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On A Coupled System of Hybrid Fractional-order Differential Equations in Banach Algebras

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Abstract

In this paper, we discuss sufficient conditions for the existence of solutions for a coupled system of hybrid fractional-order differential equation. The continuous dependence of the unique solution on the delay functions will be studied.

Keywords and Phrases: Fractional-order differential equations, existence results, Green's function, boundary value problem.

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1. INTRODUCTION AND PRELIMINARIES

Nonlinear fractional-orders differential equations are critical tools in the modeling of real nonlinear phenomena corresponds to a large variety of events, in relation to many areas of Physical Sciences and Technology. For example, it appears in the study of air movement or dynamic systems, electricity, electromagnetism, or nonlinear process control, among other things (see [9]-[22], [21]). Some differential equations representing a particular dynamic system do not have an analytical solution, so perturbation of such problems can be beneficial. Perturbed differential Equations are classified into different types. An important type of this disorder the differential equation is called hybrid (i.e., quadratic perturbation of the nonlinear differential equation) See for more details [5] and references therein. In recent years, this issue has receive of nonlinear differential equations have attracted much attention. We call such differential equations hybrid differential equations.

The significance of investigations of hybrid differential equations lies in the fact of this many dynamic systems are included as special cases. The consideration of hybrid differential equations is implicit in the works of Krasnoselskii [20] and has been addressed extensively in many papers on hybrid differential equations of different perturbations [19], [24], and [25], and references therein. This class of hybrid differential equations includes perturbations of the original differential equations in various ways. There have been many works on the theory of hybrid differential equations, and we refer the readers to the articles [4, 3, 19].
El Allaoui [7] studied the following coupled system of hybrid differential equations with perturbations of first and second type.

$$\begin{cases} \frac{d}{dt}\big(\frac{x(t)}{f_1(t,x(t),y(t))}\big) = h_1(t,x(t),y(t)),\ 0<t\leq a \\ \\ \frac{d}{dt}\big(\frac{y(t)}{f_2(t,y(t),x(t))}\big) = h_2(t,y(t),x(t)),\ 0<t\leq a \\ \\ x(0)=x_0,\ y(0)=y_0. \end{cases}$$

T. Bashiri et.al. [2] discussed the existence of solutions by using the hybrid fixed point theorems of Dhage [4] for the sum of three operators in a Banach algebra for the following fractional hybrid differential systems in Banach algebra

$$\begin{cases} {}^RD^{\alpha}\left(\frac{x(t)-u(t,x(t))}{f(t,x(t))}\right) = g(t,y(t)) \quad t\in(0,1] \\ {}^RD^{\alpha}\left(\frac{y(t)-u(t,\mu(t))}{f(t,y(t))}\right) = g(t,x(t)) \quad t\in(0,1] \\ x(0)=0,\ \ y(0)=0. \end{cases} \tag{1}$$

where D^{α} denotes the the Riemann-Liouville fractional derivative of order α.
Caballero et al [6] studied the following coupled system

$$\begin{cases} {}^RD^{\alpha}_{0+}\big(\frac{x(t)}{f(t,x(t),y(t))}\big) = g(t,x(t),y(t)) \quad 0<t\leq 1 \\ \\ {}^RD^{\alpha}_{0+}\big(\frac{y(t)}{f(t,y(t),x(t))}\big) = g(t,y(t),x(t)) \quad 0<t\leq 1, \\ \\ x(0)=y(0),\ \alpha\in(0,1), \end{cases}$$

The main tool in [6] is a fixed point theorem of Darbo type associated to measures of noncompactness.
Buvaneswari et al.[1] studied the existence of solutions for a coupled system of nonlinear hybrid differential equations of fractional order involving Hadamard

derivative with nonlocal boundary conditions.
Samina et al. [23] investigated stability results to a coupled system of nonlinear fractional hybrid differential equations.

In line with the above works, our purpose in this paper is to prove the existence result for the following an initial value problem of coupled hybrid fractional order differential equations (CHFDEs):

$$\begin{cases} {}^{R}D^{\alpha}\left(\frac{x(t)-k_1(t,x(\varphi_1(t)))}{g_1(t,x(\varphi_2(t)))}\right) = f_1(t, I^{\beta}\, u_1(t, y(\varphi_3(t)))) & t \in I = (0,T] \\ {}^{R}D^{\alpha}\left(\frac{y(t)-k_2(t,y(\varphi_1(t)))}{g_2(t,y(\varphi_2(t)))}\right) = f_2(t, I^{\beta}\, u_2(t, x(\varphi_3(t)))) & t \in I = (0,T] \\ x(0) = k_1(0, x(0)), \;\; y(0) = k_2(0, y(0)), \end{cases} \tag{2}$$

where $\alpha, \beta \in (0,1)$.
Definition By a solution of the CHFDEs system (2) we mean a function $(x, y) \in C(J, \mathbb{R} \times \mathbb{R})$ such that

(i) the functions $t \rightarrow \frac{x(t)-k_1(t,x(\varphi_1(t)))}{g_1(t,x(\varphi_2(t)))}$ and $t \rightarrow \frac{y(t)-k_2(t,y(\varphi_1(t)))}{g_2(t,y(\varphi_2(t)))}$ are continuous for each $x, y \in C(J, \mathbb{R})$, and

(ii) (x, y) satisfies the system of equations in (2).

To prove our result use the standard hybrid fixed point theory developed in [3]which includes three factors in Banach algebra to determine sufficient conditions for the existence of couple solutions for the existence of solutions for initial value problems coupled system of fractional orders hybrid functional differential equations (2). The continuous dependence of the unique solution on the delay functions will be studied.
This paper is organized as follows: In Section 2, we proved an auxiliary Theorem related to the linear variant of the problem (2) and state sufficient conditions which guarantee the existence of solutions to the problem (2). In Section 4, we present the continuous dependence on the delay functions. Our conclusion is presented in Section 5.

Lemma 1. *[4] Let S be a nonempty, closed convex and bounded subset of a Banach algebra X and let $A, C : X \rightarrow X$ and $B : S \rightarrow X$ be three operators such that:*

(a) A and C are Lipschitzian with Lipschitz constants δ and ρ, respectively.

(b) B is compact and continuous.

(c) $u = AuBv + Cu \Rightarrow u \in S$, for all $v \in S$.

(d) $\delta \aleph + \rho < r$, *for* $r > 0$ *where* $\aleph = \|B(S)\|$.

Then the operator equation $AuBu + Cu = u$ *has a solution in* S.

