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Ulam-type Stability for a Boundary Value Problem of Implicit Fractional-orders Differential Equation

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Abstract

We discuss sufficient conditions for the existence of solutions for a boundary value problem of implicit fractional-orders differential equation. Some types of Ulam stability for our problem will bee establish.

Keywords and Phrases: Implicit differential-orders differential equation, Caputo derivative, existence results, boundary value problems, Green's function, Ulam stability.

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1. INTRODUCTION

Recently, some mathematicians considered boundary value problems for fractional-orders differential equations (see [1]-[5], [7]-[22] and references therein). Let $1 < \beta < \alpha \leq 2$. Here, we establish existence, Uniqueness and stability results for the boundary value problem of the implicit fractional-orders differential problem (IFDP):

$$^cD^\alpha y(t) = f\big(t, y(t), ^cD^\beta y(t), \int_0^t k(t,s)^cD^\alpha y(s)ds\big), \ \ t \in I = [0;T], \tag{1}$$

$$y(0) = y_\circ, \ \ y(T) = y_T. \tag{2}$$

The paper is organized as follows: Section 2, we present our main result by using Schauder's fixed point theorem. Furthermore, we prove the stability of solution of our problem. In Section 3, we give examples to demonstrate the application of our main results. Finally, we present conclusion in Section 4.

2. EXISTENCE OF SOLUTIONS

Consider the problem (1)-(2) under the following assumptions

(H_1) $f : I \times R^3 \to R$ is continuous and there exists $\psi \in C(I, R_+)$, with norm $\|\psi\|$, such that:

$$|f(t, u_1, u_2, u_3) - f(t, v_1, v_2, v_3)| \leq \psi(t)\big(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|\big),$$

$\forall\, t \in I,\ u_i, v_i \in R,\ (i = 1, 2, 3).$

(H_2) $k(t, s)$ is continuous for all $(t, s) \in I \times I$, and there is a positive constant K such that

$$\max_{t,s\in[0,T]} |k(t, s)| = K.$$

Remark:
From assumption (H_1), we have

$$\begin{aligned}|f(t, u_1, u_2, u_3)| - |f(t, 0, 0, 0)| &\leq |f(t, u_1, u_2, u_3) - f(t, 0, 0, 0)|\\ &\leq \psi(t)(|u_1| + |u_2| + |u_3|),\end{aligned}$$

then

$$|f(t, u_1, u_2, u_3)| \leq \psi(t)(|u_1| + |u_2| + |u_3|) + |f(t, 0, 0, 0)|,$$

and

$$|f(t, u_1, u_2, u_3)| \leq \|\psi\|(|u_1| + |u_2| + |u_3|) + F, \text{where } \ F = \sup_{t\in I} |f(t, 0, 0, 0)|.$$

Lemma 1. *If the solution of IFDP (1)-(2) exist, it can be represented by the integral equation*

$$y(t) = h(t) + \int_0^T G(t, s)u(s)ds, \tag{3}$$

where u is the solution of the functional integral equation

$$u(t) = f\big(t, h(t) + \int_0^T G(t, s)u(s)ds, I^{\alpha-\beta}u(t), \int_0^t k(t, s)u(s)ds\big), \tag{4}$$

$G(t, s)$ is the Green's function defined by

$$G(t, s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{t(T-s)^{\alpha-1}}{T\Gamma(\alpha)} & 0 \leq s \leq t \leq T, \\ \\ \frac{-t(T-s)^{\alpha-1}}{T\Gamma(\alpha)} & 0 \leq t \leq s \leq T \end{cases} \tag{5}$$

with

$$G_0 := \max\{|G(t,s)|, (t,s) \in I \times I\},$$

and

$$h(t) = y_\circ + \frac{(y_T - y_\circ)t}{T}. \tag{6}$$

Proof. It is not difficult (see [21]) to verify that ${}^cD^\beta y(t) = I^{\alpha-\beta}{}^cD^\alpha y(t)$ for $t \in I$. If y is a solution of equation (1), then

$${}^cD^\alpha y(t) = f\big(t, y(t), I^{\alpha-\beta}\, {}^cD^\alpha y(t), \int_0^t k(t,s)\, {}^cD^\alpha y(s)ds\big), \ \ t \in J =: (0;T].$$

