-(2.1c) *hold.* .

We give two lemmas on a monotonic property and a boundedness of S_t.

Lemma 2.2. *There exists a constant* $c > 0$ *depending only on* p *and* λ *such that for all* $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$,

$$(S_t(x, |\boldsymbol{a}|^2)\boldsymbol{a} - S_t(x, |\boldsymbol{b}|^2)\boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{b}) \geq \begin{cases} c|\boldsymbol{a} - \boldsymbol{b}|^p & \text{if } p \geq 2, \\ c(|\boldsymbol{a}| + |\boldsymbol{b}|)^{p-2} |\boldsymbol{a} - \boldsymbol{b}|^2 & \text{if } 1 < p < 2. \end{cases}$$

In particular, S_t is strictly monotonic, that is,

$$(S_t(x,|\boldsymbol{a}|^2)\boldsymbol{a} - S_t(x,|\boldsymbol{b}|^2)\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b}) > 0 \textit{ for } \boldsymbol{a}\neq\boldsymbol{b}.$$

For the proof, see Aramaki [7, Lemma 3.6].

Lemma 2.3. *There exists a constants $C_1 > 0$ depending only on Λ and p such that for any $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^3$,*

$$|S_t(x,|\boldsymbol{a}|^2)\boldsymbol{a} - S_t(x,|\boldsymbol{b}|^2)\boldsymbol{b}| \le \begin{cases} C_1|\boldsymbol{a}-\boldsymbol{b}|^{p-1} & \textit{if } 1<p<2,\\ C_1(|\boldsymbol{a}|+|\boldsymbol{b}|)^{p-2}|\boldsymbol{a}-\boldsymbol{b}| & \textit{if } p\ge 2.\end{cases}$$

For the proof, see Aramaki [5]. Since we allow Ω to be a multi-connected domain in $\mathbb{R}^3$, we assume that the domain Ω satisfies the following (O1) and (O2). (cf. Amrouche and Seloula [4], Dautray and Lions [12] and Témam [16]). Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with a $C^{1,1}$ boundary Γ, and Ω be locally situated on one side of Γ.

(O1) Γ has a finite number of connected components $\Gamma_0,\Gamma_1,\dots,\Gamma_I$ with Γ_0 denoting the boundary of the infinite connected component of $\mathbb{R}^3\setminus\overline{\Omega}$.

(O2) There exist J connected open surfaces Σ_j, $(j=1,\dots,J)$, called cuts, contained in Ω such that

- (a) Σ_j is an open subset of a smooth manifold $\mathcal{M}_j$.
- (b) $\partial\Sigma_j\subset\Gamma$ $(j=1,\dots,J)$, where $\partial\Sigma_j$ denotes the boundary of Σ_j, and Σ_j is non-tangential to Γ.
- (c) $\overline{\Sigma_j}\cap\overline{\Sigma_k}=\emptyset$ $(j\neq k)$.
- (d) The open set $\Omega^\circ=\Omega\setminus(\cup_{j=1}^J\Sigma_j)$ is simply connected and pseudo $C^{1,1}$ class.

The number J is called the first Betti number and I the second Betti number. We say that Ω is simply connected if $J=0$, and Ω has no holes if $I=0$. We define two spaces.

$$\begin{aligned}\mathbb{K}_T^p(\Omega) &= \{\boldsymbol{v}\in\boldsymbol{L}^p(\Omega);\operatorname{curl}\boldsymbol{v}=\boldsymbol{0},\operatorname{div}\boldsymbol{v}=0\text{ in }\Omega,\boldsymbol{v}\cdot\boldsymbol{n}=0\text{ on }\Gamma\},\\ \mathbb{K}_N^p(\Omega) &= \{\boldsymbol{v}\in\boldsymbol{L}^p(\Omega);\operatorname{curl}\boldsymbol{v}=\boldsymbol{0},\operatorname{div}\boldsymbol{v}=0\text{ in }\Omega,\boldsymbol{v}\times\boldsymbol{n}=\boldsymbol{0}\text{ on }\Gamma\}.\end{aligned}$$

Then it is well known that $\dim\mathbb{K}_T^p(\Omega)=J$ and $\dim\mathbb{K}_N^p(\Omega)=I$. In the later, we need the basis of $\mathbb{K}_T^p(\Omega)$. Let $q_j^T\in W^{2,p}(\Omega^\circ)$ be a unique solution of the problem

$$\begin{cases} -\Delta q_j^T=0 & \text{in }\Omega^\circ,\\ \partial_{\boldsymbol{n}}q_j^T=0 & \text{on }\Gamma,\\ \left[q_j^T\right]_{\Sigma_k}=\text{const. and }\left[\partial_{\boldsymbol{n}}q_j^T\right]_{\Sigma_k}=0 & \text{for }k=1,\dots,J,\\ \langle\partial_{\boldsymbol{n}}q_j^T,1\rangle_{\Sigma_k}=\delta_{jk} & \text{for }k=1,\dots,J,\end{cases}\tag{2.3}$$

where $\left[q_j^T\right]_{\Sigma_k}$ denotes the jump of q_j^T across Σ_k. Since $\nabla q_j^T \in \boldsymbol{L}^p(\Omega^\circ)$, it can be extended to a function of $\boldsymbol{L}^p(\Omega)$, and we denote it by $\widetilde{\nabla} q_j^T$. Then the space $\mathbb{K}_T^p(\Omega)$ has a basis $\{\widetilde{\nabla} q_j^T\}_{j=1}^J$ (cf. [4, Corollary 4.1]).

We introduce some spaces of vector functions. If we define

$$\mathbb{X}^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \operatorname{curl} \boldsymbol{v} \in \boldsymbol{L}^p(\Omega), \operatorname{div} \boldsymbol{v} \in L^p(\Omega)\}$$

with the norm

$$\|\boldsymbol{v}\|_{\mathbb{X}^p(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{curl} \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{div} \boldsymbol{v}\|_{L^p(\Omega)},$$

then $\mathbb{X}^p(\Omega)$ is a Banach space. Moreover, we define two closed subspace of $\mathbb{X}^p(\Omega)$ by

$$\begin{aligned} \mathbb{X}_T^p(\Omega) &= \{\boldsymbol{v} \in \mathbb{X}^p(\Omega); \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}, \\ \mathbb{V}_T^p(\Omega) &= \{\boldsymbol{v} \in \mathbb{X}_T^p(\Omega); \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega, \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1\rangle_{\Sigma_j} = 0 \text{ for } j = 1, \dots, J\}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle_{\Sigma_j}$ denotes the duality of $W^{-1/p,p}(\Sigma_j)$ and $W^{1-1/p',p'}(\Sigma_j)$. The following inequality is used frequently (cf. [4]). If we define

$$\mathbb{X}^{1,p}(\Omega) = \{\boldsymbol{v} \in \mathbb{X}^p(\Omega); \boldsymbol{v} \cdot \boldsymbol{n} \in W^{1-1/p,p}(\Gamma)\},$$

then we can see that $\mathbb{X}^{1,p}(\Omega) \subset \boldsymbol{W}^{1,p}(\Omega)$, and there exists a constant $C > 0$ depending only on p and Ω such that for $\boldsymbol{v} \in \mathbb{X}^{1,p}(\Omega)$,

$$\begin{aligned} \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \le C(&\|\operatorname{curl} \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{div} \boldsymbol{v}\|_{L^p(\Omega)} + \|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \\ &+ \|\boldsymbol{v} \cdot \boldsymbol{n}\|_{W^{1-1/p,p}(\Gamma)}). \end{aligned} \tag{2.4}$$

Moreover, we can deduce the following inequality (cf. [4, p. 40]), for every function $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ with $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on Γ,

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \le C(\|\operatorname{curl} \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{div} \boldsymbol{v}\|_{L^p(\Omega)} + \sum_{j=1}^J |\langle \boldsymbol{v} \cdot \boldsymbol{n}, 1\rangle_{\Sigma_j}|). \tag{2.5}$$

Thus we have the following.

Lemma 2.4. *The space $\mathbb{V}_T^p(\Omega)$ is a reflexive Banach space with the norm*

$$\|\boldsymbol{v}\|_{\mathbb{V}_T^p(\Omega)} = \|\operatorname{curl} \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}$$

which is equivalent to the norm $\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}$.

2.2. The main theorem

In this subsection, we consider the following Maxwell-Stokes type system.

$$\begin{cases} \mathrm{curl}\,[S_t(x,|\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u}] + \nabla\pi = \boldsymbol{f} & \text{in } \Omega, \\ \mathrm{div}\,\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = g & \text{on } \Gamma, \\ S_t(x,|\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} & \text{on } \Gamma, \\ \langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j} = 0 & j = 1,\dots J, \end{cases} \tag{2.6}$$

where $\boldsymbol{f}, g$ and $\boldsymbol{h}$ are given functions such that $\boldsymbol{f} \in \mathbb{X}^p_T(\Omega)'$, $g \in W^{1-1/p,p}(\Gamma)$ and $\boldsymbol{h}\times\boldsymbol{n} \in \boldsymbol{W}^{-1/p',p'}(\Gamma)$. When Ω is simply connected, the last conditions of (2.6) are unnecesarry. For $g \in W^{1-1/p,p}(\Gamma)$, define

$$\mathbb{X}^p_T(g,\Omega) = \{\boldsymbol{v} \in \mathbb{X}^p(\Omega); \boldsymbol{v}\cdot\boldsymbol{n} = g\}.$$

Then it follows from (2.4) that $\mathbb{X}^p_T(g,\Omega)$ is a closed convex subset of $\boldsymbol{W}^{1,p}(\Omega)$. Moreover, define

$$\mathbb{V}^p_T(g,\Omega) = \{\boldsymbol{v} \in \mathbb{X}^p_T(g,\Omega) : \mathrm{div}\,\boldsymbol{v} = 0 \text{ in } \Omega, \langle \boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j} = 0, j = 1,\dots J\}.$$

We give the notion of a weak solution of the system (2.6).

Definition 2.5. *We say that* $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega)\times L^{p'}(\Omega)/\mathbb{R}$ *is a weak solution of* (2.6), *if* $\boldsymbol{u} \in \mathbb{V}^p_T(g,\Omega)$ *and* $(\boldsymbol{u},\pi)$ *satisfies*

$$\int_\Omega S_t(x,|\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u}\cdot\mathrm{curl}\,\boldsymbol{v}dx - \int_\Omega \pi\mathrm{div}\,\boldsymbol{v}dx = \langle \boldsymbol{f},\boldsymbol{v}\rangle_\Omega + \langle \boldsymbol{h}\times\boldsymbol{n},\boldsymbol{v}\rangle_\Gamma \tag{2.7}$$

for all $\boldsymbol{v} \in \mathbb{X}^p_T(\Omega)$, *where*

$$\langle \boldsymbol{f},\boldsymbol{v}\rangle_\Omega = \langle \boldsymbol{f},\boldsymbol{v}\rangle_{\mathbb{X}^p_T(\Omega)',\mathbb{X}^p_T(\Omega)}$$

and

$$\langle \boldsymbol{h}\times\boldsymbol{n},\boldsymbol{v}\rangle_\Gamma = \langle \boldsymbol{h}\times\boldsymbol{n},\boldsymbol{v}\rangle_{\boldsymbol{W}^{-1/p',p'}(\Gamma),\boldsymbol{W}^{1-1/p,p}(\Gamma)}.$$

We are in a position to state the main theorem.