2. MAIN RESULTS

In this section, we formulate our main result for CHFDEs (2) depending on the fixed point theorems due to Dhage [4].
Let $X = C(J, \mathbb{R})$ of the vector of all real-valued continuous functions on $J = [0, T]$. We equip the space X with the norm $||x|| = \sup_{t\in J} |x(t)|$. Clearly, $C(J, \mathbb{R})$ is a complete normed algebra with respect to this supremum norm, the multiplication in the a Banach algebra defined by $(x \cdot y)(t) = x(t) \cdot y(t)$ For the given the Banach algebra X, Assume the product space $E = X \times X$. Define a norm $\|.\|$ in the product linear space E by $\|(x, y)\| = \|x\| + \|y\|$.
Then, the normed linear space $(E, \|(., .)\|)$ is a Banach space which further becomes a Banach algebra w.r.t. the multiplication defined by

$$((x, y) \cdot (u, v))(t) = (x, y)(t) \cdot (u, v)(t) = (x(t)u(t), y(t)v(t)), \tag{3}$$

for all $t \in J$, where $(x, y), (u, v) \in X \times X = E$.
Consider the following assumptions:

(A_1) The functions $g_i : [0, T] \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$, and $k_i : [0, T] \times \mathbb{R} \to R$ are continuous and there exist two positive functions $\omega_i(t),\ L_i(t)$, with norm $\|\omega_i\|$ and $\|L_i\|$ respectively, such that

$$|k_i(t, x) - k_i(t, y)| \leq \omega_i(t)|x - y|,\ \ t \in [0, T] \quad \text{and } x, y \in \mathbb{R}$$
$$|g_i(t, x) - g_i(t, y)|\ \leq L_i(t)|x - y|,\ \ t \in [0, T] \quad \text{and } x, y \in \mathbb{R}.$$

with $\|\omega\| = \max\{\|\omega_1\|, \|\omega_2\|\}$, and $\|L\| = \max\{\|L_1\|, \|L_2\|\}$.
Remark

$$|k_i(t, x)| - |k_i(t, 0)| \leq |k_i(t, x) - k_i(t, 0)|\ \leq \omega_i(t)|x|,$$
$$|g_i(t, x)| - |g_i(t, 0)| \leq |g_i(t, x) - g_i(t, 0)|\ \leq L_i(t)|x|,$$

then

$$|ki(t, x)| \leq \|\omega\|(|x|) + H, \text{ where } H = \sup_{t\in J} |k_i(t, 0)|.$$
$$|g_i(t, x)| \leq \|L_i\|(|x|) + H, \text{ where } G = \sup_{t\in J} |k_i(t, 0)|,\ i = 1, 2, ..., m.$$

(A_2) The functions $f_i : [0,T] \times \mathbb{R} \to \mathbb{R}$, and $u_i : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfy Caratheodory condition i.e., f_i and u_i are measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in [0,T]$. There exist the bounded and measurable (on $[0,T]$) functions $a_i,\ b_i$ and $m_i,\ \ i = 1, 2.$, such that

$$|f_i(t,x)| \leq\ a_i(t) +\ b_i(t)|x|,\ \ \forall\ (t,x)\ \in [0,T] \times \mathbb{R},$$

$$|u_i(t,x)| \leq m_i(t),\ \ \forall\ (t,x)\ \in [0,T] \times \mathbb{R}$$

with $\sup |a_i(t)| \leq M,\ \ \sup |m_i(t)| \leq N$ and $\sup |b_i(t)| \leq b.$

(A_3) $\varphi_j : J \to J$, are continuous functions with $\varphi_j(0) = 0,\ j = 1, 2, 3.$

(A_4) There exists a number $r > 0$ such that

$$r \geq \frac{2H\ + 2G\big(\frac{M\ T^\alpha}{\Gamma(\alpha+1)} + \frac{bN\ T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\big)}{1 - \Big[\|\omega\| + \|L\|\ \big(\frac{MT^\alpha}{\Gamma(\alpha+1)} + \frac{bNT^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\big)\Big]}, \tag{4}$$

with

$$\|\omega\| + \|L\|\ \big(\frac{MT^\alpha}{\Gamma(\alpha+1)} + \frac{bNT^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\big) < 1. \tag{5}$$

Now, from our assumptions the following lemma can be easily proved [8].

Lemma 2. *Assume that hypotheses $(A_1) - (A_4)$ holds. Then a function $x \in C(J, \mathbb{R})$ is a solution of the hybrid differential equation*

$$\begin{cases} {}^R D^\alpha \left(\frac{x(t) - k(t, x(\varphi_1(t)))}{g(t, x(\varphi_2(t)))}\right) = f(t, I^\beta\ u(t, x(\varphi_3(t)))) \quad t \in J = [0,T] \\ x(0) = k(0, x(0)). \end{cases} \tag{6}$$

if, and only if, it satisfies the following quadratic integral equation

$$x(t) = k(t, x(\varphi_1(t))) + g(t, x(\varphi_2(t))) I^\alpha f(t, I^\beta\ u(t, x(\varphi_3(t)))). \tag{7}$$

2.1. Existence of solution

Now, our target is to prove the following existence theorems.

Theorem 1. *Assume that the hypotheses $(A_1) - (A_4)$ hold. Then the coupled system (2) has at least one solution defined on $J \times J$.*

Proof. By Lemma 2, the coupled solutions for a coupled hybrid system differential equations in (2) are the solutions to the coupled system fractional integral equations,

$$x(t) = k_1(t, x(\varphi_1(t))) + g_1(t, x(\varphi_2(t)))I^{\alpha} f_1(t, I^{\beta}\, u_1(t, y(\varphi_3(t)))). \tag{8}$$

$$y(t) = k_2(t, y(\varphi_1(t))) + g_1(t, y(\varphi_2(t)))I^{\alpha} f_1(t, I^{\beta}\, u_1(t, x(\varphi_3(t)))). \tag{9}$$

Define a subset $S = S_1 \times S_2$ of the Banach space $E = X \times X$ by

$$S := \{(x, y) \in X \times X,\ \|(x, y)\| \leq r\},$$

where r satisfies inequality (4).
Clearly S is closed, convex, and bounded subset of the Banach space $E = X \times X$. In conjunction with the functions f_i, g_i and $h_i,\ (i = 1, 2)$, we introduce the three operators $A = (A_1, A_2) : E \to E$, $C = (C_1, C_2) : E \to E$ and $B = (B_1, B_2) : S \to E$ defined by

$$\begin{aligned}
A(x, y) &= (A_1 x, A_2 y) = \big(g_1(t, x(\varphi_2(t))), g_2(t, y(\varphi_2(t)))\big),\\
B(x, y) &= (B_1 y, B_2 x) = \bigg(I^{\alpha}\ f_1(t, I^{\beta}\ u_1(t, y(\varphi_3(t))), I^{\alpha}\ f_2(t, I^{\beta}\ u_2(t, x(\varphi_3(t)))\bigg),\\
C(x, y) &= (C_1 x, C_2 y) = \big(k_1(t, x(\varphi_1(t))), k_2(t, y(\varphi_1(t)))\big).
\end{aligned}$$

Then the coupled system of hybrid integral equations (8) and (9) can be written as the system of operator equations as

$$A(x, y)(t) \cdot B(x, y)(t) + C(x, y)(t) = (x, y)(t),\ t \in J, \tag{10}$$

which further in view of the multiplication (2.4) of two elements in E yields

$$\big(A_1 x(t) \cdot B_1 y(t) + C_1 x(t), A_2 y(t) \cdot B_2 x(t) + C_2 y(t)\big) = (x, y)(t),\ t \in J. \tag{11}$$