Let ${}^cD^\alpha y(t) = u(t)$ in equation (1), then

$$u(t) = f(t, y(t), I^{\alpha-\beta} u(t), \int_0^t k(t,s)u(s)ds)$$

and

$$y(t) = c_\circ + c_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)ds$$

from which can get $c_o = y_o$ and

$$c_1 = -\frac{1}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} u(s)ds + \frac{(y_T - y_\circ)}{T}.$$

Then the solution of (1)-(2) is given by

$$\begin{aligned}
y(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} u(s)ds + (1 - \frac{t}{T})y_\circ + \frac{t}{T} y_1 \\
&= \frac{1}{\Gamma(\alpha)} \Big[\int_0^t [(t-s)^{\alpha-1} - \frac{t}{T}(T-s)^{\alpha-1}]u(s)ds - \frac{t}{T} \int_t^T (T-s)^{\alpha-1} u(s)ds \Big] \\
&\quad + (1 - \frac{t}{T})y_\circ + \frac{t}{T} y_1.
\end{aligned}$$

As a result, we have equation (3) and (4). □

Definition 1. *By a mild solution of IFDP (1)-(2), we mean a function $u \in C(I, R)$ satisfying integral equation (4).*

Our first existence result is based on Schauder's fixed point theorem.

Theorem 1. *Let (H_1)-(H_2) be hold. If*

$$\frac{\|\psi\|}{1 - \aleph} < 1, \ \ \aleph = \frac{T^{\alpha-\beta}\|\psi\|}{\Gamma(\alpha - \beta + 1)} + \|\psi\| KT.$$

Then the IFDP (1)-(2) has at least one mild solution on I.

Proof. Transform the IFDP (1)-(2) into a fixed point problem. Define the operator $A : C(I, R) \to C(I, R)$ by:

$$Ay(t) = h(t) + \int_0^T G(t,s)v(s)ds, \tag{7}$$

where $v \in C(I, R)$ satisfies the implicit functional equation

$$v(t) = f\big(t, y(t), I^{\alpha-\beta}v(t), \int_0^t k(t,s)v(s)ds\big).$$

and

$$h(t) = y_\circ + \frac{(y_T - y_\circ)t}{T}$$

with G are the functions defined by (5).
Define the ball

$$B_\varrho = \{y \in C(I,R) : \|y\| \leq \varrho\}, \ \ \varrho \geq \frac{|y_T| + \frac{G_\circ TF}{1-\aleph}}{1 - \frac{G_\circ T\|\psi\|}{1-\aleph}}.$$

It is clear that the set B_ϱ is nonempty, bounded, closed and convex. We demonstrate that the operator A defined by (7) meets the hypothesis of the fixed point theorem of Schauder. The proof will be presented in several steps.

Step.1 : The operator A is continuous.
Consider a sequence $\{x_n\} \subset B_\varrho$ such that $x_n \to x$ in B_ϱ. To show that A is continuous, we have to prove that

$$\|Ax_n - Ax\| \to 0 \ \text{ as } \ n \to \infty.$$

For this, we have

$$|Ax_n(t) - Ax(t)| \leq \int_0^T |G(t,s)||u_n(s) - u(s)|ds \tag{8}$$

where $u_n, u \in C(I, R)$, such that

$$u_n(t) = f\big(t, x_n(t), I^{\alpha-\beta}u_n(t), \int_0^t k(t,s)u_n(s)ds\big),$$
$$u(t) = f\big(t, x(t), I^{\alpha-\beta}u(t), \int_0^t k(t,s)u(s)ds\big)$$