Theorem 2.6. *Let* Ω *be a bounded domain in* $\mathbb{R}^3$ *with a* $C^{1,1}$ *boundary* Γ *satisfying (O1) and (O2), and assume that a Carathéodory function* $S(x,t)$ *satisfies the structure conditions* (2.1a)-(2.1c). *Let* $\boldsymbol{f} \in \mathbb{X}^p_T(\Omega)'$, $g \in W^{1-1/p,p}(\Gamma)$ *and* $\boldsymbol{h}\times\boldsymbol{n} \in \boldsymbol{W}^{-1/p',p'}(\Gamma)$. *Then the following compatibility conditions*

$$\langle \boldsymbol{f},\boldsymbol{v}\rangle_\Omega + \langle \boldsymbol{h}\times\boldsymbol{n},\boldsymbol{v}\rangle_\Gamma = 0 \textit{ for all } \boldsymbol{v} \in \mathbb{K}^p_T(\Omega), \tag{2.8}$$

$$\int_\Gamma g dS = 0, \tag{2.9}$$

where dS *denotes the surface measure on* Γ *are necessary and sufficient for the existence of a weak solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^{p'}(\Omega)/\mathbb{R}$ *to the Maxwell-Stokes problem* (2.6). *In this situation, the weak solution is unique and there exists a constant* $C > 0$ *such that*

$$\begin{aligned} &\|\boldsymbol{u}\|^p_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|^{p'}_{L^{p'}(\Omega)/\mathbb{R}} \\ &\qquad \le C(\|\boldsymbol{f}\|^{p'}_{\mathbb{X}^p_T(\Omega)'} + \|g\|^p_{W^{1-1/p,p}(\Gamma)} + \|\boldsymbol{h}\times\boldsymbol{n}\|^{p'}_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}). \end{aligned} \tag{2.10}$$

Remark 2.7. *When* $S(x,t) = t$*, the system* (2.6) *becomes to the system* (1.1), (1.2) *and* (1.5). *Thus in the case where* $p = 2$*, Theorem 2.6 is an extension of the result of [3, Theorem 4.4]. In fact, the authors in [3] assumed that* $\boldsymbol{f} \in \boldsymbol{H}^2_0(\mathrm{div}\,,\Omega)'$*, where*

$$\boldsymbol{H}^2_0(\mathrm{div}\,,\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega); \mathrm{div}\,\boldsymbol{v} \in L^2(\Omega), \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\}.$$

However, since we can easily see that $\mathbb{X}^2_T(\Omega) \hookrightarrow \boldsymbol{H}^2_0(\mathrm{div}\,,\Omega)$*, our result is also an extension to the case where* $\boldsymbol{f} \in \mathbb{X}^2_T(\Omega)'$.

3. PROOF OF THEOREM 2.6

In this section, we derive the proof of Theorem 2.6.

From now on, we write various positive constants depending only on p, λ, Λ and Ω by C which may vary from line to line.

Before beginning to prove Theorem 2.6, we consider the case without the pressure, under more stronger assumptions. Let $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$ satisfy $\mathrm{div}\,\boldsymbol{f} = 0$ in Ω, $g \in W^{1-1/p,p}(\Gamma)$ and $\boldsymbol{h}\times\boldsymbol{n} \in \boldsymbol{W}^{-1/p',p'}(\Gamma)$. We consider the following problem: Find $\boldsymbol{\xi} \in \boldsymbol{W}^{1,p}(\Omega)$ such that

$$\begin{cases} \mathrm{curl}\,[S_t(x,|\mathrm{curl}\,\boldsymbol{\xi}|^2)\mathrm{curl}\,\boldsymbol{\xi}] = \boldsymbol{f} & \text{in } \Omega, \\ \mathrm{div}\,\boldsymbol{\xi} = 0 & \text{in } \Omega, \\ \boldsymbol{\xi}\cdot\boldsymbol{n} = g & \text{on } \Gamma, \\ S_t(x,|\mathrm{curl}\,\boldsymbol{\xi}|^2)\mathrm{curl}\,\boldsymbol{\xi}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} & \text{on } \Gamma, \\ \langle \boldsymbol{\xi}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j} = 0 & \text{for } j = 1,\ldots,J. \end{cases} \tag{3.1}$$

We introduce the compatibility conditions.

$$\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} dx + \langle \boldsymbol{h}\times\boldsymbol{n}, \boldsymbol{v}\rangle_\Gamma = 0 \text{ for all } \boldsymbol{v} \in \mathbb{K}^p_T(\Omega), \tag{3.2}$$

$$\int_\Gamma g dS = 0, \tag{3.3}$$

$$\boldsymbol{f}\cdot\boldsymbol{n} - \operatorname{div}_\Gamma(\boldsymbol{h}\times\boldsymbol{n}) = 0 \text{ on } \Gamma, \tag{3.4}$$

where $\operatorname{div}_\Gamma$ denotes the surface divergence (cf. Mitreau et al. [15, p. 143]).

We say $\boldsymbol{\xi}\in\mathbb{V}_T^p(g,\Omega)$ is a weak solution of (3.1), if $\boldsymbol{\xi}$ satisfies

$$\int_\Omega S_t(x,|\operatorname{curl}\boldsymbol{\xi}|^2)\operatorname{curl}\boldsymbol{\xi}\cdot\operatorname{curl}\boldsymbol{v}dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}dx + \langle\boldsymbol{h}\times\boldsymbol{n},\boldsymbol{v}\rangle_\Gamma \text{ for all } \boldsymbol{v}\in\mathbb{X}_T^p(\Omega). \tag{3.5}$$

We have the following proposition.

Proposition 3.1. *Assume that* $\boldsymbol{f}\in\boldsymbol{L}^{p'}(\Omega)$ *satisfies* $\operatorname{div}\boldsymbol{f}=0$ *in* Ω, $g\in W^{1-1/p,p}(\Gamma)$ *and* $\boldsymbol{h}\times\boldsymbol{n}\in\boldsymbol{W}^{-1/p',p'}(\Gamma)$. *Then the compatibility conditions* (3.2)-(3.4) *are necessary and sufficient for the existence of a weak solution to* (3.1). *In this situation, the solution* $\boldsymbol{\xi}$ *is unique and there exists a constant* $C>0$ *such that*

$$\|\boldsymbol{\xi}\|^p_{\boldsymbol{W}^{1,p}(\Omega)} \le C(\|\boldsymbol{f}\|^{p'}_{\boldsymbol{L}^{p'}(\Omega)} + \|g\|^p_{W^{1-1/p,p}(\Gamma)} + \|\boldsymbol{h}\times\boldsymbol{n}\|^{p'}_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}).$$

Proof. Step 1 (Necessity)

Let $\boldsymbol{\xi}\in\mathbb{V}_T^p(g,\Omega)$ be a weak solution of (3.1). Since

$$\operatorname{curl}[S_t(x,|\operatorname{curl}\boldsymbol{\xi}|^2)\operatorname{curl}\boldsymbol{\xi}] = \boldsymbol{f}\in\boldsymbol{L}^{p'}(\Omega)$$

and

$$\operatorname{div}\operatorname{curl}[S_t(x,|\operatorname{curl}\boldsymbol{\xi}|^2)\operatorname{curl}\boldsymbol{\xi}] = 0 \text{ in } \Omega,$$

the normal trace

$$\boldsymbol{n}\cdot\operatorname{curl}[S_t(x,|\operatorname{curl}\boldsymbol{\xi}|^2)\operatorname{curl}\boldsymbol{\xi}]\in W^{-1/p',p'}(\Gamma)$$

is well defined. By [15, (4.5)], for any $\phi\in W^{2,p}(\Omega)$, we have

$$\begin{aligned}
&\langle\boldsymbol{f}\cdot\boldsymbol{n},\phi\rangle_{W^{-1-1/p',p'}(\Gamma),W^{2-1/p,p}(\Gamma)}\\
&\quad= \langle\boldsymbol{n}\cdot\operatorname{curl}[S_t(x,|\operatorname{curl}\boldsymbol{\xi}|^2)\operatorname{curl}\boldsymbol{\xi}],\phi\rangle_{W^{-1-1/p',p'}(\Gamma),W^{2-1/p,p}(\Gamma)}\\
&\quad= \langle\operatorname{div}_\Gamma(\boldsymbol{h}\times\boldsymbol{n},\phi\rangle_{W^{-1-1/p',p'}(\Gamma),W^{1+1/p',p}(\Gamma)}.
\end{aligned}$$

Thus we have $\boldsymbol{f}\cdot\boldsymbol{n} = \operatorname{div}_\Gamma(\boldsymbol{h}\times\boldsymbol{n})$ in $W^{-1-1/p',p'}(\Gamma)$. Since $\boldsymbol{f}\in\boldsymbol{L}^{p'}(\Omega)$ satisfies $\operatorname{div}\boldsymbol{f}=0$ in Ω, we have $\boldsymbol{f}\cdot\boldsymbol{n}\in W^{-1/p',p'}(\Gamma)$, so (3.4) holds in $W^{-1/p',p'}(\Gamma)$. Since $\operatorname{div}\boldsymbol{\xi}=0$ in Ω and $\boldsymbol{\xi}\cdot\boldsymbol{n}=g$ on Γ, (3.3) easily follows from the divergence theorem. To show (3.2), we consider the following Neumann problem

$$\begin{cases}\Delta\theta = 0 & \text{in } \Omega,\\ \frac{\partial\theta}{\partial\boldsymbol{n}} = g & \text{on } \Gamma.\end{cases} \tag{3.6}$$

Thanks to (3.3) and Girault and Raviart [14, Theorem 1.10], the problem (3.6) has a unique solution $\theta \in W^{2,p}(\Omega)$, up to an additive constant, with the estimate

$$\|\theta\|_{W^{2,p}(\Omega)} \leq C(p,\Omega)\|g\|_{W^{1-1/p,p}(\Gamma)}. \tag{3.7}$$

Define $\boldsymbol{z} = \boldsymbol{\xi} - \nabla\theta \in \boldsymbol{W}^{1,p}(\Omega)$. Since $\operatorname{curl}\boldsymbol{z} = \operatorname{curl}\boldsymbol{\xi}$ in Ω and $\boldsymbol{z}\cdot\boldsymbol{n} = 0$ on Γ, $\boldsymbol{z}$ satisfies the following system.

$$\begin{cases} \operatorname{curl}[S_t(x,|\operatorname{curl}\boldsymbol{z}|^2)\operatorname{curl}\boldsymbol{z}] = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{z} = 0 & \text{in } \Omega, \\ \boldsymbol{z}\cdot\boldsymbol{n} = 0 & \text{on } \Gamma, \\ S_t(x,|\operatorname{curl}\boldsymbol{\xi}|^2)\operatorname{curl}\boldsymbol{\xi}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} & \text{on } \Gamma. \end{cases} \tag{3.8}$$

Hence

$$\int_\Omega S_t(x,|\operatorname{curl}\boldsymbol{z}|^2)\operatorname{curl}\boldsymbol{z}\cdot\operatorname{curl}\boldsymbol{v}dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}dx + \langle\boldsymbol{h}\times\boldsymbol{n},\boldsymbol{v}\rangle_\Gamma \text{ for all } \boldsymbol{v}\in\mathbb{X}^p_T(\Omega). \tag{3.9}$$

In particular, if we take $\boldsymbol{v}\in\mathbb{K}^p_T(\Omega)$ as a test function of (3.9), we see that (3.2) holds.

Step 2 (Sufficiency).