This further implies that

$$\begin{cases} A_1 x(t) \cdot B_1 y(t) + C_1 x(t) = x(t), & t \in J, \\ \\ A_2 y(t) \cdot B_2 x(t) + C_2 y(t) = y(t), & t \in J. \end{cases} \tag{12}$$

We shall show that $A,\ B$ and C satisfy all the conditions of Lemma 1. This will be achieved in the following series of steps.
Step 1. We will show that $A = (A_1, A_2)$ and $C = (C_1, C_2)$ are Lipschitzian on E. So, we will show that $A_i,\ C_i$ are lipschitzian on $E,\ i = 1, 2$. Let $x, y \in X$. Then by (A_1), we have

$$\begin{aligned}
|A_i x(t) - A_i y(t)| &= |g_i(t, x(\varphi_2(t))) - g_i(t, y(\varphi_2(t)))| \\
&\leq L_i(t)\ |x(\varphi_2(t)) - y(\varphi_2(t))| \leq \|L_i\|\ \|x - y\|.
\end{aligned}$$

Consequently, for all $t \in [0, T]$. Taking the supremum over t, we get

$$\|A_i x - A_i y\| \le \|L_i\| \, \|x - y\|.$$

Therefore, A_i are Lipschitzian on X with Lipschitz constant $\|L_i\|$.
Similarly, for any $x, y \in X$, we have

$$\begin{aligned} |C_i x(t) - C_i y(t)| &= |k_i(t, x(\varphi_1(t))) - k_i(t, y(\varphi_1(t)))| \\ &\le \omega_i(t)|x(\varphi_1(t)) - y(\varphi_1(t))| \le \|\omega_i\| \, \|x - y\|. \end{aligned}$$

Consequently, for all $t \in [0, T]$. Taking the supremum over t, we get

$$\|C_i x - C_i y\| \le \|\omega_i\| \, \|x - y\|.$$

This shows that C_i are Lipschitz mapping on X with the Lipschitz constant $\|\omega_i\|$.
Hence, for all $u = (x, y), \ v = (\bar{x}, \bar{y}) \in E$, by definition of the operator A, we obtain

$$Au - Av = A(x, y) - A(\bar{x}, \bar{y}) = (A_1 x, A_2 y) - (A_1 \bar{x}, A_2 \bar{y}) = (A_1 x - A_1 \bar{x} \ , \ A_2 y - A_2 \bar{y}),$$

then

$$\begin{aligned} \|Au - Av\| &\le \|A_1 x - A_1 \bar{x}\| + \|A_2 y - A_2 \bar{y})\| \\ &\le \|L_1\| \, \|x - \bar{x}\| + \|L_2\| \, \|y - \bar{y}\| \\ &\le \|L\| \left(\|x - \bar{x}\| + \ \|y - \bar{y}\|\right) = \|L\| \, \|u - v\|, \end{aligned}$$

which shows that A is Lipschitzian with Lipschitz constant $\|L\|$. Similarly, we can deduce C is Lipschitzian with Lipschitz constant $\|\omega\|$ as well.

Step 2. To prove that $B = (B_1, B_2)$ is a compact and continuous operator on S into E. First we show that B is continuous on E. let (x_n, y_n) be a sequence of points in S converging to a point $(x, y) \in E$ which satisfies $\{(x_n, y_n)\} \to (x, y)$ as $(n \to \infty)$. Then, by Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} I^{\beta} \, u_1(s, y_n(\varphi_3(s))) = I^{\beta} \, u_1(s, y(\varphi_3(s))).$$

Also, since $f_1(t, y(t))$ is continuous in y, then using the properties of the fractional-order integral and applying Lebesgue Dominated Convergence Theorem, we get

$$\lim_{n\to\infty} B_1 y_n(t) = \lim_{n\to\infty} I^{\alpha} \, f(t, I^{\beta} \, u(t, y_n(\varphi_3(t))) = I^{\alpha} \, f_1(t, I^{\beta} \, u(t, y(\varphi_3(t))) = By(t).$$

Thus, $B_1 y_n \to B_1 y$ as $n \to \infty$ uniformly on $\mathbb{R}_+$, and hence, B_1 is a continuous operator. Similarly B_2 can be verified continuously as well.

$$\lim_{n\to\infty} B_2 x_n(t) = B_1 x(t).$$

Hence,

$$\lim_{n\to\infty} Bu_n(t) = (\lim_{n\to\infty} B_1 y_n, \lim_{n\to\infty} B_2 x_n) = (B_1 y(t), B_2 x(t)) = B(x,y) = Bu(t).$$

Thus, $Bu_n \to Bu$ as $n \to \infty$ uniformly on $\mathbb{R}_+$, and hence B is a continuous operator on S into S.

Now, we show that B is a compact operator on S. It is enough to show that $B(S)$ is a uniformly bounded and equicontinuous set in E. To see that, let $(x,y) \in S$ be arbitrary. Then by hypothesis (A_2),

$$\begin{aligned}
|B_1 y(t)| &\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \, |f_1(s, I^\beta u_1(s, y(\varphi_3(s)))| ds \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \, [a_1(s) + b_1(s) I^\beta |u_1(s, y(\varphi_3(s))|] ds \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} a_1(s) ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |b_1(s)| I^\beta |u_1(s, y(\varphi_3(s)))| ds \\
&\leq I^\alpha a_1(t) + b I^{\alpha+\beta} m_1(t) \\
&\leq M \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + bN \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \, ds \\
&\leq M \frac{T^\alpha}{\Gamma(\alpha+1)} + b\, N \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)},
\end{aligned}$$

for all $t \in J$. Taking supremum over t,

$$\|B_1 y\| \leq M \frac{T^\alpha}{\Gamma(\alpha+1)} + b\, N \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)},$$

for all $x \in S_1$. This shows that B_1 is uniformly bounded on S_1.
In the same way, we can conclude

$$\|B_2 x\| \leq M \frac{T^\alpha}{\Gamma(\alpha+1)} + b\, N \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)},$$

Let $u = (x,y) \in S$. Then we get

$$\begin{aligned}
\|Bu\| &= \|B(x,y)\| = \|(B_1 y, B_2 x)\| = \|B_1 y\| + \|B_2 x\| \\
&\leq 2\, M \frac{T^\alpha}{\Gamma(\alpha+1)} + 2\, b\, N \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)},
\end{aligned}$$

for all $u \in S$, we can get the fact that B is uniformly bounded on S.