and by (H_1), we have

$$\begin{aligned}
&|u_n(t) - u(t)| \\
&= |f\big(t, x_n(t), I^{\alpha-\beta}u_n(t), \int_0^t k(t,s)u_n(s)ds\big) - f\big(t, x(t), I^{\alpha-\beta}u(t), \int_0^t k(t,s)u(s)ds\big)| \\
&\leq \ \psi(t)\ \big(|x_n(t) - x(t)| + \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}|u_n(s) - u(s)|\, ds + \int_0^t k(t,s)|u_n(s) - u(s)|ds\big),
\end{aligned}$$

then

$$||u_n(t) - u(t)|| \leq \|\psi\| \big(\|x_n - x\| + \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\|u_n - u\| + K\,T\|u_n - u\|\big).$$

Thus

$$\|u_n - u\| \leq \frac{\|\psi\|}{1 - \|\psi\|\big(K\,T + \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\big)}\|x_n - x\|.$$

Since $x_n \to x$, then we get $u_n(t) \to u(t)$ as $n \to \infty$ for each $t \in I$. And let $\varepsilon > 0$ be such that, for each $t \in I$, we have $|u_n(t)| \leq \varepsilon$, and $|u(t)| \leq \varepsilon$. Then, we have

$$\begin{aligned} |G(t,s)||u_n(s) - u(s)| &\leq |G(t,s)|\,(|u_n(s)| + |u(s)|) \\ &\leq 2\varepsilon|G(t,s)|. \end{aligned}$$

For each $t \in I$, the function $s \to 2\varepsilon|G(t;s)|$ is integrable on I. Then applying Lebesgue Dominated Convergence Theorem, then (8) implies that

$$\|Ax_n - Ax\| \to 0 \ \text{ as } \ n \to \infty.$$

Consequently, A is continuous.

Step.2: A maps bounded sets into bounded sets in B_ρ, i.e., $A(B_\rho) \subset B_\rho$. Indeed, it is sufficient to demonstrate that there is a positive constant ρ for each $y \in B_\rho$, we have $\|Ay\| \leq \rho$, show that $Ay \in B_\rho$.

We have that for each $t \in I$, by the condition (H_2),

$$|Ay(t)| = |h(t) + \int_0^T G(t,s)v(s)ds| \leq |h(t)| + \int_0^T |G(t,s)||v(s)|ds, \tag{9}$$

where $v(t) = f\big(t, y(t), I^{\alpha-\beta}v(t), \int_0^t k(t,s)v(s)ds\big)$

$$\begin{aligned} |v(t)| &= \big|f\big(t, y(t), I^{\alpha-\beta}v(t), \int_0^t k(t,s)v(s)ds\big)\big| \\ &\leq \ F + \psi(t)|y(t)| + \psi(t) \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}|v(s)|\,ds + \psi(t)\int_0^t |k(t,s)||v(s)|ds. \end{aligned}$$

Taking supermum for $t \in I$, we have

$$\|v\| \leq \ F + \|\psi\|\,\|y\| + \|\psi\|\frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\|v\| + \|\psi\|K\|v\|T.$$

Thus

$$\|v\| \leq \ \frac{F + \|\psi\|\,\varrho}{1 - \big(\frac{\|\psi\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \|\psi\|KT\big)},$$

and

$$
\begin{aligned}
|h(t)| &= \left| y_\circ + \frac{(y_T - y_\circ)t}{T} \right| \\
&\leq \frac{|y_\circ|(t-T)}{T} + \frac{|y_T|t}{T} \leq |y_T|.
\end{aligned}
$$

Thus (9) implies that, for each $t \in I$,

$$
|Ay(t)| \leq |y_T| + \frac{G_\circ\, T(F + \|\psi\|\ \varrho)}{1 - \ \aleph} \leq \ \rho.
$$

Taking supermum for $t \in I$, we have

$$
||Ay\| \leq \rho.
$$

Then $A(B_\rho) \subset B_\rho$.