We assume that the compatibility conditions (3.2)-(3.4) hold. We show that (3.8) has a unique weak solution $\boldsymbol{z}\in\boldsymbol{W}^{1,p}(\Omega)$. Then if we define

$$\boldsymbol{\xi} = \boldsymbol{z} + \nabla\theta - \sum_{j=1}^J \langle(\boldsymbol{z}+\nabla\theta)\cdot\boldsymbol{n},1\rangle_{\Sigma_j}\widetilde{\nabla}q_j^T, \tag{3.10}$$

where θ is a unique solution of (3.6), up to an additive constant, then $\operatorname{curl}\boldsymbol{\xi} = \operatorname{curl}\boldsymbol{z}$ in Ω, $\operatorname{div}\boldsymbol{\xi} = 0$ in Ω, $\boldsymbol{\xi}\cdot\boldsymbol{n} = g$ on Γ, $S_t(x,|\operatorname{curl}\boldsymbol{\xi}|^2)\operatorname{curl}\boldsymbol{\xi}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n}$ on Γ, and

$$\begin{aligned} \langle\boldsymbol{\xi}\cdot\boldsymbol{n},1\rangle_{\Sigma_k} &= \langle(\boldsymbol{z}+\nabla\theta)\cdot\boldsymbol{n},1\rangle_{\Sigma_k} - \sum_{j=1}^J\langle(\boldsymbol{z}+\nabla\theta)\cdot\boldsymbol{n},1\rangle_{\Sigma_j}\langle\boldsymbol{n}\cdot\widetilde{\nabla}q_j^T,1\rangle_{\Sigma_k} \\ &= \langle(\boldsymbol{z}+\nabla\theta)\cdot\boldsymbol{n},1\rangle_{\Sigma_k} - \sum_{j=1}^J\delta_{jk}\langle(\boldsymbol{z}+\nabla\theta)\cdot\boldsymbol{n},1\rangle_{\Sigma_j} \\ &= 0 \text{ for every } k = 1,\dots,J. \end{aligned}$$

Hence $\boldsymbol{\xi}$ is a weak solution of (3.1).

In order to show that (3.8) has a unique weak solution, we derive that the following problem has a unique solution: Find $\boldsymbol{z}\in\mathbb{V}^p_T(\Omega)$ such that

$$\int_\Omega S_t(x,|\operatorname{curl}\boldsymbol{z}|^2)\operatorname{curl}\boldsymbol{z}\cdot\operatorname{curl}\boldsymbol{v}dx = \int_\Omega\boldsymbol{f}\cdot\boldsymbol{v}dx + \langle\boldsymbol{h}\times\boldsymbol{n},\boldsymbol{v}\rangle_\Gamma$$

for all $\boldsymbol{v} \in \mathbb{V}^p_T(\Omega)$.

Though we use a minimization problem, taking the proof of Theorem 2.6 into consideration, we introduce a more general minimization problem. We assume that $\boldsymbol{f} \in \mathbb{X}^p_T(\Omega)'$ and $\boldsymbol{h} \times \boldsymbol{n} \in \boldsymbol{W}^{-1/p',p'}(\Gamma)$. We note that if $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$, then $\boldsymbol{f} \in \mathbb{X}^p_T(\Omega)'$. Define a functional on $\mathbb{V}^p_T(\Omega)$ by

$$F[\boldsymbol{v}] = \frac{1}{2}\int_\Omega S(x, |\mathrm{curl}\,\boldsymbol{v}|^2)dx - \langle \boldsymbol{f}, \boldsymbol{v}\rangle_\Omega - \langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{v}\rangle_\Gamma,$$

and put

$$f_* = \inf_{\boldsymbol{v}\in\mathbb{V}^p_T(\Omega)} F[\boldsymbol{v}]. \tag{3.11}$$

Lemma 3.2. *Assume that* $\boldsymbol{f} \in \mathbb{X}^p_T(\Omega)'$ *and* $\boldsymbol{h} \times \boldsymbol{n} \in \boldsymbol{W}^{-1/p',p'}(\Gamma)$. *Then the minimization problem* (3.11) *has a unique minimizer* $\boldsymbol{z} \in \mathbb{V}^p_T(\Omega)$*, that is,*

$$F[\boldsymbol{z}] = f_* = \inf_{\boldsymbol{v}\in\mathbb{V}^p_T(\Omega)} F[\boldsymbol{v}].$$

The minimizer $\boldsymbol{z}$ *is a unique solution of the following variational problem.*

$$\int_\Omega S_t(x, |\mathrm{curl}\,\boldsymbol{z}|^2)\mathrm{curl}\,\boldsymbol{z} \cdot \mathrm{curl}\,\boldsymbol{v} dx = \langle \boldsymbol{f}, \boldsymbol{v}\rangle_\Omega + \langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{v}\rangle_\Gamma \tag{3.12}$$

for all $\boldsymbol{v} \in \mathbb{V}^p_T(\Omega)$.

Moreover, there exists a constant $C > 0$ *such that*

$$\|\boldsymbol{z}\|^p_{\mathbb{V}^p_T(\Omega)} \le C(\|\boldsymbol{f}\|^{p'}_{\mathbb{X}^p_T(\Omega)'} + \|\boldsymbol{h} \times \boldsymbol{n}\|^{p'}_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}). \tag{3.13}$$

Proof. From Lemma 2.4, we know that $\mathbb{V}^p_T(\Omega)$ is a reflexive Banach space with the norm

$$\|\boldsymbol{v}\|_{\mathbb{V}^p_T(\Omega)} = \|\mathrm{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \approx \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}.$$

For any $\boldsymbol{v} \in \mathbb{V}^p_T(\Omega)$, we have

$$|\langle \boldsymbol{f}, \boldsymbol{v}\rangle_\Omega| \le \|\boldsymbol{f}\|_{\mathbb{X}^p_T(\Omega)'}\|\boldsymbol{v}\|_{\mathbb{X}^p_T(\Omega)} = \|\boldsymbol{f}\|_{\mathbb{X}^p_T(\Omega)'}\|\boldsymbol{v}\|_{\mathbb{V}^p_T(\Omega)},$$

and

$$\begin{aligned}
|\langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{v}\rangle_\Gamma| &\le \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}\|\boldsymbol{v}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} \\
&\le C\|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \\
&\le C\|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}\|\boldsymbol{v}\|_{\mathbb{V}^p_T(\Omega)}
\end{aligned}$$

Thus

$$\mathbb{V}_T^p(\Omega) \ni \boldsymbol{v} \mapsto \langle \boldsymbol{f}, \boldsymbol{v} \rangle_\Omega + \langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{v} \rangle_\Gamma$$

defines a functional in $\mathbb{V}_T^p(\Omega)'$. Using the structure condition (2.1b), we see that F is a strictly convex and proper functional. By Aramaki [6], we can see that F is lower semi-continuous. So F is weakly lower semi-continuous. By (2.2) and the Young inequality, for any $\varepsilon > 0$, we have

$$F[\boldsymbol{v}] \geq \frac{\lambda}{p} \|\boldsymbol{v}\|^p_{\mathbb{V}_T^p(\Omega)} - C(\varepsilon)(\|\boldsymbol{f}\|^{p'}_{\mathbb{X}_T^p(\Omega)'} + \|\boldsymbol{h} \times \boldsymbol{n}\|^{p'}_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}) - \varepsilon \|\boldsymbol{v}\|^p_{\mathbb{V}_T^p(\Omega)}.$$

If we choose $\varepsilon > 0$ so that $\varepsilon < 2\lambda/p$, then we can see that F is coercive. Thus it follows from Ekeland and Témam [13, Proposition 1.2] that the minimization problem has a unique minimizer $\boldsymbol{z} \in \mathbb{V}_T^p(\Omega)$. By the Euler-Lagrange equation, the minimizer $\boldsymbol{z} \in \mathbb{V}_T^p(\Omega)$ satisfies (3.12). Taking $\boldsymbol{v} = \boldsymbol{z}$ as a test function of (3.12) and using (2.1a), we get the estimate (3.13). We note that from the strict monotonicity of S_t (Lemma 2.2), the solution of (3.12) is unique. □

We continue the proof of Proposition 3.1. We show that we can extend the space $\mathbb{V}_T^p(\Omega)$ of test function of (3.12) to $\mathbb{X}_T^p(\Omega)$, that is, it shows that $\boldsymbol{z}$ satisfies (3.9). In fact, for any $\widetilde{\boldsymbol{v}} \in \mathbb{X}_T^p(\Omega)$, choose $\chi \in W^{2,p}(\Omega)$ such that

$$\begin{cases} \Delta\chi = \operatorname{div} \widetilde{\boldsymbol{v}} & \text{in } \Omega, \\ \frac{\partial \chi}{\partial \boldsymbol{n}} = 0 & \text{on } \Gamma. \end{cases} \tag{3.14}$$

Since $\int_\Omega \operatorname{div} \widetilde{\boldsymbol{v}} dx = \int_\Gamma \widetilde{\boldsymbol{v}} \cdot \boldsymbol{n} dS = 0$, (3.14) has a unique solution $\chi \in W^{2,p}(\Omega)$, up to an additive constant, Define

$$\boldsymbol{v} = \widetilde{\boldsymbol{v}} - \nabla\chi - \sum_{j=1}^{J} \langle (\widetilde{\boldsymbol{v}} - \nabla\chi) \cdot \boldsymbol{n}, 1 \rangle_{\Sigma_j} \widetilde{\nabla} q_j^T \in \mathbb{V}_T^p(\Omega).$$

Since $\operatorname{curl} \boldsymbol{v} = \operatorname{curl} \widetilde{\boldsymbol{v}}$ in Ω, and $\operatorname{div} \boldsymbol{f} = 0$ in Ω, we have

$$\int_\Omega \boldsymbol{f} \cdot \nabla\chi dx + \langle \boldsymbol{h} \times \boldsymbol{n}, \nabla\chi \rangle_\Gamma = \langle \boldsymbol{f} \cdot \boldsymbol{n} - \operatorname{div}_\Gamma(\boldsymbol{h} \times \boldsymbol{n}), \chi \rangle_\Gamma = 0$$

from (3.3), and

$$\int_\Omega \boldsymbol{f} \cdot \widetilde{\nabla} q_j^T dx + \langle \boldsymbol{h} \times \boldsymbol{n}, \widetilde{\nabla} q_j^T \rangle_\Gamma = 0$$

from (3.1), so we can see that $\boldsymbol{z}$ satisfies (3.9).

Step 3 (The uniqueness of a weak solution).

Let $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ in $\mathbb{V}_T^p(g,\Omega)$ be two weak solutions of (3.1). Then we can take $\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 \in \mathbb{V}_T^p(\Omega) \subset \mathbb{X}_T^p(\Omega)$ as a test function of (3.5). Thus

$$\int_\Omega S_t(x, |\mathrm{curl}\,\boldsymbol{\xi}_i|^2)\mathrm{curl}\,\boldsymbol{\xi}_i \cdot \mathrm{curl}\,(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)dx = \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)dx + \langle \boldsymbol{h}\times\boldsymbol{n}, \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\rangle_\Gamma$$

for $i = 1, 2$. Hence

$$\int_\Omega \big(S_t(x, |\mathrm{curl}\,\boldsymbol{\xi}_1|^2)\mathrm{curl}\,\boldsymbol{\xi}_1 - S_t(x, |\mathrm{curl}\,\boldsymbol{\xi}_2|^2)\mathrm{curl}\,\boldsymbol{\xi}_2\big) \cdot \mathrm{curl}\,(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)dx = 0.$$

From the strict monotonicity (Lemma 2.2), we have $\mathrm{curl}\,(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) = \boldsymbol{0}$ in Ω. Since $\mathrm{div}\,(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) = 0$ in Ω, $(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)\cdot\boldsymbol{n} = 0$ on Γ and $\langle \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\rangle_{\Sigma_j} = 0$ for $j = 1, \dots, J$, it follows from (2.5) that $\boldsymbol{\xi}_1 = \boldsymbol{\xi}_2$.