Next, we show that $B(S)$ is equi-continuous sequence of functions in E. Choose $t_1,\ t_2 \in J$, such that $t_1 < t_2$ and $u = (x, y) \in S$, then we get

$$
\begin{aligned}
(B_1 y)(t_2) \;-\; & (B_1 y)(t_1) \\
\leq\; & \int_0^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, I^\beta u_1(s, y(\varphi_3(s))))ds \\
-\; & \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, I^\beta u_1(s, y(\varphi_3(s))))ds \\
\leq\; & \int_0^{t_1} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, I^\beta u_1(s, y(\varphi_3(s))))ds \\
+\; & \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, I^\beta u_1(s, y(\varphi_3(s))))ds \\
-\; & \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, I^\beta u_1(s, y(\varphi_3(s))))ds \\
\leq\; & \int_0^{t_1} \frac{(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, I^\beta u_1(s, y(\varphi_3(s))))ds \\
+\; & \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, I^\beta u_1(s, y(\varphi_3(s))))ds,
\end{aligned}
$$

and

$$
\begin{aligned}
|(B_1 y)(t_2) \;-\; & (B_1 y)(t_1)| \\
\leq\; & \int_0^{t_1} \frac{(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} |f_1(s, I^\beta u_1(s, y(\varphi_3(s))))|ds \\
+\; & \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} |f_1(s, I^\beta u_1(s, y(\varphi_3(s))))|ds \\
\leq\; & \int_0^{t_1} \frac{(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} \big[a_1(s) + b_1(s) I^\beta |u_1(s, y(\varphi_3(s)))|\big]ds \\
+\; & \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} \big[|a_1(s)| + |b_1(s)| I^\beta |u_1(s, y(\varphi_3(s)))|\big]\, ds \\
\leq\; & M\Big[\int_0^{t_1} \frac{(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} ds + \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} ds\Big] \\
+\; & b\Big[\int_0^{t_1} \frac{(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} I^\beta |u_1(s, y(\varphi_3(s)))|\, ds \\
+\; & \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} I^\beta |u_1(s, y(\varphi_3(s)))|\, ds\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\leq M\left(\frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|}{\Gamma(\alpha+1)}\right)\\
&+ b\left[\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}I^\beta m_1(s)ds+\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}I^\beta m_1(s)ds\right]\\
&\leq M\left(\frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|}{\Gamma(\alpha+1)}\right)+b\bigg[\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}I^\beta m_1(s)ds\\
&+ \int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}I^\beta m_1(s)ds\bigg]\\
&\leq M\left(\frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|}{\Gamma(\alpha+1)}\right)\\
&+ bN\bigg[\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\int_0^s\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}d\tau ds\\
&+ \int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\int_0^s\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}d\tau ds\bigg]\\
&\leq M\left(\frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|}{\Gamma(\alpha+1)}\right)+b\,N\bigg[\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{(s)^\beta}{\Gamma(\beta+1)}ds\\
&+ \int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{(s)^\beta}{\Gamma(\beta+1)}ds\bigg]\\
&\leq M\left(\frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|}{\Gamma(\alpha+1)}\right)+b\,N\,\left(\frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|\,T^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right),
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
|(B_1y)(t_2)-(B_1y)(t_1)| \leq a&\left(\frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|}{\Gamma(\alpha+1)}\right)\\
&+b\,N\,\left(\frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|\,T^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right),
\end{aligned}
$$

which is independent on $y \in S_2$. Hence, for $\epsilon_1 > 0$, there exists a $\delta_1 > 0$, such that

$$|t_2 - t_1| < \delta_1 \quad \Longrightarrow \quad |(B_1y)(t_2) - (B_1y)(t_1)| < \epsilon_1.$$

Similarly, one can get that for $\epsilon_2 > 0$, there exists a $\delta_2 > 0$, such that

$$|t_2 - t_1| < \delta_2 \quad \Longrightarrow \quad |(B_2x)(t_2) - (B_2x)(t_1)| < \epsilon_2.$$

Hence, for $\epsilon > 0$, there exists a $\delta > 0$, such that

$$|t_2 - t_1| < \delta \quad \Longrightarrow \quad |Bu(t_2) - Bu(t_1)| < \epsilon.$$

Let $t_2, t_1 \in J$ and for all $u \in S$. This shows that $B(S)$ is an equi-continuous set in E. Now, the set $B(S)$ is a uniformly bounded and equi-continuous set in E, so it is compact

by the Arzela-Ascoli theorem. As a result, B is a complete continuous operator on S.
Step 3. The hypothesis (c) of Lemma 1 is satisfied. let for $u \in E$,

$$u = (x, y) = (A_1xB_1y + C_1x, A_2yB_2x + C_2y).$$

Let $x \in S_1$ and $y \in S_2$ be arbitrary elements such that $x = A_1xB_1y + C_1x$. Then we have

$$\begin{aligned}
|x(t)| \leq\ & |A_1x(t)||B_1y(t)| + |C_1x(t)| \\
\leq\ & |g_1(t, x(\varphi_2(t)))| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f_1(s, I^\beta\, u_1(s, y(\varphi_3(s))))| ds + |k_1(t, x(\varphi_1(s)))| \\
\leq\ & \left[\|L_1\||x(\varphi_2(t))| + G_1\right] \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[a_1(s) + b_1(s)I^\beta|u_1(s, y(\varphi_3(s)))|\right] ds \\
& + \|\omega_1\||x(\varphi_1(t))| + H_1 \\
\leq\ & [\|L_1\||x(\varphi_2(t))| + G_1] + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [a_1(s) + |b_1(s)|I^\beta m_1(s)] ds \\
& + \|\omega_1\||x(\varphi_1(t))| + H_1 \\
\leq\ & [\|L_1\|\ \|x\| + G_1]\ +\ I^\alpha a_1(t) +\ \|b_1\|I^{\alpha+\beta} m_1(t) + \|\omega_1\|\ \|x\| + H_1 \\
\leq\ & [\|L_1\|\|x\| + G_1] \left(M \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + bN \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} ds \right) + \|\omega_1\|\|x\| + H_1 \\
\leq\ & [\|L_1\|\|x\| + G_1] \left(M \frac{s^\alpha}{\Gamma(\alpha+1)} + bN \frac{s^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \right) + \|\omega_1\|\|x\| + H_1.
\end{aligned}$$

Therefore

$$|x(t)| \leq \|\omega_1\|\|x\| + H_1 + [\|L_1\|\|x\| + G_1] \left(M \frac{s^\alpha}{\Gamma(\alpha+1)} + bN \frac{s^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \right).$$

Taking supremum over $t \in I$,

$$\|x\| \leq \|\omega_1\|\|x\| + H_1 + [\|L_1\|\|x\| + G_1] \left(M \frac{T^\alpha}{\Gamma(\alpha+1)} + bN \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \right). \quad (13)$$

Hence, $x \in S_1$.