Step.3: $A(S)$ is relatively compact.
we prove that A maps bounded sets into equicontinuous sets of $C(I, R)$, i.e, $B\varrho$ is equicontinuous. Now, Let $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ and $t_1, t_2 \in I,\ t_1 < t_2,\ \ |t_2 - t_1| < \delta$. Then we have

$$
\begin{aligned}
|Ay(t_2) - Ay(t_1)| &\leq \int_0^T |G(t_2, s) - G(t_1, s)|\ |v(s)| ds \\
&\leq \|v\| \int_0^T |G(t_2, s) - G(t_1, s)|\ ds \\
&\leq \frac{F + \|\psi\|\ \varrho}{1 - \aleph} \int_0^T |G(t_2, s) - G(t_1, s)|\ ds.
\end{aligned}
$$

As $t_1 \to t_2$, the right-hand side of the above inequality is not dependent on y and tends to zero. Consequently,

$$
|Ay(t_2) - Ay(t_1)| \to 0,\ \ \forall\, |t_2 - t_1| \to 0.
$$

Thus, $\{Ay\}$ is equi-continuous on B_ϱ, and A is compact operator by the Arzela-Ascoli Theorem [6] and in view of the above three steps. Therefore, Operator $A : C(I, R) \to C(I, R)$ is continuous and completely continuous.
Hence, all the hypotheses of Schauder's fixed point Theorem [6] holds and shows that A has a fixed point on B_ϱ. Therefore, the IFDP (1)-(2) has a mild solution. The proof is complete. □

Our second result is based on Banach's fixed point Theorem to obtain the existence of a unique solution of the IFDP (1)-(2).

Theorem 2. *Let the assumptions of Theorem 1 holds, with*

$$\frac{G_\circ\, \|\psi\|\, T}{1-\|\psi\|\big(K\, T+\frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\big)} < 1. \tag{10}$$

Then the IFDP (1)-(2) has a unique mild solution on I.

Proof. It follows, from Theorem 1, that IFDP (1)-(2) has at least one solution. Therefore, we only need to show that the operator A described in (7) is a contraction.

Now take $x, y \in C(I, R)$. Then for $t \in I$, we have

$$Ax(t) - Ay(t) = \int_0^T G(t,s)u(s)ds - \int_0^T G(t,s)v(s)ds, \tag{11}$$

where $u, v \in C(I, R)$ be such that

$$u(t) = f\big(t, x(t), I^{\alpha-\beta}u(t), \int_0^t k(t,s)u(s)ds\big),$$
$$v(t) = f\big(t, y(t), I^{\alpha-\beta}v(t), \int_0^t k(t,s)v(s)ds\big).$$

Then, for $t \in I$

$$|Ax(t) - Ay(t)| \leq \int_0^T G(t,s)\, |u(s) - v(s)|ds, \tag{12}$$

but by condition (H_2), we have

$$\begin{aligned}
&|u(t)-v(t)| \\
&= |f\big(t, x(t), I^{\alpha-\beta}u(t), \int_0^t k(t,s)u(s)ds\big) - f\big(t, y(t), I^{\alpha-\beta}v(t), \int_0^t k(t,s)v(s)ds\big)| \\
&\leq\ \psi(t)\ \big(|x(t)-y(t)| + \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}|u(s)-v(s)|\, ds + \int_0^t k(t,s)|u(s)-v(s)|ds\big) \\
&\leq\ \|\psi\|\ \big(\|x-y\| + \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\|u-v\| + K\|u-v\|\, T\big).
\end{aligned}$$

Thus

$$\|u-v\| \leq \frac{\|\psi\|}{1-\|\psi\|\big(K\, T+\frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\big)}\|x-y\|.$$

Retrain to (12) , we have

$$|Ax(t) - Ay(t)| \leq \frac{G_\circ\, \|\psi\|\, T}{1-\|\psi\|\big(K\, T+\frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\big)}\|x-y\|.$$

Taking supermum for $t \in I$, we have

$$\|Ax - Ay\| \leq \big(\frac{G_\circ \ \|\psi\| \ T}{1 - \|\psi\|\big(K\ T + \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\big)}\big)\|x - y\|.$$