Step 4 (Estimate).

From (3.10) and (3.7), we have

$$\begin{aligned}\|\boldsymbol{\xi}\|^p_{\boldsymbol{W}^{1,p}(\Omega)} &\le C(\|\boldsymbol{z}\|^p_{\boldsymbol{W}^{1,p}(\Omega)} + \|\nabla\theta\|^p_{\boldsymbol{W}^{1,p}(\Omega)})\\ &\le C(\|\boldsymbol{z}\|^p_{\boldsymbol{W}^{1,p}(\Omega)} + \|g\|^p_{W^{1-1/p,p}(\Gamma)}).\end{aligned}$$

From (3.9) with $\boldsymbol{v} = \boldsymbol{z}$, we have

$$\lambda\|\mathrm{curl}\,\boldsymbol{z}\|^p_{\boldsymbol{L}^p(\Omega)} \le C(\varepsilon)(\|\boldsymbol{f}\|^{p'}_{\boldsymbol{L}^{p'}(\Omega)} + \|\boldsymbol{h}\times\boldsymbol{n}\|^{p'}_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}) + \varepsilon\|\boldsymbol{z}\|^p_{\mathbb{V}_T^p(\Omega)}.$$

If we choose $\varepsilon = \lambda/2$, then we have

$$\frac{\lambda}{2}\|\mathrm{curl}\,\boldsymbol{z}\|^p_{\boldsymbol{L}^p(\Omega)} \le C(\varepsilon)(\|\boldsymbol{f}\|^{p'}_{\boldsymbol{L}^{p'}(\Omega)} + \|\boldsymbol{h}\times\boldsymbol{n}\|^{p'}_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}).$$

Thus we get the estimate. This completes the proof of Proposition 3.1. $\square$

Proof of Theorem 2.6.

Step 1 (Necessity).

Let $(\boldsymbol{u}, \pi) \in \mathbb{V}_T^p(g,\Omega) \times L^{p'}(\Omega)/\mathbb{R}$ be a weak solution of (2.6). Since $\mathrm{div}\,\boldsymbol{u} = 0$ in Ω and $\boldsymbol{u}\cdot\boldsymbol{n} = g$ on Γ, it follows from the divergence theorem that (2.9) holds. For (2.8), since $(\boldsymbol{u}, \pi) \in \mathbb{V}_T^p(g,\Omega) \times L^{p'}(\Omega)/\mathbb{R}$ is a weak solution of (2.6), we can see that (2.7) holds. If we choose $\boldsymbol{v} \in \mathbb{K}_T^p(\Omega)$ as a test function of (2.7), we get (2.8).

Step 2 (Sufficiency).

Assume that $\boldsymbol{f} \in \mathbb{X}_T^p(\Omega)', g \in W^{1-1/p,p}(\Gamma)$ and $\boldsymbol{h}\times\boldsymbol{n} \in \boldsymbol{W}^{-1/p',p'}(\Gamma)$ satisfy (2.8) and (2.9). For any $\boldsymbol{v} \in \mathbb{V}_T^p(\Omega)$,

$$|\langle \boldsymbol{f}, \boldsymbol{v}\rangle_\Omega| \le \|\boldsymbol{f}\|_{\mathbb{X}_T^p(\Omega)'}\|\boldsymbol{v}\|_{\mathbb{X}_T^p(\Omega)} = \|\boldsymbol{f}\|_{\mathbb{X}_T^p(\Omega)'}\|\boldsymbol{v}\|_{\mathbb{V}_T^p(\Omega)}.$$

We consider the following problem: Find $\boldsymbol{z} \in \mathbb{V}^p_T(\Omega)$ such that

$$\int_\Omega S_t(x, |\operatorname{curl} \boldsymbol{z}|^2)\operatorname{curl} \boldsymbol{z} \cdot \operatorname{curl} \boldsymbol{v} dx = \langle \boldsymbol{f}, \boldsymbol{v}\rangle_\Omega + \langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{v}\rangle_\Gamma \tag{3.15}$$

for all $\boldsymbol{v} \in \mathbb{V}^p_T(\Omega)$. It follows from Lemma 3.2 and (3.12) that (3.15) has a unique solution $\boldsymbol{z} \in \mathbb{V}^p_T(\Omega)$, and there exists a constant $C > 0$ such that

$$\|\boldsymbol{z}\|^p_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(\|\boldsymbol{f}\|^{p'}_{\mathbb{X}^p_T(\Omega)'} + \|\boldsymbol{h} \times \boldsymbol{n}\|^{p'}_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}). \tag{3.16}$$

We extend the space $\mathbb{V}^p_T(\Omega)$ of test functions in (3.15) to any $\widetilde{\boldsymbol{v}} \in \mathbb{X}^p_T(\Omega)$ with $\operatorname{div} \widetilde{\boldsymbol{v}} = 0$ in Ω. Indeed, for any $\widetilde{\boldsymbol{v}} \in \mathbb{X}^p_T(\Omega)$ with $\operatorname{div} \widetilde{\boldsymbol{v}} = 0$ in Ω, define

$$\boldsymbol{v} = \widetilde{\boldsymbol{v}} - \sum_{j=1}^J \langle \widetilde{\boldsymbol{v}} \cdot \boldsymbol{n}, 1\rangle_{\Sigma_j} \widetilde{\nabla} q^T_j.$$

Then $\boldsymbol{v} \in \mathbb{V}^p_T(\Omega)$ and $\operatorname{curl} \boldsymbol{v} = \operatorname{curl} \widetilde{\boldsymbol{v}}$ in Ω. From the compatibility condition (2.8), we have

$$\langle \boldsymbol{f}, \widetilde{\nabla} q^T_j\rangle_\Omega + \langle \boldsymbol{h} \times \boldsymbol{n}, \widetilde{\nabla} q^T_j\rangle_\Gamma = 0.$$

Therefore, we have

$$\int_\Omega S_t(x, |\operatorname{curl} \boldsymbol{z}|^2)\operatorname{curl} \boldsymbol{z} \cdot \operatorname{curl} \widetilde{\boldsymbol{v}} dx = \langle \boldsymbol{f}, \widetilde{\boldsymbol{v}}\rangle_\Omega + \langle \boldsymbol{h} \times \boldsymbol{n}, \widetilde{\boldsymbol{v}}\rangle_\Gamma \tag{3.17}$$

for all $\widetilde{\boldsymbol{v}} \in \mathbb{X}^p_T(\Omega)$ with $\operatorname{div} \widetilde{\boldsymbol{v}} = 0$ in Ω. Taking $\boldsymbol{\phi} \in \boldsymbol{D}_\sigma(\Omega) = \{\boldsymbol{v} \in \boldsymbol{D}(\Omega); \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\}$, where $\boldsymbol{D}(\Omega)$ is the space of functions in $\boldsymbol{C}^\infty(\Omega)$ with compact support in Ω, as a space of test functions of (3.17), we have

$$\langle \operatorname{curl} [S_t(x, |\operatorname{curl} \boldsymbol{z}|^2)\operatorname{curl} \boldsymbol{z}] - \boldsymbol{f}, \boldsymbol{\phi}\rangle_{\boldsymbol{D}'(\Omega),\boldsymbol{D}(\Omega)} = 0 \text{ for all } \boldsymbol{\phi} \in \boldsymbol{D}_\sigma(\Omega).$$

for all $\boldsymbol{\phi} \in \boldsymbol{D}_\sigma(\Omega)$. Hence, from the De Rham theorem (cf. Boyer and Fabrie [10, Theorem IV.2.4]), there exists $\pi \in L^{p'}(\Omega)$ such that

$$\operatorname{curl} [S_t(x, |\operatorname{curl} \boldsymbol{z}|^2)\operatorname{curl} \boldsymbol{z}] + \nabla\pi = \boldsymbol{f} \text{ in } \Omega.$$

Since $\nabla\pi \in \mathbb{X}^p_T(\Omega)'$ from (A.1) in Appendix, we have

$$\operatorname{curl} [S_t(x, |\operatorname{curl} \boldsymbol{z}|^2)\operatorname{curl} \boldsymbol{z}] = \boldsymbol{f} - \nabla\pi \in \mathbb{X}^p_T(\Omega)'.$$

Hence, for any $\boldsymbol{v} \in \mathbb{X}^p_T(\Omega)$ with $\operatorname{div} \boldsymbol{v} = 0$ in Ω, it follows from the Green formula (cf. (A.2) in Appendix) that

$$\begin{aligned}
&\int_\Omega S_t(x, |\operatorname{curl} \boldsymbol{z}|^2)\operatorname{curl} \boldsymbol{z} \cdot \operatorname{curl} \boldsymbol{v} dx \\
&\qquad = \langle \operatorname{curl} [S_t(x, |\operatorname{curl} \boldsymbol{z}|^2)\operatorname{curl} \boldsymbol{z}], \boldsymbol{v}\rangle_\Omega + \langle S_t(x, |\operatorname{curl} \boldsymbol{z}|^2)\operatorname{curl} \boldsymbol{z} \times \boldsymbol{n}, \boldsymbol{v}\rangle_\Gamma \\
&\qquad = \langle \boldsymbol{f}, \boldsymbol{v}\rangle_\Omega - \langle \nabla\pi, \boldsymbol{v}\rangle_\Omega + \langle S_t(x, |\operatorname{curl} \boldsymbol{z}|^2)\operatorname{curl} \boldsymbol{z} \times \boldsymbol{n}, \boldsymbol{v}\rangle_\Gamma.
\end{aligned}$$

Since $\langle \nabla\pi, \boldsymbol{v}\rangle_\Omega = 0$ for any $\boldsymbol{v} \in \mathbb{X}^p_T(\Omega)$ with $\operatorname{div} \boldsymbol{v} = 0$ in Ω, from (3.17), we have

$$\langle S_t(x, |\operatorname{curl} \boldsymbol{z}|^2)\operatorname{curl} \boldsymbol{z} \times \boldsymbol{n}, \boldsymbol{v}\rangle_\Gamma = \langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{v}\rangle_\Gamma$$

for all $\boldsymbol{v} \in \mathbb{X}^p_T(\Omega)$ with $\operatorname{div} \boldsymbol{v} = 0$ in Ω. This implies that

$$S_t(x, |\operatorname{curl} \boldsymbol{z}|^2)\operatorname{curl} \boldsymbol{z} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} \text{ on } \Gamma.$$

Thus $(\boldsymbol{z}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^{p'}(\Omega)/\mathbb{R}$ is a weak solution of

$$\begin{cases} \operatorname{curl}[S_t(x, |\operatorname{curl} \boldsymbol{z}|^2)\operatorname{curl} \boldsymbol{z}] + \nabla\pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{z} = 0 & \text{in } \Omega, \\ \boldsymbol{z} \cdot \boldsymbol{n} = 0 & \text{on } \Gamma, \\ S_t(x, |\operatorname{curl} \boldsymbol{z}|^2)\operatorname{curl} \boldsymbol{z} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} & \text{on } \Gamma. \end{cases} \tag{3.18}$$

Define

$$\boldsymbol{u} = \boldsymbol{z} + \nabla\theta - \sum_{j=1}^{J} \langle(\boldsymbol{z} + \nabla\theta) \cdot \boldsymbol{n}, 1\rangle_{\Sigma_j} \widetilde{\nabla} q_j^T, \tag{3.19}$$

where $\theta \in W^{2,p}(\Omega)$ is a solution of (3.6). Here we note that we use (2.9) for the existence of solution θ. Then $\operatorname{curl} \boldsymbol{u} = \operatorname{curl} \boldsymbol{z}$ in Ω, $\operatorname{div} \boldsymbol{u} = 0$ in Ω and $\boldsymbol{u} \cdot \boldsymbol{n} = \frac{\partial\theta}{\partial\boldsymbol{n}} = g$ on Γ. Since $\langle \boldsymbol{n} \cdot \widetilde{\nabla} q_j^T, 1\rangle_{\Sigma_k} = \delta_{jk}$, we have $\langle \boldsymbol{u} \cdot \boldsymbol{n}, 1\rangle_{\Sigma_k} = 0$ for $k = 1, \ldots, J$. Therefore, $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^{p'}(\Omega)/\mathbb{R}$ is a weak solution of (2.6).