Similarly, proceeding with the analogous arguments, and if we let $x \in S_1$ and $y \in S_2$ be arbitrary elements such that $y = A_2yB_2x + C_2y$. Then we have

$$\begin{aligned}
|y(t)| &\leq |A_2y(t)||B_2x(t)| + |C_2y(t)| \\
&\leq \|\omega_2\|\ \ \|y\|\ + H_2 + [\|L_2\|\ \|y\| + G_2] \left(M \frac{s^\alpha}{\Gamma(\alpha+1)} + bN \frac{s^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \right).
\end{aligned}$$

Taking supremum over $t \in I$,

$$\|y\| \leq \|\omega_2\|\ \|y\|\ + H_2 + [\|L_2\|\ \|y\| + G_2] \left(M \frac{T^\alpha}{\Gamma(\alpha+1)} + bN \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \right). \quad (14)$$

Therefore, $y \in S_2$. By assumption (c), we can conclude that $u = (x, y) \in S$.
Adding the inequalities (13) and (14), we obtain

$$\begin{aligned}
\|x\| + \|y\| \quad &\leq \quad \|\omega_1\| \ \|x\| \ + H_1 + \ 2 \left[\|L_1\| \ \|x\| + G_1\right] \left(M \frac{T^\alpha}{\Gamma(\alpha+1)} + bN \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right) \\
&+ \quad \|\omega_{\|} \ \|y\| \ + H_2 + \ \left[\|L_2\| \ \|y\| + G_2\right] \left(M \frac{T^\alpha}{\Gamma(\alpha+1)} + bN \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right) \\
&\leq \quad \|\omega\| \ \left[\|x\| + \|y\|\right] \ + H_1 + H_2 \\
&+ \quad \left[\|L\| \ \left[\|x\| + \|y\|\right] + G_1 + G_2\right] \left(M \frac{T^\alpha}{\Gamma(\alpha+1)} + bN \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right).
\end{aligned}$$

Consequently

$$\begin{aligned}
\|(x,y)\| \quad &\leq \quad \|\omega\| \ \|(x,y)\| \ + H_1 + H_2 \\
&+ \quad \left[\|L\| \ \|(x,y)\| + G_1 + G_2\right] \left(M \frac{T^\alpha}{\Gamma(\alpha+1)} + bN \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right) \\
&\leq \quad H_1 + H_2 + \left[G_1 + G_2\right] \left(M \frac{T^\alpha}{\Gamma(\alpha+1)} + bN \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right) \\
&+ \quad \left[\|\omega\| + \|L\| \left(M \frac{T^\alpha}{\Gamma(\alpha+1)} + bN \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right)\right] \|(x,y)\|.
\end{aligned}$$

As $\|(x, y)\| = \|x\| + \|y\|$, we have that $\|(x, y)\| \leq r$ and so the hypothesis (c) of Lemma 1 is satisfied.
Step 4. Finally, we show that $\delta\aleph + \rho < 1$, that is, (d) of Lemma 1 holds.
Since

$$\begin{aligned}
\aleph \quad &= \quad \|B(S)\| = \sup\left\{\|B(x,y)\| : \ (x,y) \in S\right\} = \ \sup\left\{\|B_1(y)\| + \|B_2(x)\| : \ (x,y) \in S\right\} \\
&\leq \quad \frac{2M \ T^\alpha}{\Gamma(\alpha+1)} + \frac{2 \ bN \ T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}.
\end{aligned}$$

and by (A_4), we have $\frac{\|L\|}{2} \ \aleph \ + \ \|k\| \ < 1$. with $\delta = \frac{\|L\|}{2}$ and $\rho = \|k\|$.

Thus all the conditions of Lemma 1 are satisfied and hence the operator equation $(x, y) = A(x, y)B(x, y)+C(x, y)$ has a solution in S. In consequence, a coupled hybrid system of the fractional differential equations (2) has a solution defined on J. □

By a similar way as done above, we can prove an existence result for the following fractional hybrid differential equation

$$\begin{cases}
D^\alpha \left(\frac{x(t) - k_1(t, x(\varphi_1(t)))}{g_1(t, x(\varphi_2(t)))}\right) = f_1(t, I^\beta \ u_1(t, y(\varphi_3(t)))), \quad t \in I = [0, T] \\
D^\alpha \left(\frac{y(t) - k_2(t, y(\varphi_1(t)))}{g_2(t, y(\varphi_2(t)))}\right) = f_2(t, I^\beta \ u_2(t, x(\varphi_3(t)))), \quad t \in I = [0, T]. \\
x(0) = 0, \ \ y(0) = 0.
\end{cases} \tag{15}$$

Lemma 3. *Assume that hypotheses $(A_1) - (A_4)$ hold. If a function $x \in C(J, \mathbb{R})$ is a solution of the CHFDEs* (15)*, then it satisfies the quadratic fractional integral equation* (12)

Theorem 2. *Assume that the hypotheses $(A_1) - (A_4)$ of Theorem 1 holds. Then the CHFDEs* (15) *has at least one solution defined on $J \times J$.*

3. PARTICULAR CASES AND REMARK

The CHFDEs (15) involves many particular cases:

(i) When $u_i(t,x) = x, \ \ i = 1,2, \ \varphi_j(t) = t, \ j = 1,2,3., \ T = 1,$ and letting $\beta \to 0,$ we have the following fractional hybrid differential systems in Banach algebra which is studied in [2].

$$\begin{cases} {}^R D^\alpha \left(\frac{x(t) - k_1(t,x(t))}{g_1(t,x(t))} \right) = f_1(t, y(t)), & t \in [0,1] \\ {}^R D^\alpha \left(\frac{y(t) - K_2(t,y(t))}{g_2(t,y(t))} \right) = f_2(t, x(t)), & t \in [0,1] \\ x(0) = 0, \ y(0) = 0. \end{cases} \tag{16}$$

(ii) When $k_i(t,x) = 0, \ u_i(t,x) = x, \ \ i = 1,2, \ \varphi_j(t) = t, \ j = 1,2,3., \ T = 1,$ and letting $\beta \to 0,$ we have the following hybrid fractional differential system which is coupled system of hybrid fractional differential equation studied in [24]

$$\begin{cases} {}^R D^\alpha \left(\frac{x(t)}{g_1(t,x(t))} \right) = f_1(t, y(t)) & t \in [0,T] \\ {}^R D^\alpha \left(\frac{y(t)}{g_2(t,y(t))} \right) = f_2(t, x(t)) & t \in [0,T] \\ x(0) = 0, \ y(0) = 0. \end{cases} \tag{17}$$

(v) When $k_i(t,x) = 0, \ g_i(t,x) = 1, \ i = 1,2.$ Taking $f_1(t,x(t)) = p(t) + x(t),$ we can deduce existence results for the following coupled system of fractional differential equations

$$\begin{cases} {}^R D^\alpha x(t) = p(t) + I^\beta \, u_1(t, y(t)), & t \in [0,T] \\ {}^R D^\alpha y(t) = p(t) + I^\beta \, u_2(t, x(t)), & t \in [0,T] \\ x(0) = 0, \quad y(0) = 0. \end{cases} \tag{18}$$

Remark 1. *The existence results for the CFHDEs (7) can be proved under another sequence of assumptions.*
Let the assumptions of Theorem 1 be satisfied, with replace (A_1) and (A_4) by the following assumptions:

(A_1^*) *The functions* $k_i : J \times R \to R$, *and* $g_i : J \times R \to R$, $g_i(0,0) \neq 0$, $i = 1, 2$, *are continuous and there exist positive functions* $\omega_i(t)$ $p_i(t)$ *and* $\Phi_i(t)$, *with norms* $\|\omega_i\|$ $\|p_i\|$, $\|\Phi_i\|$, *and respectively such that*

$$|k_i(t,x) - k_i(t,y)| \leq \omega_i(t)|x - y|,$$
$$|g_i(t,x) - g_i(t,y)| \leq p_i(t)\Phi_i(|x|).$$

(A_4^*) *There exists a number* $r > 0$ *such that*

$$r \geq \frac{H_1 + H_2 \ + (\|p_1\|\Phi_1(r) + \|p_2\|\Phi_2(r))\big(\frac{M\ T^\alpha}{\Gamma(\alpha+1)} + \frac{bN\ T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\big)}{1 - \Big[\|\omega\| + \|L\|\ \big(\frac{MT^\alpha}{\Gamma(\alpha+1)} + \frac{b\ NT^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\big)\Big]},$$

$$\textit{where}\ \ H_i = \sup_{t\in J}|k_i(t,0)|,\ \ \textit{and}\ \ \|\omega\| + \|L\|\ \big(\frac{MT^\alpha}{\Gamma(\alpha+1)} + \frac{b\ NT^{\alpha+\beta}}{\Gamma(\alpha+1)}\big) < 1.$$

4. CONTINUOUS DEPENDENCE

In this section, we give sufficient conditions for the uniqueness of the solution for a coupled hybrid system (2) and study the continuous dependence of solution on the delay functions $\varphi_i(t)$.

4.1. Uniqueness of the solution

Let us state the following assumption

(A_2^*) Let $f_i : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $u_1 : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous functions satisfying the Lipschitz condition and there exists four positive functions $\lambda_i(t)$, $\theta_i(t)$ with bounded $\|\lambda_i\|$ and $\|\theta_i\|$, $i = 1, 2.$, such that

$$|f_i(t,x) - f_i(t,y)| \leq \lambda_i(t)|x - y|,$$
$$|u_i(t,x) - u_i(t,y)| \leq \theta_i(t)|x - y|.$$

Remark

$$|f_i(t,x)| \leq |\lambda_i(t)||x| + F_i,\ \text{where}\ \ F_i = \sup_{t\in J}|f_i(t,0)|.$$
$$|u_i(t,x)| \leq |\theta_i(t)||x| + U_i,\ \text{where}\ \ U_i = \sup_{t\in J}|u_i(t,0)|,\ i = 1, 2, ..., m.$$

where $F_i = \sup_{t\in I}|f_i(t,0)|$, $U_i = \sup_{t\in I}|u_i(t,0)|$, $i = 1, 2.$, with $F = \max\{F_1, F_2\}$, $U = \max\{U_1, U_2\}$, and $\|\lambda\| = \max\{\|\lambda_1\|, \|\lambda_2\|\}$.

Theorem 3. *Let the assumptions of Theorem 1 be satisfied with replace condition* (A_2) *by* (A_2^*)*, if*

$$\|K\|+\|L\|\frac{\|\lambda\|[\|\theta\|\|u\|+U]T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\|L\|\frac{F\,T^{\alpha}}{\Gamma(\alpha+1)}+\frac{[\|L\|\|u\|+G]\|\lambda\|\|\theta\|T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}<1,$$

then the CHFDEs (2) has a unique solution.

Proof. Let $u_1=(x_1,y_1)$ and $u_2=(x_2,y_2)$ be two solutions of (2). Then

$$\begin{aligned}
&|x_1(t)-x_2(t)|\\
&\le |k_1(t,x_1(\varphi_1(t)))-k_1(t,x_2(\varphi_1(t)))|+|g_1(t,x_1(\varphi_2(t)))I^{\alpha}f_1(t,I^{\beta}u_1(t,y_1(\varphi_3(t)))))\\
&-g_1(t,x_2(\varphi_2(t)))I^{\alpha}f_1(t,I^{\beta}u_1(t,y_2(\varphi_3(t))))|\\
&\le k_1(t)|x_1(\varphi_1(t))-x_2(\varphi_1(t))|\\
&+\left|g_1(t,x_1(\varphi_2(t)))-g_2(t,x_2(\varphi_2(t)))\right|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f_1(s,I^{\beta}u_1(s,y_1(\varphi_3(s)))|ds\\
&+|g_1(t,x_1(\varphi_2(t)))|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f_1(s,I^{\beta}u(s,y_1(\varphi_3(s))))-f_1(s,I^{\beta}u_1(s,y_2(\varphi_3(s))))\right|ds\\
&\le\ |k_1(t)|\ |x_1(\varphi_1(t))-x_2(\varphi_1(t))|\\
&+\ |L_1(t)||x_1(\varphi_2(t))-x_2(\varphi_2(t))|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big[|\lambda_1(s)|I^{\beta}|u_1(s,y_1(\varphi_3(s)))|+F_1\big]ds\\
&+\big[L_1(t)|x_1(\varphi_2(t))|+G_1\big]\\
&\times\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|\lambda_1(s)|\int_0^s\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}\big|u_1(\tau,y_1(\varphi_3(\tau)))-u_1(\tau,y_2(\varphi_3(\tau)))\big|\,ds\,d\tau\\
&\le\ \|k_1\|\ \|x_1-x_2\|+\|L_1\|\|x_1-x_2\|\\
&\times\left[\|\lambda_1\|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\int_0^s\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}\big[|\theta(\tau)||y_1(\varphi_3(\tau))|+U_1\big]ds\,d\tau+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}F_1ds\right]\\
&+[\|L_1\|\|x_1\|+G_1]\|\lambda_1\|\ \|\theta_1\|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\int_0^s\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}|y_1(\varphi_3(\tau))-y_2(\varphi_3(\tau))|dsd\tau\\
&\le\ \ \|k_1\|\|x_1-x_2\|\\
&+\ \|L_1\|\|x_1-x_2\|\left[\|\lambda_1\|[\|\theta\|\|y_1\|+U_1]\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+F_1\ \frac{T^{\alpha}}{\Gamma(\alpha+1)}\right]\\
&+[\|L_1\|\|x_1\|+G_1]\|\lambda_1\|\|\theta_1\|\|y_1-y_2\|\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}.
\end{aligned}$$

Taking the supremum $t\in I$, we get

$$\begin{aligned}
\|x_1-x_2\|\le\ \ &\|k\|\|x_1-x_2\|+\ \|L\|\|x_1-x_2\|\left[\|\lambda\|[\|\theta\|\|y_1\|+U]\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+F\ \frac{T^{\alpha}}{\Gamma(\alpha+1)}\right]\\
&+[\|L\|\|x_1\|+G]\|\lambda\|\|\theta\|\|y_1-y_2\|\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}.
\end{aligned}$$