By $\big(\frac{G_\circ \ \|\psi\| \ T}{1 - \|\psi\|\big(K\ T + \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\big)}\big) < 1$, the operator A is a contraction. Hence, by Banach's contraction principle, A has a unique fixed point which is a mild solution of the IFDP (1)-(2) on I. □

Now, we consider the Ulam stability for IFDP (1)-(2). Let $\epsilon > 0$ and $\Phi : I \rightarrow R_+$ be a continuous function. We consider the following inequalities:

$$|^c D^\alpha y(t) - f\big(t, y(t), ^c D^\beta y(t), \int_0^t k(t,s)^c D^\alpha y(s)ds\big)| \leq \epsilon(t), \ t \in I \tag{13}$$

$$|^c D^\alpha y(t) - f\big(t, y(t), ^c D^\beta y(t), \int_0^t k(t,s)^c D^\alpha y(s)ds\big)| \leq \Phi(t), \ t \in I \tag{14}$$

$$|^c D^\alpha y(t) - f\big(t, y(t), ^c D^\beta y(t), \int_0^t k(t,s)^c D^\alpha y(s)ds\big)| \leq \epsilon\, \Phi(t), \ \ t \in I. \tag{15}$$

Definition 2. *[22] The IFDP (1)-(2) is Ulam−Hyers stable if there exists a real number $c_f > 0$ such that there exists a solution $x \in C(I, R)$ of (1)-(2) for*

$$|y(t) - x(t)| \leq \epsilon\, c_f, \ \ t \in I.$$

for each solution $y \in C(I, R)$ of the inequality (13).

Definition 3. *[22] The IFDP (1)-(2) is generalized to be Ulam−Hyers stable if there is $c_f \in C(R_+, R_+)$ with $c_f(0) = 0$ so that there is a solution $x \in C(I, R)$ of (1)-(2) with*

$$|y(t) - x(t)| \leq c_f(\epsilon), \ \ t \in I.$$

for each $\epsilon > 0$ and for each solution $y \in C(I, R)$ of the inequality (13).

Definition 4. *[22] The IFDP (1)-(2) is Ulam−Hyers−Rassias stable with respect to Φ if there exists a real number $c_{f,\Phi} > 0$ such that there is a solution $x \in C(I, R)$ of (1)-(2) with*

$$|y(t) - x(t)| \leq \epsilon\, c_{f,\Phi}\Phi(t), \ \ t \in I.$$

for each $\epsilon > 0$ and for each solution $y \in C(I, R)$ of the inequality (15).

Definition 5. *[22] The IFDP (2) is generalized Ulam−Hyers−Rassias stable with respect to Φ if the actual number $c_{f,\Phi} > 0$ exists in such a way that for each solution $y \in C(I, R)$ of the inequality (14) there is a solution $x \in C(I, R)$ of (1)-(2) with the solution $x \in C(I, R)$ of the inequality.*

$$|y(t) - x(t) > | \leq c_{f,\Phi}\Phi(t), \ \ t \in I.$$

2.1 Ulam-Hyers Stability

Next, we present the following Ulam−Hyers stable result.

Theorem 3. *Let the assumptions of Theorem 2 be satisfied. Then IFDP (1)-(2) is Ulam−Hyers stable.*

Proof. Let $\epsilon > 0$ and let $z \in C(I, R)$ be a function which satisfies the inequality (13),

$$|^c D^\alpha z(t) - f\big(t, z(t),^c D^\beta z(t), \int_0^t k(t,s)^c D^\alpha z(s)ds\big)| \leq \epsilon, \quad t \in I$$

and let $y \in C(I, R)$ be the unique solution of IFDE (1)-(2) which is by lemma 1 the IFDP (1)-(2) equivalence to fractional order integral equation

$$y(t) = h(t) + \int_0^T G(t,s)u(s)ds,$$

where u is the solution of the functional integral equation

$$u(t) = f\big(t, h(t) + \int_0^T G(t,s)u(s)ds, I^{\alpha-\beta}u(t), \int_0^t k(t,s)u(s)ds\big).$$