Step 3 (Uniqueness of a weak solution).

Let $(\boldsymbol{u}_1, \pi_1), (\boldsymbol{u}_2, \pi_2) \in \mathbb{V}^p_T(g, \Omega) \times L^{p'}(\Omega)/\mathbb{R}$ be two weak solutions of (2.6). Then, from (2.7), we have

$$\begin{aligned} &\int_\Omega S_t(x, |\operatorname{curl} \boldsymbol{u}_i|^2)\operatorname{curl} \boldsymbol{u}_i \cdot \operatorname{curl} \boldsymbol{v} dx - \int_\Omega \pi_i \operatorname{div} \boldsymbol{v} dx \\ &\qquad = \langle \boldsymbol{f}, \boldsymbol{v}\rangle_\Omega + \langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{v}\rangle_\Gamma \text{ for all } \boldsymbol{v} \in \mathbb{X}^p_T(\Omega) \text{ and } i = 1, 2. \end{aligned} \tag{3.20}$$

Since $(\boldsymbol{u}_i, \pi_i)$ satisfies the first equation of (2.6) in the distribution sense, we have

$$\begin{cases} \Delta\pi_i = \operatorname{div} \boldsymbol{f} & \text{in } \Omega, \\ \frac{\partial\pi_i}{\partial\boldsymbol{n}} = \boldsymbol{f} \cdot \boldsymbol{n} - \operatorname{div}_\Gamma(\boldsymbol{h} \times \boldsymbol{n}) & \text{on } \Gamma, \end{cases}$$

hence,

$$\begin{cases} \Delta(\pi_1 - \pi_2) = 0 & \text{in } \Omega, \\ \frac{\partial}{\partial\boldsymbol{n}}(\pi_1 - \pi_2) = 0 & \text{on } \Gamma. \end{cases}$$

Therefore, $\pi_1 - \pi_2$ is a constant, i.e., $\pi_1 = \pi_2$ in $L^{p'}(\Omega)/\mathbb{R}$. Moreover, we have

$$\int_\Omega (\pi_1 - \pi_2)\operatorname{div} \boldsymbol{v} dx = \text{const.} \int_\Omega \operatorname{div} \boldsymbol{v} dx = \text{const.} \int_\Gamma \boldsymbol{v} \cdot \boldsymbol{n} dS = 0$$

for all $\boldsymbol{v} \in \mathbb{X}^p_T(\Omega)$. Taking $\boldsymbol{u}_1 - \boldsymbol{u}_2 \in \mathbb{V}^p_T(\Omega) \subset \mathbb{X}^p_T(\Omega)$ as a test function of (3.20), we can see that

$$\int_\Omega \big(S_t(x, |\mathrm{curl}\,\boldsymbol{u}_1|^2)\mathrm{curl}\,\boldsymbol{u}_1 - S_t(x, |\mathrm{curl}\,\boldsymbol{u}_2|^2)\mathrm{curl}\,\boldsymbol{u}_2\big) \cdot (\mathrm{curl}\,\boldsymbol{u}_1 - \mathrm{curl}\,\boldsymbol{u}_2)dx = 0$$

Since S_t is strictly monotone (Lemma 2.2), we have $\mathrm{curl}\,(\boldsymbol{u}_1 - \boldsymbol{u}_2) = \boldsymbol{0}$ in Ω. Since $\boldsymbol{u}_1 - \boldsymbol{u}_2 \in \mathbb{V}^p_T(\Omega)$, we have $\boldsymbol{u}_1 = \boldsymbol{u}_2$ in Ω.

Step 4 (Estimate).

According to (3.19), we can write

$$\boldsymbol{u} = \boldsymbol{z} + \nabla\theta - \sum_{j=1}^{J} \langle(\boldsymbol{z} + \nabla\theta) \cdot \boldsymbol{n}, 1\rangle_{\Sigma_j} \widetilde{\nabla} q_j^T,$$

where $\boldsymbol{z}$ is a solution of (3.18) and θ is a solution of (3.6). Therefore, from (3.16) and (3.7), we have

$$\|\boldsymbol{u}\|^p_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(\|\boldsymbol{f}\|^{p'}_{\mathbb{X}^p_T(\Omega)} + \|\boldsymbol{h} \times \boldsymbol{n}\|^{p'}_{\boldsymbol{W}^{-1/p',p'}(\Gamma)} + \|g\|^p_{W^{1-1/p,p}(\Gamma)}).$$

On the other hand, from Amrouche and Girault [1, p. 114], we can see that

$$\|\pi\|_{L^{p'}(\Omega)/\mathbb{R}} \leq C\|\nabla\pi\|_{\boldsymbol{W}^{-1,p'}(\Omega)}.$$

Since $\mathbb{X}^p_T(\Omega)' \hookrightarrow W^{-1,p'}(\Omega)$ and $\nabla\pi \in \mathbb{X}^p_T(\Omega)'$, using the first equation of (2.6), we have

$$\begin{aligned}
\|\pi\|_{L^{p'}(\Omega)/\mathbb{R}} &\leq C(\|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,p'}(\Omega)} + \|\mathrm{curl}\,[S_t(x, |\mathrm{curl}\,\boldsymbol{z}|^2)\mathrm{curl}\,\boldsymbol{z}]\|_{\boldsymbol{W}^{-1,p'}(\Omega)}) \\
&\leq C(\|\boldsymbol{f}\|_{\mathbb{X}^p_T(\Omega)'} + \|S_t(x, |\mathrm{curl}\,\boldsymbol{z}|^2)\mathrm{curl}\,\boldsymbol{z}\|_{\boldsymbol{L}^{p'}(\Omega)}) \\
&\leq C(\|\boldsymbol{f}\|_{\mathbb{X}^p_T(\Omega)'} + \|\boldsymbol{z}\|^{p-1}_{\boldsymbol{W}^{1,p}(\Omega)}).
\end{aligned}$$

Hence, using again (3.16), we can see that

$$\|\pi\|^{p'}_{L^{p'}(\Omega)/\mathbb{R}} \leq C(\|\boldsymbol{f}\|^{p'}_{\mathbb{X}^p_T(\Omega)'} + \|\boldsymbol{z}\|^p_{\boldsymbol{W}^{1,p}(\Omega)}) \leq C(\|\boldsymbol{f}\|^{p'}_{\mathbb{X}^p_T(\Omega)'} + \|\boldsymbol{h} \times \boldsymbol{n}\|^{p'}_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}).$$

This completes the proof of Theorem 2.6.

Remark 3.3. *If we suppose in Theorem 2.6 that $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$ with $\mathrm{div}\,\boldsymbol{f} = 0$ in Ω and we add the compatibility conditions* (3.4)*, then the pressure π is constant. Indeed, from the first equation of* (2.6)*, we have*

$$\begin{cases} \Delta\pi = 0 & \text{in } \Omega, \\ \frac{\partial\pi}{\partial\boldsymbol{n}} = \boldsymbol{f} \cdot \boldsymbol{n} - \mathrm{div}_\Gamma(\boldsymbol{h} \times \boldsymbol{n}) = 0 & \text{on } \Gamma. \end{cases}$$

This implies that π is a constant and the Maxwell-Stokes problem (2.6) *in nothing other than problem* (3.1).

Finally, let $\boldsymbol{f} \in \mathbb{X}_T^p(\Omega)', g \in W^{1-1/p,p}(\Gamma), \boldsymbol{h} \times \boldsymbol{n} \in \boldsymbol{W}^{1-/p',p'}(\Gamma)$ and $\chi \in L^p(\Omega)$. We consider a slightly more general equation than (2.6).

$$\begin{cases} \operatorname{curl}[S_t(x, |\operatorname{curl}\boldsymbol{u}|^2)\operatorname{curl}\boldsymbol{u}] + \nabla\pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = \chi & \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = g & \text{on } \Gamma, \\ S_t(x, |\operatorname{curl}\boldsymbol{u}|^2)\operatorname{curl}\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} & \text{on } \Gamma, \\ \langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j} = 0 & j = 1, \ldots, J. \end{cases} \tag{3.21}$$

We impose the compatibility conditions (2.8) and

$$\int_\Gamma g dS = \int_\Omega \chi dx. \tag{3.22}$$

Then we have the following proposition.

Proposition 3.4. *Assume that* $\boldsymbol{f} \in \mathbb{X}_T^p(\Omega)', g \in W^{1-1/p,p}(\Gamma), \boldsymbol{h} \times \boldsymbol{n} \in \boldsymbol{W}^{-1/p',p'}(\Gamma)$ *and* $\chi \in L^p(\Omega)$. *Then the compatibility conditions* (2.8) *and* (3.22) *are necessary and sufficient for the existence of a weak solution* $(\widetilde{\boldsymbol{u}}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^{p'}(\Omega)/\mathbb{R}$. *In this situation, the weak solution is unique, and there exists a constant* $C > 0$ *such that*

$$\begin{aligned} &\|\widetilde{\boldsymbol{u}}\|^p_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|^{p'}_{L^{p'}(\Omega)} \\ &\qquad \le C(\|\boldsymbol{f}\|^{p'}_{\mathbb{X}_T^p(\Omega)'} + \|\boldsymbol{h}\times\boldsymbol{n}\|^{p'}_{\boldsymbol{W}^{-1/p',p'}(\Gamma)} + \|\chi\|^p_{L^p(\Omega)} + \|g\|^p_{W^{1-1/p,p}(\Gamma)}). \end{aligned}$$

Proof. According to the proof of Theorem 2.6, it suffices to prove the existence of a weak solution. Let $(\boldsymbol{u}, \pi)$ be a unique solution of (2.6) with $g = 0$. Thanks to (3.22), the following Neumann problem

$$\begin{cases} \Delta\phi = \chi & \text{in } \Omega, \\ \frac{\partial\phi}{\partial\boldsymbol{n}} = g & \text{on } \Gamma \end{cases}$$

has a unique solution $\phi \in W^{2,p}(\Omega)$, up to an additive constant, and there exists a constant $C > 0$ such that

$$\|\phi\|_{W^{2,p}(\Omega)} \le C(\|\chi\|_{L^p(\Omega)} + \|g\|_{W^{1-1/p,p}(\Gamma)}). \tag{3.23}$$

Define

$$\widetilde{\boldsymbol{u}} = \boldsymbol{u} + \nabla\phi - \sum_{j=1}^J \langle(\boldsymbol{u} + \nabla\phi)\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j}\widetilde{\nabla}q_j^T.$$

Then we have $\operatorname{curl}\widetilde{\boldsymbol{u}} = \operatorname{curl}\boldsymbol{u}$ in Ω, $\operatorname{div}\widetilde{\boldsymbol{u}} = \operatorname{div}\boldsymbol{u} + \Delta\phi = \chi$ in Ω, $\widetilde{\boldsymbol{u}}\cdot\boldsymbol{n} = \boldsymbol{u}\cdot\boldsymbol{n} + \frac{\partial\phi}{\partial\boldsymbol{n}} = g$ on Γ and $\langle\widetilde{\boldsymbol{u}}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j} = 0$ for $j = 1, \ldots, J$. The estimate follows from Theorem 2.6 and (3.23). $\square$