Similarly

$$\begin{aligned}\|y_1-y_2\| \leq\ & \|k\|\|y_1-y_2\| \\ & + \|L\|\|y_1-y_2\|\left[\|\lambda\|[\|\theta\|\|x_1\|+U]\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+F\,\frac{T^{\alpha}}{\Gamma(\alpha+1)}\right] \\ & + [\|L\|\|y_1\|+G]\|\lambda\|\|\theta\|\|x_1-x_2\|\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|u_1-u_2\| &= \|(x_1,y_1)-(x_2,y_2)\| \\ &= \|x_1-x_2\|+\|y_1-y_2\| \\ &\leq \|k\|\|x_1-x_2\| + \|L\|\|x_1-x_2\|\left[\|\lambda\|[\|\theta\|\|y_1\|+U]\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+F\,\frac{T^{\alpha}}{\Gamma(\alpha+1)}\right] \\ &+ [\|L\|\|x_1\|+G]\|\lambda\|\|\theta\|\|y_1-y_2\|\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\ &+ \|k\|\|y_1-y_2\| + \|L\|\|y_1-y_2\|\left[\|\lambda\|[\|\theta\|\|x_1\|+U]\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+F\,\frac{T^{\alpha}}{\Gamma(\alpha+1)}\right] \\ &+ [\|L\|\|y_1\|+G]\|\lambda\|\|\theta\|\|x_1-x_2\|\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\ &\leq \Bigg(\|k\| + \|L\|\frac{\|\lambda\|[\|\theta\|\left[\|x\|+\|y\|\right]+U]\,T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\|L\|\frac{F\,T^{\alpha}}{\Gamma(\alpha+1)} \\ &+ \frac{[\|L\|\left[\|x\|+\|y\|\right]+G]\|\lambda\|\|\theta\|\,T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Bigg)[\|x_1-x_2\|+\|y_1-y_2\|.]\end{aligned}$$

Then

$$\begin{aligned}\Bigg[1-\Bigg(&\|k\|+\|L\|\frac{\|\lambda\|[\|\theta\|\|u\|+U]T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\ &+\|L\|\frac{F\,T^{\alpha}}{\Gamma(\alpha+1)}+\frac{[\|L\|\|u\|+G]\|\lambda\|\|\theta\|T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Bigg)\Bigg]\|u_1-u_2\| \qquad \leq 0,\end{aligned}$$

which implies

$$\|u_1-u_2\|=0 \implies u_1=u_2.$$

This proves the uniqueness of the solution of CHFDEs (2). □

4.2. Continuous dependence on the delay functions

Now we show that the solution of the CHFDEs (2) is depending continuously on the delay functions $\varphi_i(t)$

Definition 1. *The solution of the initial value problem (2) depends continuously on the delay functions $\varphi_1(t)$ if $\forall\, \epsilon>0$, $\exists\, \delta>0$, such that*

$$|\varphi_1(t)-\varphi_1^*(t)|\leq\delta \quad\implies ||u-u^*||\leq\epsilon.$$

Theorem 4. *Let the assumptions of Theorem 3 be satisfied, then the solution of the CHFDEs (2) depends continuously on the delay function $\varphi_1(t)$.*

Proof. Let $u = (x, y), u^* = (x^*, y^*)$ be two solutions of CSHDE (2). Then
Let $\delta > 0$ be given such that $|\varphi_1(t) - \varphi_1^*(t)| \leq \delta, \forall\, t \geq 0$, then

$$
\begin{aligned}
&|x(t) - x^*(t)| \\
&\leq |k_1(t, x(\varphi_1(t))) - k_1(t, x^*(\varphi_1^*(t)))| \\
&+ \Big|g_1(t, x(\varphi_2(t))) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, I^\beta u_1(s, y(\varphi_3(s)))ds \\
&- g_1(t, x^*(\varphi_2(t))) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, I^\beta u_1(s, y^*((\varphi_3(s)))))ds\Big| \\
&\leq |k_1(t, x(\varphi_1(t))) - k_1(t, x^*(\varphi_1^*(t)))| \\
&+ \Big|g_1(t, x(\varphi_2(t))) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, I^\beta u_1(s, y(\varphi_3(s))))ds \\
&- g_1(t, x^*(\varphi_2(t))) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, I^\beta u(s, y(\varphi_3(s))))ds\Big| \\
&+ \Big|g_1(t, x^*(\varphi_2(t))) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, I^\beta u_1(s, y(\varphi_3(s))))ds \\
&- g_1(t, x^*(\varphi_2(t))) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, I^\beta u_1(s, y^*(\varphi_3(s))))ds\Big| \\
&\leq\; k_1(t)|x(\varphi_1(t)) \;- x^*(\varphi_1^*(t))| \\
&+ \Big|g_1(t, x(\varphi_2(t))) - g_1(t, x^*(\varphi_2(t)))\Big| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f_1(s, I^\beta\; u_1(s, y(\varphi_3(s))))|\; ds \\
&+ |g_1(t, x^*(\varphi_2(t)))| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \big|f_1(s, I^\beta u_1(s, y(\varphi_3(s)))) - f_1(s, I^\beta u_1(s, y^*(\varphi_3(s))))\big| ds \\
&\leq \|k_1\| \big[|x(\varphi_1(t)) - x^*(\varphi_1(t))| + |x^*(\varphi_1(t)) - x^*(\varphi_1^*(t))|\big] \\
&+ |L_1(t)||x(\varphi_2(t)) - x^*(\varphi_2(t))| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [\lambda_1(t)|I^\beta|u_1(s, y(\varphi_3(s)))| + F_1]ds \\
&+ (|L_1(t)||x^*(\varphi_2(t))| + G_1) \\
&\times \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |\lambda_1(t)| \int_0^\tau \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} \big|u_1(\tau, y(\varphi_3(\tau))) - u_1(\tau, y^*(\varphi_3(\tau)))\big|\; ds\; d\tau \\
&+ \|L_1\|\|x - x^*\| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [|\lambda_1(t)| I^\beta \big[|\theta_1(t)|y(\varphi_3(s))| + U_1] + F_1\big] ds \\
&+\; (\|L_1\|\|x^*\| + G_1)\; \|\lambda_1\| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |\theta_1(t)| \int_0^\tau \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} |y(\varphi_3(\tau)) - y^*(\varphi_3(\tau))|\; ds\; d\tau \\
&\leq \|k_1\| \big[\|x - x^*\| + |x^*(\varphi_1(t)) - x^*(\varphi_1^*(t))|\big] \\
&+ \|L_1\|\|x - x^*\|\|\lambda_1\|[\|\theta_1\|\|y\| + U_1] \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}
\end{aligned}
$$

$$\begin{aligned}
&\int_0^\tau \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} ds d\tau \\
&\quad + \|L_1\| \|x - x^*\| \, F_1 \, \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&\quad + \; (\|L_1\| \|x_1\| + G_1) \, \|\lambda_1\| \, \|y - y^*\| \, \|\theta_1\| \, \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^\tau \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} ds \, d\tau \\
&\quad \leq \|k_1\| \big[\|x - x^*\| + |x^*(\varphi_1(t)) - x^*(\varphi_1^*(t))| \big] \\
&\quad + \|L_1\| \|x - x^*\| \left[\frac{\|\lambda_1\| [\|\theta_1\| \|y\| + U_1] \, T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{F_1 \, T^\alpha}{\Gamma(\alpha+1)} \right] \\
&\quad + (\|L_1\| \|x^*\| + G_1) \|\lambda_1\| \|\theta_1\| \|y - y^*\| \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}.
\end{aligned}$$