Operating by I^α on both sides of (13), and then integrating, we get

$$\big|z(t) - h(t) - \int_0^T G(t,s)v(s)ds\big| \leq \frac{\epsilon\, T^\alpha}{\Gamma(\alpha+1)}. \tag{16}$$

For each $t \in I$, we have

$$\begin{aligned}
|z(t) - y(t)| &= \big|z(t) - h(t) - \int_0^T G(t,s)u(s)ds\big| \\
&\leq \big|z(t) - h(t) - \int_0^T G(t,s)v(s)ds\big| \\
&+ \big|h(t) + \int_0^T G(t,s)v(s)ds - h(t) - \int_0^T G(t,s)u(s)ds\big| \\
&\leq \frac{\epsilon\, T^\alpha}{\Gamma(\alpha+1)} + \int_0^T G(t,s)|v(s) - u(s)|ds \\
&\leq \frac{\epsilon\, T^\alpha}{\Gamma(\alpha+1)} + G_\circ \|u - v\|\, T.
\end{aligned}$$

Indeed, from proof of Theorem 2, we have

$$\|u - v\| \leq \frac{\|\psi\|}{1 - \|\psi\|\big(K\,T + \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\big)} \|z - y\|.$$

Then, for each $t \in I$

$$\|z-y\| \leq \frac{\epsilon\, T^{\alpha}}{\Gamma(\alpha+1)} + \frac{G_{\circ}\, \|\psi\|\, T}{1-\|\psi\|\big(K\, T + \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\big)}\|z-y\|.$$

Thus

$$\|z-y\| \;\; \leq \;\; \frac{\epsilon\, T^{\alpha}}{\Gamma(\alpha+1)}\Bigg[1-\Big(\frac{G_{\circ}\, \|\psi\|\, T}{1-\|\psi\|\big(KT + \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}}\Big)\Bigg]^{-1} = \varsigma\, \epsilon,$$

for let $\varsigma = \frac{T^{\alpha}}{\Gamma(\alpha+1)}\Bigg[1-\Big(\frac{G_{\circ}\, \|\psi\|\, T}{1-\|\psi\|\big(KT+\frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\big)}\Bigg]^{-1}$. So, the IFDE (1)-(2) is Ulam-Hyers stable. □

By putting $\Phi(\epsilon) = \varsigma\, \epsilon,\ \Phi(0) = 0$ yields that the IFDE (1)-(2) is generalized Ulam-Hyers stable.

2.2 Ulam-Hyers-Rassias Stability.

Now, we state the following Ulam−Hyers−Rassias stable result.

Theorem 4. *Assume assumptions $(H_1) - (H_3)$ and*

(H_4) *The function $\Phi \in C(I, R_+)$ is increasing and there exists $\lambda_{\Phi} > 0$ such that, for each $t \in J$, we have*

$$I^{\alpha}\, \Phi(t) \leq \lambda_{\Phi}\, \Phi(t).$$

are hold. Then IFDP (1)-(2) is Ulam−Hyers−Rassias stable with respect to Φ.

Proof. Let $z \in C(I, R)$ be a solution of the inequation (15), i.e.,

$$\Big|{}^{c}D^{\alpha} z(t) - f\big(t, z(t), {}^{c}D^{\beta} z(t), \int_0^t k(t,s)\, {}^{c}D^{\alpha} z(s) ds\big)\Big| \leq \epsilon\, \Phi, \;\; t \in I$$

and let us assume that y is a solution of the problem (1)−(2). Thus, we have

$$y(t) = h(t) + \int_0^T G(t,s) u(s) ds,$$

where $u \in C(I, R)$ such that

$$u(t) = f\big(t, y(t), I^{\alpha-\beta} u(t), \int_0^t k(t,s) u(s) ds\big).$$

Operating by I^α on both sides of the inequality (15) and then integrating, we get