4. CONTINUOUS DEPENDENCE OF A WEAK SOLUTION ON THE DATA

In this section, we consult the continuous dependence of a weak solution of (2.6) on the data. In order to do so, for every $n = 0, 1, \ldots$, assume that $S^{(n)}(x,t)$ satisfies (2.1a)-(2.1c) with the same constants λ and Λ. Moreover, assume that $\boldsymbol{f}_n \in \mathbb{X}^p_T(\Omega)'$, $g_n \in W^{1-1/p,p}(\Gamma)$ and $\boldsymbol{h}_n \times \boldsymbol{n} \in \boldsymbol{W}^{-1/p',p'}(\Gamma)$ satisfy the compatibility conditions (2.8) and (2.9) with $\boldsymbol{f} = \boldsymbol{f}_n$, $g = g_n$ and $\boldsymbol{h} = \boldsymbol{h}_n$ for $n = 0, 1, \ldots$. Let $(\boldsymbol{u}_n, \pi_n) \in \boldsymbol{W}^{1,p}(\Omega) \times L^{p'}(\Omega)/\mathbb{R}$ be a unique weak solution of (2.6), i.e.,

$$\begin{cases} \operatorname{curl}[S_t^{(n)}(x, |\operatorname{curl}\boldsymbol{u}_n|^2)\operatorname{curl}\boldsymbol{u}_n] + \nabla\pi_n = \boldsymbol{f}_n & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u}_n = 0 & \text{in } \Omega, \\ \boldsymbol{u}_n \cdot \boldsymbol{n} = g_n & \text{on } \Gamma, \\ S_t^{(n)}(x, |\operatorname{curl}\boldsymbol{u}_n|^2)\operatorname{curl}\boldsymbol{u}_n \times \boldsymbol{n} = \boldsymbol{h}_n \times \boldsymbol{n} & \text{on } \Gamma, \\ \langle \boldsymbol{u}_n \cdot \boldsymbol{n}, 1\rangle_{\Sigma_j} = 0 & j = 1, \ldots, J. \end{cases} \tag{4.1}$$

for every $n = 0, 1, \ldots$. Thus $(\boldsymbol{u}_n, \pi_n)$ satisfies

$$\int_\Omega S_t^{(n)}(x, |\operatorname{curl}\boldsymbol{u}_n|^2)\operatorname{curl}\boldsymbol{u}_n \cdot \operatorname{curl}\boldsymbol{v}dx - \int_\Omega \pi_n \operatorname{div}\boldsymbol{v}dx \\ = \langle \boldsymbol{f}_n, \boldsymbol{v}\rangle_\Omega + \langle \boldsymbol{h}_n \times \boldsymbol{n}, \boldsymbol{v}\rangle_\Gamma \text{ for all } \boldsymbol{v} \in \mathbb{X}^p_T(\Omega). \tag{4.2}$$

Then we have the following theorem on the continuous dependence on the data.

Theorem 4.1. *We assume that for every $n = 0, 1, \ldots$, a Carathéodory function $S^{(n)}(x,t)$ satisfies* (2.1a)-(2.1c) *with the same constants λ and Λ, and assume that $\boldsymbol{f}_n \in \mathbb{X}^p_T(\Omega)'$, $g_n \in W^{1-1/p,p}(\Omega)$ and $\boldsymbol{h}_n \times \boldsymbol{n} \in \boldsymbol{W}^{-1/p',p'}(\Gamma)$ satisfy the compatibility conditions* (2.8) *and* (2.9) *with $\boldsymbol{f} = \boldsymbol{f}_n, g = g_n$ and $\boldsymbol{h} = \boldsymbol{h}_n$. Let $(\boldsymbol{u}_n, \pi_n) \in \boldsymbol{W}^{1,p}(\Omega) \times L^{p'}(\Omega)/\mathbb{R}$ be a unique weak solution of* (4.1)*. If $S_t^{(n)}(x,t) \to S_t^{(0)}(x,t)$ a.e. in $\Omega \times [0,\infty)$, and $\boldsymbol{f}_n \to \boldsymbol{f}_0$ in $\mathbb{X}^p_T(\Omega)'$, $g_n \to g_0$ in $W^{1-1/p,p}(\Gamma)$ and $\boldsymbol{h}_n \times \boldsymbol{n} \to \boldsymbol{h}_0 \times \boldsymbol{n}$ in $\boldsymbol{W}^{-1/p',p'}(\Gamma)$ as $n \to \infty$, then $\boldsymbol{u}_n \to \boldsymbol{u}_0$ in $\boldsymbol{W}^{1,p}(\Omega)$ and $\pi_n \to \pi_0$ in $L^{p'}(\Omega)/\mathbb{R}$ as $n \to \infty$.*

In the particular case where $S^{(n)}(x,t) = S^{(0)}(x,t)$ for all $n = 1.2, \ldots$, there exists a constant $C > 0$ depending only on $p, \lambda, \Lambda, \Omega$, $\|\boldsymbol{f}_0\|_{\mathbb{X}^p_T(\Omega)'}$ and $\|\boldsymbol{h}_0 \times \boldsymbol{n}\|_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}$ such that for large n,

$$\|\boldsymbol{u}_n - \boldsymbol{u}_0\|^{p\vee p'}_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi_n - \pi_0\|^{p\vee p'}_{L^{p'}(\Omega)/\mathbb{R}} \\ \le C(\|\boldsymbol{f}_n - \boldsymbol{f}\|^{p'}_{\mathbb{X}^p_T(\Omega)'} + \|\boldsymbol{f}_n - \boldsymbol{f}\|^{p}_{\mathbb{X}^p_T(\Omega)'} + \|g_n - g_0\|^{p\vee p'}_{W^{1-1/p,p}(\Gamma)} \\ + \|\boldsymbol{h}_n \times \boldsymbol{n} - \boldsymbol{h}_0 \times \boldsymbol{n}\|^{p'}_{\boldsymbol{W}^{-1/p',p'}(\Gamma)} + \|\boldsymbol{h}_n \times \boldsymbol{n} - \boldsymbol{h}_0 \times \boldsymbol{n}\|^{p}_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}),$$

where $p \vee p' = \max\{p, p'\}$.

Proof. First we consider the following system.

$$\begin{cases} \mathrm{curl}\,[S_t^{(n)}(x,|\mathrm{curl}\,\boldsymbol{w}_n|^2)\mathrm{curl}\,\boldsymbol{w}_n]+\nabla\pi_n=\boldsymbol{f}_n & \text{in }\Omega,\\ \mathrm{div}\,\boldsymbol{w}_n=0 & \text{in }\Omega,\\ \boldsymbol{w}_n\cdot\boldsymbol{n}=0 & \text{on }\Gamma,\\ S_t^{(n)}(x,|\mathrm{curl}\,\boldsymbol{w}_n|^2)\mathrm{curl}\,\boldsymbol{w}_n\times\boldsymbol{n}=\boldsymbol{h}_n\times\boldsymbol{n} & \text{on }\Gamma,\\ \langle\boldsymbol{w}_n\cdot\boldsymbol{n},1\rangle_{\Sigma_j}=0 & j=1,\dots,J. \end{cases} \tag{4.3}$$

By Theorem 2.6, the system (4.4) has a unique weak solution $(\boldsymbol{w}_n,\pi_n)\in\boldsymbol{W}^{1,p}(\Omega)\times L^{p'}(\Omega)/\mathbb{R}$ satisfying (4.2). Taking $\boldsymbol{v}=\boldsymbol{w}_n-\boldsymbol{w}_0\in\mathbb{V}_T^p(\Omega)\hookrightarrow\mathbb{X}_T^p(\Omega)$ as a test function of (4.2) and noting that $\mathrm{div}\,(\boldsymbol{w}_n-\boldsymbol{w}_0)=0$ in Ω, we have

$$\int_\Omega\big(S_t^{(n)}(x,|\mathrm{curl}\,\boldsymbol{w}_n|^2)\mathrm{curl}\,\boldsymbol{w}_n-S_t^{(0)}(x,|\mathrm{curl}\,\boldsymbol{w}_0|^2)\mathrm{curl}\,\boldsymbol{w}_0\big)\cdot\mathrm{curl}\,(\boldsymbol{w}_n-\boldsymbol{w}_0)dx$$
$$=\langle\boldsymbol{f}_n-\boldsymbol{f}_0,\boldsymbol{w}_n-\boldsymbol{w}_0\rangle_\Omega+\langle(\boldsymbol{h}_n-\boldsymbol{h}_0)\times\boldsymbol{n},\boldsymbol{w}_n-\boldsymbol{w}_0\rangle_\Gamma.$$

We write this equality into the form

$$\int_\Omega\big(S_t^{(n)}(x,|\mathrm{curl}\,\boldsymbol{w}_n|^2)\mathrm{curl}\,\boldsymbol{w}_n$$
$$-S_t^{(n)}(x,|\mathrm{curl}\,\boldsymbol{w}_0|^2)\mathrm{curl}\,\boldsymbol{w}_0\big)\cdot\mathrm{curl}\,(\boldsymbol{w}_n-\boldsymbol{w}_0)dx=I_1-I_2, \tag{4.4}$$

where

$$I_1=\langle\boldsymbol{f}_n-\boldsymbol{f}_0,\boldsymbol{w}_n-\boldsymbol{w}_0\rangle_\Omega+\langle(\boldsymbol{h}_n-\boldsymbol{h}_0)\times\boldsymbol{n},\boldsymbol{w}_n-\boldsymbol{w}_0\rangle_\Gamma,$$
$$I_2=\int_\Omega\big(S_t^{(n)}(x,|\mathrm{curl}\,\boldsymbol{w}_0|^2)\mathrm{curl}\,\boldsymbol{w}_0$$
$$-S_t^{(0)}(x,|\mathrm{curl}\,\boldsymbol{w}_0|^2)\mathrm{curl}\,\boldsymbol{w}_0\big)\cdot\mathrm{curl}\,(\boldsymbol{w}_n-\boldsymbol{w}_0)dx.$$

We estimate $|I_1|$ and $|I_2|$ from above. We have

$$\begin{aligned}|I_1| &\le \|\boldsymbol{f}_n-\boldsymbol{f}_0\|_{\mathbb{X}_T^p(\Omega)'}\|\boldsymbol{w}_n-\boldsymbol{w}_0\|_{\mathbb{X}_T^p(\Omega)}\\ &\quad+\|(\boldsymbol{h}_n-\boldsymbol{h}_0)\times\boldsymbol{n}\|_{\boldsymbol{W}^{-1/p',p'}(\Gamma)})\|\boldsymbol{w}_n-\boldsymbol{w}_0\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}\\ &\le C(\|\boldsymbol{f}_n-\boldsymbol{f}_0\|_{\mathbb{X}_T^p(\Omega)'}+\|(\boldsymbol{h}_n-\boldsymbol{h}_0)\times\boldsymbol{n}\|_{\boldsymbol{W}^{-1/p',p'}(\Gamma)})\\ &\quad\times\|\boldsymbol{w}_n-\boldsymbol{w}_0\|_{\mathbb{V}_T^p(\Omega)}\end{aligned}$$

and

$$|I_2|\le\|S_t^{(n)}(x,|\mathrm{curl}\,\boldsymbol{w}_0|^2)\mathrm{curl}\,\boldsymbol{w}_0-S_t^{(0)}(x,|\mathrm{curl}\,\boldsymbol{w}_0|^2)\mathrm{curl}\,\boldsymbol{w}_0\|_{\boldsymbol{L}^{p'}(\Omega)}\times\|\boldsymbol{w}_n-\boldsymbol{w}_0\|_{\mathbb{V}_T^p(\Omega)}.$$

For the brevity of notation, we put

$$G_n = \|\boldsymbol{f}_n - \boldsymbol{f}_0\|_{\mathbb{X}^p_T(\Omega)'} + \|(\boldsymbol{h}_n - \boldsymbol{h}_0) \times \boldsymbol{n}\|_{\boldsymbol{W}^{-1/p',p'}(\Gamma)} \\ + \|S^{(n)}_t(x, |\mathrm{curl}\, \boldsymbol{w}_0|^2)\mathrm{curl}\, \boldsymbol{w}_0 - S^{(0)}_t(x, |\mathrm{curl}\, \boldsymbol{w}_0|^2)\mathrm{curl}\, \boldsymbol{w}_0\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

Thus we have

$$|I_1| + |I_2| \le G_n \|\boldsymbol{w}_n - \boldsymbol{w}_0\|_{\mathbb{V}^p_T(\Omega)}.$$

Next we estimate the left hand side of (4.4) from below, using Lemma 2.2.