Taking the supremum over $t \in I$, we get

$$\begin{aligned}
\|x \, - x^*\| \leq \|k\| \big[\|x - x^*\| + |x^*(\varphi_1(t)) - x^*(\varphi_1^*(t))| \big] \\
+ \|L\| \|x - x^*\| \left[\frac{\|\lambda\| [\|\theta\| \|y\| + U] \, T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{F \, T^\alpha}{\Gamma(\alpha+1)} \right] \\
+ (\|L\| \|x^*\| + G) \|\lambda\| \|\theta\| \|y - y^*\| \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}.
\end{aligned}$$

In a similar manna, one can drive that

$$\begin{aligned}
|y(t) - y^*(t)\| \leq \|k_2\| \big[\|y - y^*\| + |y^*(\varphi_1(t)) - y^*(\varphi_1^*(t))| \big] \\
+ \|L_2\| \|y - y^*\| \left[\frac{\|\lambda_2\| [\|\theta_2\| \|x\| + U_2] \, T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{F_2 \, T^\alpha}{\Gamma(\alpha+1)} \right] \\
+ (\|L_2\| \|y^*\| + G_2) \|\lambda_2\| \|\theta_2\| \|x - x^*\| \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}.
\end{aligned}$$

Taking the supremum over $t \in J$, we get

$$\begin{aligned}
\|y - y^*\| \leq \|k\| \big[\|y - y^*\| + |y^*(\varphi_1(t)) - y^*(\varphi_1^*(t))| \big] \\
+ \|L\| \|y - y^*\| \left[\frac{\|\lambda\| [\|\theta\| \|x\| + U] \, T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{F \, T^\alpha}{\Gamma(\alpha+1)} \right] \\
+ (\|L\| \|y^*\| + G) \|\lambda\| \|\theta\| \|x - x^*\| \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}.
\end{aligned}$$

Hence

$$
\begin{aligned}
\|u-u^*\| &= \|x-x^*\|+\|y-y^*\| \\
&\leq \|k\|\big[\|x-x^*\|+|x^*(\varphi_1(t))-x^*(\varphi_1^*(t))|\big] \\
&+\|L\|\|x-x^*\|\left[\frac{\|\lambda\|[\|\theta\|\|y\|+U]T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{F\,T^{\alpha}}{\Gamma(\alpha+1)}\right] \\
&+(\|L\|\|x^*\|+G)\|\lambda\|\|\theta\|\|y-y^*\|\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\
&+\|k\|\big[\|y-y^*\|+|y^*(\varphi_1(t))-y^*(\varphi_1^*(t))|\big] \\
&+\|L\|\|y-y^*\|\left[\frac{\|\lambda\|[\|\theta\|\|x\|+U]T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{F\,T^{\alpha}}{\Gamma(\alpha+1)}\right] \\
&+(\|L\|\|y^*\|+G)\|\lambda\|\|\theta\|\|x-x^*\|\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\
&\leq \big[\|k\|+\|L\|\left[\frac{\|\lambda\|[\|\theta\|[\|x\|+\|y\|]+U]\,T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{F\,T^{\alpha}}{\Gamma(\alpha+1)}\right] \\
&+(\|L\|[\|x^*\|+\|y^*\|]+G)\|\lambda\|\|\theta\|\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\big][\|x-x^*\|+\|y-y^*\|] \\
&+\|k\|\big[|x^*(\varphi_1(t))-x^*(\varphi_1^*(t))|+|y^*(\varphi_1(t))-y^*(\varphi_1^*(t))|\big] \\
&\leq \big[\|k\|+\|L\|\left[\frac{\|\lambda\|[\|\theta\|\|u\|+U]\,T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{F\,T^{\alpha}}{\Gamma(\alpha+1)}\right] \\
&+(\|L\|\|u\|+G)\|\lambda\|\|\theta\|\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\big]\|u-u^*\| \\
&+\|k\|\big[|x^*(\varphi_1(t))-x^*(\varphi_1^*(t))|+|y^*(\varphi_1(t))-y^*(\varphi_1^*(t))|\big]
\end{aligned}
$$

$$
\|u-u^*\| \leq \frac{\|k\|\big[|x^*(\varphi_1(t))-x^*(\varphi_1^*(t))|+|y^*(\varphi_1(t))-y^*(\varphi_1^*(t))|}{1-\left(\|k\|+\|L\|\left[\frac{\|\lambda\|[\|\theta\|\|u\|+U]\,T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{F\,T^{\alpha}}{\Gamma(\alpha+1)}\right]+(\|L\|\|u\|+G)\|\lambda\|\|\theta\|\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right)}.
$$

But from the continuity of solution x^* and y^*, we have

$$
\begin{aligned}
|\varphi_1(t)-\varphi_1^*(t)|\leq\delta \quad &\Longrightarrow \quad |x^*(\varphi_1(t))-x^*(\varphi_1^*(t))|\leq\epsilon_1 \\
&\Longrightarrow \quad |y^*(\varphi_1(t))-y^*(\varphi_1^*(t))|\leq\epsilon_2.
\end{aligned}
$$

Then

$$
\begin{aligned}
\|u-u^*\| &\leq \frac{\|k\|\big[\epsilon_1+\epsilon_2\big]}{1-\left(\|k\|+\|L\|\left[\frac{\|\lambda\|[\|\theta\|\|u\|+U]T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{FT^{\alpha}}{\Gamma(\alpha+1)}\right]+(\|L\|\|u\|+G)\|\lambda\|\|\theta\|\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right)} \\
&\leq \epsilon.
\end{aligned}
$$

This means that the solution of the CHFDEs (2) depends continuously on delay function φ_1. This completes the proof. ■

By a similar way as done above, the continuous dependence of the solution of the CHFDEs (2) on delay functions φ_2 and φ_3 can be studied. $\square$

5. CONCLUSION

We have proven an auxiliary lemma related to the linear variant of the CHFDEs (2) and stated sufficient conditions which guarantee the existence of solutions to the CHFDEs (2) in a Banach algebra due to Dhage [4], Results on the existence and continuous dependence of solutions for CHFDEs on delay function φ_1 were also studied. It should be noted that in the same way, the reader can get the continuous dependence of solutions for CHFDEs on the other delay functions.

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