$$\begin{aligned}\Big|z(t)-h(t)-\int_0^T G(t,s)v(s)ds\Big| &\leq \frac{\epsilon}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,\Phi(s)ds\\ &\leq \epsilon\lambda_\Phi\Phi(t),\end{aligned}$$

where $v\in C(I,R)$ such that

$$v(t)=f\big(t,z(t),I^{\alpha-\beta}v(t),\int_0^t k(t,s)v(s)ds\big).$$

For each $t\in I$, we have

$$\begin{aligned}|z(t)-y(t)| &= \Big|z(t)-h(t)-\int_0^T G(t,s)u(s)ds\Big|\\ &\leq \Big|z(t)-h(t)-\int_0^T G(t,s)v(s)ds\Big|\\ &+ \Big|h(t)+\int_0^T G(t,s)v(s)ds-h(t)-\int_0^T G(t,s)u(s)ds\Big|\\ &\leq \epsilon\,\lambda_\Phi\,\Phi(t)+\int_0^T G(t,s)|v(s)-u(s)|ds\\ &\leq \epsilon\,\lambda_\Phi\,\Phi(t)+\int_0^T G(t,s)|v(s)-u(s)|ds\\ &\leq \epsilon\,\lambda_\Phi\,\Phi(t)+G_\circ\|v-u\|\,T.\end{aligned}$$

Indeed, from proof of Theorem 2, we have

$$\|u-v\|\leq\frac{\|\psi\|}{1-\|\psi\|\big(K\,T+\frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\big)}\|z-y\|.$$

Then, for each $t\in I$

$$\|z-y\|\leq\epsilon\,\lambda_\Phi\,\Phi(t)+\frac{G_\circ\,\|\psi\|\,T}{1-\|\psi\|\big(K\,T+\frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\big)}\|z-y\|.$$

Thus

$$\|z-y\|\leq\left[1-\frac{G_\circ\,\|\psi\|\,T}{1-\|\psi\|\big(K\,T+\frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\big)}\right]^{-1}\epsilon\,\lambda_\Phi\,\Phi(t)=c_\Phi\epsilon\,\Phi(t),$$

where

$$c_\Phi=\left[1-\frac{G_\circ\,\|\psi\|\,T}{1-\|\psi\|\big(K\,T+\frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\big)}\right]^{-1}\lambda_\Phi.$$

So, the problem IFDP (1)-(2) is Ulam-Hyers-Rassias stable with respect to Φ. □

3. EXAMPLE

Example 1. Given the following IFDP:

$$ {}^cD^{\frac{1}{2}} = \frac{2 + y(t) +^c D^{\frac{4}{3}} + \int_0^1 e^{t-s}\, {}^cD^{\frac{1}{2}} y(s) ds}{2e^{t+1}\big(1 + y(t) +^c D^{\frac{4}{3}} + \int_0^1 e^{t-s}\, {}^cD^{\frac{1}{2}} y(s) ds\big)}, \qquad t \in [0,1] \tag{17} $$

$$ y(0) = 1 \quad \text{and} \quad y(1) = 1. \tag{18} $$

Set

$$ f(t,u,v,w) \quad = \quad \frac{2 + |u| + |v| + |w|}{2e^{t+1}\big(1 + |u| + |v| + |w|\big)}. $$

Obviously, f is mutually continuous function. In fact, for any $u_1, v_1, w_1, u_2, v_2, w_2 \in R \ and \ t \in [0,1]$

$$ |f(t,u,v,w) - f(t,u_1,v_1,w_1)| \le \frac{1}{2e^2}\big(|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|\big). $$

Hence condition (H_2) is satisfied with $\psi(t) = \frac{1}{2e^{t+1}}$.
Also,

$$ |f(t,u,v,w)| \quad = \quad \frac{1}{2e^{t+1}}\big(2 + |u| + |v| + |w|\big), \ \text{where} \ F = \frac{1}{e^2}, \ \text{and} \ \|\psi\| = \frac{1}{2e^2}. $$