When $p \ge 2$, we have

$$c \int_\Omega |\mathrm{curl}\, (\boldsymbol{w}_n - \boldsymbol{w}_0)|^p dx \le |I_1| + |I_2| \le G_n \|\boldsymbol{w}_n - \boldsymbol{w}_0\|_{\mathbb{V}^p_T(\Omega)}.$$

Hence we have

$$\|\boldsymbol{w}_n - \boldsymbol{w}_0\|^p_{\mathbb{V}^p_T(\Omega)} \le C G^{p'}_n. \tag{4.5}$$

When $1 < p < 2$, we have

$$c \int_\Omega (|\mathrm{curl}\, \boldsymbol{w}_n| + |\mathrm{curl}\, \boldsymbol{w}_0|)^{p-2} |\mathrm{curl}\, (\boldsymbol{w}_n - \boldsymbol{w}_0)|^2 dx \le G_n \|\boldsymbol{w}_n - \boldsymbol{w}_0\|_{\mathbb{V}^p_T(\Omega)}.$$

In this case, we use the reverse Hölder inequality (cf. Sobolev [17, p. 8]) with $0 < s = p/2 < 1$ and $s' = p/(p-2) < 0$. Then we have

$$\int_\Omega \big(|\mathrm{curl}\, \boldsymbol{w}_n| + |\mathrm{curl}\, \boldsymbol{w}_0|)^{p-2} |\mathrm{curl}\, (\boldsymbol{w}_n - \boldsymbol{w}_0)|^2 dx \\ \ge 2^{p-1} (\|\mathrm{curl}\, \boldsymbol{w}_n\|^p_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{curl}\, \boldsymbol{w}_0\|^p_{\boldsymbol{L}^p(\Omega)})^{(p-2)/p} \|\mathrm{curl}\, (\boldsymbol{w}_n - \boldsymbol{w}_0)\|^2_{\boldsymbol{L}^p(\Omega)}.$$

Thus using the estimate (2.10) of Theorem 2.6, we have

$$\begin{aligned} &\|\mathrm{curl}\, (\boldsymbol{w}_n - \boldsymbol{w}_0)\|^2_{\boldsymbol{L}^p(\Omega)} \le C(\|\mathrm{curl}\, \boldsymbol{w}_n\|^p_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{curl}\, \boldsymbol{w}_0\|^p_{\boldsymbol{L}^p(\Omega)})^{(2-p)/p} \\ &\quad \times G_n \|\boldsymbol{w}_n - \boldsymbol{w}_0\|_{\mathbb{V}^p_T(\Omega)} \\ &\le C_1 (\|\boldsymbol{f}_n\|^{p'}_{\mathbb{X}^p_T(\Omega)'} + \|\boldsymbol{f}_0\|^{p'}_{\mathbb{X}^p_T(\Omega)'} + \|\boldsymbol{h}_n \times \boldsymbol{n}\|^{p'}_{\boldsymbol{W}^{-1/p',p'}(\Gamma)} \\ &\quad + \|\boldsymbol{h}_0 \times \boldsymbol{n}\|^{p'}_{\boldsymbol{W}^{-1/p',p'}(\Gamma)})^{(2-p)/p} G_n \|\boldsymbol{w}_n - \boldsymbol{w}_0\|_{\mathbb{V}^p_T(\Omega)}. \end{aligned}$$

Hence, for large n,

$$\|\boldsymbol{w}_n - \boldsymbol{w}_0)\|_{\mathbb{V}^p_T(\Omega)} \le C(\|\boldsymbol{f}_0\|^{p'}_{\mathbb{X}^p_T(\Omega)'} + \|\boldsymbol{h}_0 \times \boldsymbol{n}\|^{p'}_{\boldsymbol{W}^{-1/p',p'}(\Gamma)} + 1)^{(2-p)/p} G_n. \tag{4.6}$$

On the other hand, from (4.2), we have

$$\int_\Omega \big(S_t^{(n)}(x,|\mathrm{curl}\,\boldsymbol{w}_n|^2)\mathrm{curl}\,\boldsymbol{w}_n - S_t^{(0)}(x,|\mathrm{curl}\,\boldsymbol{w}_0|^2)\mathrm{curl}\,\boldsymbol{w}_0\big)\cdot \mathrm{curl}\,\boldsymbol{v}dx \\ - \int_\Omega (\pi_n - \pi_0)\mathrm{div}\,\boldsymbol{v}dx = \langle \boldsymbol{f}_n - \boldsymbol{f}_0, \boldsymbol{v}\rangle_\Omega + \langle (\boldsymbol{h}_n - \boldsymbol{h}_0)\times \boldsymbol{n}, \boldsymbol{v}\rangle_\Gamma \quad (4.7)$$

for any $\boldsymbol{v} \in \mathbb{X}_T^p(\Omega)$. We write the mean value of a function φ by c_φ, i.e.,

$$c_\varphi = \frac{1}{|\Omega|}\int_\Omega \varphi dx.$$

For any $\boldsymbol{v} \in \mathbb{X}_T^p(\Omega)$, it follows from the divergence theorem that

$$\int_\Omega (\pi_n - \pi_0 - c_{\pi_n-\pi_0})\mathrm{div}\,\boldsymbol{v}dx = \int_\Omega (\pi_n - \pi_0)\mathrm{div}\,\boldsymbol{v}dx - c_{\pi_n-\pi_0}\int_\Omega \mathrm{div}\,\boldsymbol{v}dx \\ = \int_\Omega (\pi_n - \pi_0)\mathrm{div}\,\boldsymbol{v}dx.$$

Thus we may assume that $\pi_n - \pi_0 \in L_0^{p'}(\Omega)$ where

$$L_0^{p'}(\Omega) := \{\varphi \in L^{p'}(\Omega); \int_\Omega \varphi dx = 0\}.$$

Hence, for any $\phi \in L^p(\Omega)$, we see that

$$\int_\Omega (\pi_n - \pi_0)\phi dx = \int_\Omega (\pi_n - \pi_0)(\phi - c_\phi)dx.$$

By [1, Corollary 3.1], there exists $\boldsymbol{z} \in \boldsymbol{W}_0^{1,p}(\Omega)$ such that $\mathrm{div}\,\boldsymbol{z} = \phi - c_\phi$, and there exists a constant $C > 0$ depending only on p and Ω such that

$$\|\boldsymbol{z}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} \le C\|\phi\|_{L^p(\Omega)}.$$

Taking $\boldsymbol{v} = \boldsymbol{z} \in \boldsymbol{W}_0^{1,p}(\Omega) \subset \mathbb{X}_T^p(\Omega)$ as a test function of (4.7),

$$\int_\Omega \big(S_t^{(n)}(x,|\mathrm{curl}\,\boldsymbol{w}_n|^2)\mathrm{curl}\,\boldsymbol{w}_n - S_t^{(0)}(x,|\mathrm{curl}\,\boldsymbol{w}_0|^2)\mathrm{curl}\,\boldsymbol{w}_0\big)\cdot \mathrm{curl}\,\boldsymbol{z}dx \\ - \int_\Omega (\pi_n - \pi_0)\phi dx = \langle \boldsymbol{f}_n - \boldsymbol{f}, \boldsymbol{z}\rangle + \langle (\boldsymbol{h}_n - \boldsymbol{h}_0)\times \boldsymbol{n}, \boldsymbol{z}\rangle_\Gamma.$$

We write this equality in the following form.

$$\int_\Omega (\pi_n - \pi_0)\phi dx = J_1 + J_2 + J_3. \quad (4.8)$$

where

$$
\begin{aligned}
J_1 &= \int_\Omega \big(S_t^{(n)}(x, |\operatorname{curl} \boldsymbol{w}_n|^2)\operatorname{curl} \boldsymbol{w}_n \\
&\qquad - S_t^{(n)}(x, |\operatorname{curl} \boldsymbol{w}_0|^2)\operatorname{curl} \boldsymbol{w}_0\big) \cdot \operatorname{curl} \boldsymbol{z} dx, \\
J_2 &= \int_\Omega \big(S_t^{(n)}(x, |\operatorname{curl} \boldsymbol{w}_0|^2)\operatorname{curl} \boldsymbol{w}_0 \\
&\qquad - S_t^{(0)}(x, |\operatorname{curl} \boldsymbol{w}_0|^2)\operatorname{curl} \boldsymbol{w}_0\big) \cdot \operatorname{curl} \boldsymbol{z} dx, \\
J_3 &= -\langle \boldsymbol{f}_n - \boldsymbol{f}_0, \boldsymbol{z}\rangle_\Omega - \langle (\boldsymbol{h}_n - \boldsymbol{h}_0) \times \boldsymbol{n}, \boldsymbol{z}\rangle_\Gamma.
\end{aligned}
$$

We have

$$
\begin{aligned}
|J_2| &\le C\|S_t^{(n)}(x, |\operatorname{curl} \boldsymbol{w}_0|^2)\operatorname{curl} \boldsymbol{w}_0 \\
&\qquad - S_t^{(0)}(x, |\operatorname{curl} \boldsymbol{w}_0|^2)\operatorname{curl} \boldsymbol{w}_0\|_{\boldsymbol{L}^{p'}(\Omega)} \|\boldsymbol{z}\|_{W_0^{1,p}(\Omega)} \\
&\le CG_n \|\phi\|_{L^p(\Omega)},
\end{aligned}
$$

Clearly we have

$$
\begin{aligned}
|J_3| \le C(\|\boldsymbol{f}_n - \boldsymbol{f}_0\|_{\mathbb{X}_T^p(\Omega)'} + \|\boldsymbol{h}_n \times \boldsymbol{n} - \boldsymbol{h}_0 \times \boldsymbol{n}\|_{\boldsymbol{W}^{-1/p',p'}(\Gamma)})\|\phi\|_{L^p(\Omega)} \\
\le CG_n \|\phi\|_{L^p(\Omega)}.
\end{aligned}
$$

When $1 < p < 2$, using Lemma 2.3, the Hölder inequality and (4.7), we have

$$
\begin{aligned}
|J_1| &\le C\Big\{\int_\Omega |\operatorname{curl} (\boldsymbol{w}_n - \boldsymbol{w}_0)|^{p-1} |\operatorname{curl} \boldsymbol{z}| dx \\
&\le C\|\boldsymbol{w}_n - \boldsymbol{w}_0\|_{\mathbb{V}_T^p(\Omega)}^{p-1} \|\boldsymbol{z}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} \\
&\le CG_n^{p-1} \|\phi\|_{L^p(\Omega)}.
\end{aligned}
$$