Thus condition

$$ \frac{\|\psi\|}{1 - \aleph} \approx 0.393913 < 1. $$

where $\aleph = \frac{T^{\alpha-\beta}\|\psi\|}{\Gamma(\alpha-\beta+1)} + \|\psi\|KT$, with $T = 1$, $\alpha = \frac{1}{2}$, $\beta = \frac{4}{3}$, $F = \frac{1}{e^2}$, $\|\psi\| = \frac{1}{2e}$ and $K = e$. It follows from Theorem 1 that the IFDP (17)-(18) has at least one mild solution on I.

Example 2 Consider the following IFDP:

$$ {}^cD^{\frac{3}{2}} = \frac{1}{2e^{t+1}\big(1 + y(t) +^c D^{\frac{4}{3}} + \int_0^1 e^{t-s}\, {}^cD^{\frac{3}{2}} y(s) ds\big)} \tag{19} $$

$$ y(0) = 1 \quad \text{and} \quad y(1) = 2. \tag{20} $$

Set

$$ f(t,u,v,w) \quad = \quad \frac{1}{2e^{t+1}\big(1 + |u| + |v| + |w|\big)}, \quad t \in [0,1], \ u,v,w \in R. $$

Clearly, the function f is jointly continuous.
For any $u_1.v_1, w_1, u_2.v_2, w_2 \in R$ and $t \in [0,1]$

$$\begin{aligned} |f(t,u_1,v_1,w_1) \; &- \; f(t,u_2,v_2,w_2)| \\ &\leq \left| \frac{1}{2e^{t+1}} \left[\frac{1}{\big(1+|u_1|+|v_1|+|w_1|\big)} - \frac{1}{\big(1+|u_2|+|v_2|+|w_2|\big)} \right] \right| \\ &\leq \left| \frac{1}{2e^{t+1}} \left[\frac{|u_1-u_2|+|v_1-v_2|+|w_1-w_2|}{\big(1+|u_1|+|v_1|+|w_1|\big)\big(1+|u_2|+|v_2|+|w_2|\big)} \right] \right| \\ &\leq \frac{1}{2e^2}\big(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|\big). \end{aligned}$$

Hence condition (H_2) is hold with $||\psi|| = \frac{1}{2e^2} < 1$. From (5) the function G is given by

$$G(t,s) = \begin{cases} \frac{(t-s)^{\frac{1}{2}}}{\Gamma(\alpha)} - \frac{t(1-s)^{\frac{1}{2}}}{T\Gamma(\alpha)} & 0 \leq s \leq t \leq 1, \\ \\ \frac{-t(1-s)^{\frac{1}{2}}}{T\Gamma(\alpha)} & 0 \leq t \leq s \leq 1, \end{cases}$$

Clearly $G_0 < 2$. Thus We shall check that condition 10 thus

$$\frac{G_\circ \, \|\psi\| \, T}{1 - \|\psi\|\big(K\,T + \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\big)} = 0.1821179198 < 1.$$

is hold with $T = 1$, $\alpha = \frac{3}{2}$, $\beta = \frac{4}{3}$, $||\psi|| = \frac{1}{2e^2}$ and $K = e$. It follows from Theorem 2 that the problem (19)-(20) as a unique solution on I.

4. CONCLUSION

In this current research paper. First, the equivalence between IFDP (1)-(2) and the Volterra integration equation (3) was developed in our study. Secondly, the existence and uniqueness of mild solutions for boundary value problems of implicit fractional order differential equations were established based on Schauder's fixed point theorem, and Banach contraction principle. We found the Ulam Hayers stability and the generalized Ulam Hayers stability, the stability of Ulam - Hyers - Rassias and the generalized stability of Ulam - Hyers - Rassias allowed on the implicit differential equation of fractional order, supplemented with fractional integral type boundary conditions. Finally, we end the article with illustrations examples to prove the applicability of the result obtained.

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