When $p \ge 2$, similarly, we have

$$
\begin{aligned}
|J_1| &\le C\Big\{\int_\Omega |\operatorname{curl} \boldsymbol{w}_n| + |\operatorname{curl} \boldsymbol{w}_0|)^{p-2} |\operatorname{curl} (\boldsymbol{w}_n - \boldsymbol{w}_0)| |\operatorname{curl} \boldsymbol{z}| dx \\
&\le C(\|\boldsymbol{w}_n\|_{\mathbb{V}_T^p(\Omega)} + \|\boldsymbol{w}_0\|_{\mathbb{V}_T^p(\Omega)})^{p-2} \|\boldsymbol{w}_n - \boldsymbol{w}_0\|_{\mathbb{V}_T^p(\Omega)} \|\boldsymbol{z}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} \\
&\le CG_n^{p'/p} \|\phi\|_{L^p(\Omega)}.
\end{aligned}
$$

Therefore, we have

$$
\|\pi_n - \pi_0\|_{L^{p'}(\Omega)/\mathbb{R}} \le \begin{cases} C_4 G_n + C_5 G_n^{p-1} & \text{if } 1 < p < 2, \\ C_4 G_n + C_5 G_n^{p'-1} & \text{if } p \ge 2. \end{cases} \tag{4.9}
$$

Taking (4.7) into consideration, we have

$$\|\boldsymbol{w}_n - \boldsymbol{w}_0\|_{\mathbb{V}_T^p(\Omega)}^{p\vee p'} + \|\pi_n - \pi_0\|_{L^{p'}(\Omega)/\mathbb{R}}^{p\vee p'} \le C(G_n^{p'} + G_n^p).$$

We show that $G_n \to 0$ as $n \to \infty$ if $\boldsymbol{f}_n \to \boldsymbol{f}_0$ in $\mathbb{X}_T^p(\Omega)'$ and $\boldsymbol{h}_n \times \boldsymbol{n} \to \boldsymbol{h}_0 \times \boldsymbol{n}$ in $\boldsymbol{W}^{-1/p',p'}(\Gamma)$ and $S_t^{(n)}(x,t) \to S_t^{(0)}(x,t)$, a.e. in $\Omega \times [0,\infty)$.

Since

$$\begin{aligned}|S_t^{(n)}(x, |\mathrm{curl}\,\boldsymbol{w}_0|^2)\mathrm{curl}\,\boldsymbol{w}_0 - S_t^{(0)}(x, |\mathrm{curl}\,\boldsymbol{w}_0|^2)\mathrm{curl}\,\boldsymbol{w}_0|^{p'}& \\ \le (2\Lambda)^{p'}|\mathrm{curl}\,\boldsymbol{w}_0|^p \in L^1(\Omega)&\end{aligned}$$

and $S_t^{(n)}(x, |\mathrm{curl}\,\boldsymbol{w}_0|^2)\mathrm{curl}\,\boldsymbol{w}_0 \to S_t^{(0)}(x, |\mathrm{curl}\,\boldsymbol{w}_0|^2)\mathrm{curl}\,\boldsymbol{w}_0$ a.e. in Ω, it follows from the Lebesgue dominated theorem that

$$\|S_t^{(n)}(x, |\mathrm{curl}\,\boldsymbol{w}_0|^2)\mathrm{curl}\,\boldsymbol{w}_0 - S_t^{(0)}(x, |\mathrm{curl}\,\boldsymbol{w}_0|^2)\mathrm{curl}\,\boldsymbol{w}_0\|_{\boldsymbol{L}^{p'}(\Omega)} \to 0$$

as $n \to \infty$. Hence we have $G_n \to 0$ as $n \to \infty$.

End of the proof of Theorem 4.1.

Let $(\boldsymbol{w}_n, \pi_n) \in \mathbb{V}_T^p(\Omega) \times L^{p'}(\Omega)/\mathbb{R}$ be a unique weak solution of (4.4). From the compatibility condition (2.9) with $g = g_n$, the following Neumann problem

$$\begin{cases} \Delta\theta_n = 0 & \text{in } \Omega, \\ \frac{\partial\theta_n}{\partial\boldsymbol{n}} = g_n & \text{on } \Gamma \end{cases}$$

has a unique solution $\theta_n \in W^{2,p}(\Omega)$, up to an additive constant, and there exists a constant $C > 0$ such that

$$\|\theta_n\|_{W^{2,p}(\Omega)} \le C\|g_n\|_{W^{1-1/p,p}(\Gamma)}.$$

Define

$$\boldsymbol{u}_n = \boldsymbol{w}_n + \nabla\theta_n - \sum_{j=1}^J \langle(\boldsymbol{w}_n + \nabla\theta_n)\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j}\widetilde{\nabla}q_j^T.$$

Since

$$\boldsymbol{u}_n - \boldsymbol{u}_0 = \boldsymbol{w}_n - \boldsymbol{w}_0 + \nabla(\theta_n - \theta_0) - \sum_{j=1}^J \langle(\boldsymbol{w}_n - \boldsymbol{w}_0 + \nabla(\theta_n - \theta_0))\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j}\widetilde{\nabla}q_j^N,$$

we have

$$\begin{aligned}\|\boldsymbol{u}_n - \boldsymbol{u}_0\|_{\boldsymbol{W}^{1,p}(\Omega)} &\le C(\|\boldsymbol{w}_n - \boldsymbol{w}_0\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\theta_n - \theta_0\|_{W^{2,p}(\Omega)}) \\ &\le C_1(\|\boldsymbol{w}_n - \boldsymbol{w}_0\|_{\mathbb{V}_T^p(\Omega)} + \|g_n - g_0\|_{W^{1-1/p,p}(\Gamma)}).\end{aligned}$$

This completes the proof of Theorem 4.1. □

A. THE GREEN FORMULA

First, we show that if $\pi \in L^{p'}(\Omega)/\mathbb{R}$, then $\nabla\pi \in \mathbb{X}^p_T(\Omega)'$ is well defined and

$$\langle \nabla\pi, \boldsymbol{\varphi}\rangle_\Omega = \langle \nabla\pi, \boldsymbol{\varphi}\rangle_{\mathbb{X}^p_T(\Omega)', \mathbb{X}^p_T(\Omega)} = -\int_\Omega \pi \operatorname{div} \boldsymbol{\varphi} dx, \text{ for all } \boldsymbol{\varphi} \in \mathbb{X}^p_T(\Omega).$$

To show this, define a functional T_π on $\mathbb{X}^p_T(\Omega)$ by

$$\langle T_\pi, \boldsymbol{\varphi}\rangle = -\int_\Omega \pi \operatorname{div} \boldsymbol{\varphi} dx, \text{ for } \boldsymbol{\varphi} \in \mathbb{X}^p_T(\Omega).$$

Since $\boldsymbol{\varphi}\cdot\boldsymbol{n} = 0$ on Γ, the above definition is independent of the choice of representative of $\pi \in L^{p'}(\Omega)/\mathbb{R}$. Moreover, we have

$$|\langle T_\pi, \boldsymbol{\varphi}\rangle| \leq \|\pi\|_{L^{p'}(\Omega)/\mathbb{R}} \|\boldsymbol{\varphi}\|_{\mathbb{X}^p_T(\Omega)}.$$

Thus $T_\pi \in \mathbb{X}^p_T(\Omega)'$ and

$$\|T_\pi\|_{\mathbb{X}^p_T(\Omega)'} \leq \|\pi\|_{L^{p'}(\Omega)/\mathbb{R}}.$$

If $\pi \in \boldsymbol{D}(\Omega)/\mathbb{R}$, then we can clearly see that

$$\langle T_\pi, \boldsymbol{\varphi}\rangle = \langle \nabla\pi, \boldsymbol{\varphi}\rangle = -\int_\Omega \pi \operatorname{div} \boldsymbol{\varphi} dx \text{ for all } \boldsymbol{\varphi} \in \boldsymbol{D}(\Omega).$$

Since $\boldsymbol{D}(\Omega)/\mathbb{R}$ is dense in $L^{p'}(\Omega)/\mathbb{R}$, T_π is a unique extension of $\nabla\pi$ and we have $T_\pi = \nabla\pi$ in $\mathbb{X}^p_T(\Omega)'$. Thus

$$\langle \nabla\pi, \boldsymbol{\varphi}\rangle_\Omega = -\int_\Omega \pi \operatorname{div} \boldsymbol{\varphi} dx \text{ for } \boldsymbol{\varphi} \in \mathbb{X}^p_T(\Omega). \tag{A.1}$$

Next, let $\boldsymbol{w} \in \boldsymbol{L}^{p'}(\Omega)$ and $\operatorname{curl}\boldsymbol{w} \in \mathbb{X}^p_T(\Omega)'$. For any $\boldsymbol{\psi} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$ with $\boldsymbol{\psi}\cdot\boldsymbol{n} = 0$ on Γ, there exists $\widetilde{\boldsymbol{\psi}} \in \boldsymbol{W}^{1,p}(\Omega)$ such that $\widetilde{\boldsymbol{\psi}} = \boldsymbol{\psi}$ on Γ, and

$$\|\widetilde{\boldsymbol{\psi}}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}.$$

Define

$$\langle \boldsymbol{w}\times\boldsymbol{n}, \boldsymbol{\psi}\rangle = -\langle \operatorname{curl}\boldsymbol{w}, \widetilde{\boldsymbol{\psi}}\rangle_\Omega + \int_\Omega \boldsymbol{w}\cdot\operatorname{curl}\widetilde{\boldsymbol{\psi}} dx.$$

Then we have

$$\begin{aligned} |\langle \boldsymbol{w}\times\boldsymbol{n}, \boldsymbol{\psi}\rangle| &\leq (\|\operatorname{curl}\boldsymbol{w}\|_{\mathbb{X}^p_T(\Omega)'} + \|\boldsymbol{w}\|_{\boldsymbol{L}^{p'}(\Omega)})\|\widetilde{\boldsymbol{\psi}}\|_{\mathbb{X}^p_T(\Omega)} \\ &\leq C(\|\operatorname{curl}\boldsymbol{w}\|_{\mathbb{X}^p_T(\Omega)'} + \|\boldsymbol{w}\|_{\boldsymbol{L}^{p'}(\Omega)})\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}. \end{aligned}$$

Thus $\boldsymbol{w} \times \boldsymbol{n} \in \boldsymbol{W}^{-1/p',p'}(\Gamma)$ is well defined, and for any $\boldsymbol{\varphi} \in \mathbb{X}_T^p(\Omega)$, the following Green formula holds.

$$\langle \operatorname{curl} \boldsymbol{w}, \boldsymbol{\varphi} \rangle_\Omega = -\langle \boldsymbol{w} \times \boldsymbol{n}, \boldsymbol{\varphi} \rangle_\Gamma + \int_\Omega \boldsymbol{w} \cdot \operatorname{curl} \boldsymbol{\varphi} dx. \tag{A.2}$$

Of course, if $\boldsymbol{w} \in \boldsymbol{W}^{1,p'}(\Omega)$, we have

$$\int_\Omega \operatorname{curl} \boldsymbol{w} \cdot \boldsymbol{\varphi} dx = -\int_\Gamma (\boldsymbol{w} \times \boldsymbol{n}) \cdot \boldsymbol{\varphi} dS + \int_\Omega \boldsymbol{w} \cdot \operatorname{curl} \boldsymbol{\varphi} dx \text{ for all } \boldsymbol{\varphi} \in \mathbb{X}_T^p(\Omega).$$

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