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Some Definite Integral Involving Some Valuable Special Functions

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Abstract

In this paper we have established some Definite Integral formulae involving Bessel function of the first kind, Hypergeometric function, Struve function and Sine function.

Key Words : Bessel Function,Hypergeometric Function, Boolean Algebra.

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1. INTRODUCTION

Yurry A. Brychkov[Brychkov p.201(4.7.5)] has derived the following formulae

$$\int_0^1 x^0 cos^{-1}x\ J_0(ax)dx = \frac{1}{a}Si(a) \tag{1.1}$$

$$\int_0^1 x^1 cos^{-1}x\ J_0(ax)dx = \frac{\pi}{2a}J_0(\frac{a}{2})J_1(\frac{a}{2}) \tag{1.2}$$

$$\int_0^1 x^2 cos^{-1}x\ J_0(ax)dx = \frac{1}{a^3}[2sin\ a - acos\ a - Si(a)] \tag{1.3}$$

$$\int_0^1 cos^{-1}x\ J_1(ax)dx = \frac{\pi}{2a}[1 - J_0^2(\frac{a}{2})] \tag{1.4}$$

$$\int_0^1 xcos^{-1}x\ J_1(ax)dx = \frac{1}{a^2}[Si(a) - sin\ a] \tag{1.5}$$

$$\int_0^1 \frac{1}{x}cos^{-1}x\ J_1(ax)dx = Si(a) + \frac{1}{a^2}(cos\ a - 1) \tag{1.6}$$

Bessel functions of the first kind, denoted as $J_\alpha(x)$, are solutions of Bessel's differential equation that are finite at the origin ($x = 0$) for integer or positive α, and diverge as x approaches zero for negative non-integer α. It is possible to define the function by its Taylor series expansion around $x = 0$.

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha} \tag{1.7}$$

where $\Gamma(z)$ is the gamma function, a shifted generalization of the factorial function to non-integer values. The Bessel function of the first kind is an entire function if α is an integer.

A generalized hypergeometric function ${}_pF_q(a_1, ..., a_p; b_1, ..., b_q; z)$ is a function which can be defined in the form of a hypergeometric series, i.e., a series for which the ratio of successive terms can be written

$$\frac{c_{k+1}}{c_k} = \frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)...(k+a_p)}{(k+b_1)(K+b_2)...(k+b_q)(k+1)}\, z. \tag{1.8}$$

Where $k+1$ in the denominator is present for historical reasons of notation[Koepf p.12(2.9)], and the resulting generalized hypergeometric function is written

$${}_pF_q\left[\begin{array}{lcl} a_1, a_2, \cdots, a_p & ; & \\ & & z \\ b_1, b_2, \cdots, b_q & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k \cdots (a_p)_k z^k}{(b_1)_k(b_2)_k \cdots (b_q)_k k!} \tag{1.9}$$

where the parameters $b_1, b_2, \cdots, b_q$ are positive integers.

The ${}_pF_q$ series converges for all finite z if $p \leq q$, converges for $\mid z \mid < 1$ if $p = q+1$, diverges for all z , $z \neq 0$ if $p > q+1$[Luke p.156(3)].

The function ${}_2F_1(a, b; c; z)$ corresponding to $p = 2, q = 1$, is the first hypergeometric function to be studied (and, in general, arises the most frequently in physical problems), and so is frequently known as "the" hypergeometric equation or, more explicitly, Gauss's hypergeometric function [Gauss p.123-162]. To confuse matters even more, the term "hypergeometric function" is less commonly used to mean closed form, and "hypergeometric series" is sometimes used to mean hypergeometric function.

In mathematics, the falling factorial or Pochhammer symbol (sometimes called the descending factorial,falling sequential product, or lower factorial) is defined as the polynomial[Steffensen p.8]

$$(x)_n = x(x-1)(x-2).......(x-n+1) = \prod_{k=1}^{n}(x-k+1) = \prod_{k=0}^{n-1}(x-k) \tag{1.11}$$

Struve functions are solutions of the non-homogeneous Bessel's differential equation:

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\alpha^2)y=\frac{4\left(\frac{x}{2}\right)^{\alpha+1}}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})} \tag{1.12}$$

and are defined as:

$$H_\alpha(x)=\frac{2\left(\frac{x}{2}\right)^{\alpha}}{\Gamma(\alpha+\frac{1}{2})\Gamma\left(\frac{1}{2}\right)}\int_0^{\frac{\pi}{2}} sin(x\ cos\ \theta)sin^{2\alpha}(\theta)\ d\theta \tag{1.13}$$

The basic operations of Boolean algebra are as follows:
AND (conjunction), denoted $x\wedge y$, satisfies $x\wedge y=1$ if $x=y=1$, and $x\wedge y=0$ otherwise.
OR (disjunction), denoted $x\vee y$, satisfies $x\vee y=0$ if $x=y=0$, and $x\vee y=1$ otherwise.
NOT (negation), denoted $\neg x$, satisfies $\neg x=0$ if $x=1$ and $\neg x=1$ if $x=0$.

2. MAIN FORMULAE OF THE INTEGRATION

For $Re(a)\geq 0\wedge Re(n)<2\wedge(Im(a)>0\vee Re(a)>0)$

$$\int_0^1\frac{sin^{-1}x\ J_0(ax)}{x^n}dx=\frac{1}{4(n-1)}\pi\Big[2^n\Gamma(2-n)\,{}_2F_3\Big(\frac{1}{2}-\frac{n}{2},1-\frac{n}{2};1,\frac{3}{2}-\frac{n}{2},\frac{3}{2}-\frac{n}{2};-\frac{a^2}{4}\Big)-$$
$$-2\,{}_1F_2\Big(\frac{1}{2}-\frac{n}{2};1,\frac{3}{2}-\frac{n}{2};-\frac{a^2}{4}\Big)\Big] \tag{2.1}$$

For $Re(a)\geq 0\wedge Re(n)<3\wedge(Im(a)>0\vee Re(a)>0)$

$$\int_0^1\frac{sin^{-1}x\ J_1(ax)}{x^n}dx=\frac{1}{16(n-2)}\pi a\Big[2^n\Gamma(3-n)\,{}_2F_3\Big(\frac{3}{2}-\frac{n}{2},1-\frac{n}{2};2,2-\frac{n}{2},2-\frac{n}{2};-\frac{a^2}{4}\Big)-$$
$$-4\,{}_1F_2\Big(1-\frac{n}{2};2,2-\frac{n}{2};-\frac{a^2}{4}\Big)\Big] \tag{2.2}$$

For $Re(a)\geq 0\wedge Re(n)<4\wedge(Im(a)>0\vee Re(a)>0)$

$$\int_0^1\frac{sin^{-1}x\ J_2(ax)}{x^n}dx=\frac{1}{64(n-3)}\pi a^2\Big[2^n\Gamma(4-n)\,{}_2F_3\Big(\frac{3}{2}-\frac{n}{2},2-\frac{n}{2};3,\frac{5}{2}-\frac{n}{2},\frac{5}{2}-\frac{n}{2};-\frac{a^2}{4}\Big)-$$
$$-4\,{}_1F_2\Big(\frac{3}{2}-\frac{n}{2};3,\frac{5}{2}-\frac{n}{2};-\frac{a^2}{4}\Big)\Big] \tag{2.3}$$

For $Re(a)\geq 0\wedge Re(n)<5\wedge(Im(a)>0\vee Re(a)>0)$

$$\int_0^1 \frac{sin^{-1}x\ J_3(ax)}{x^n}dx = \frac{1}{256}\pi a^3\Big[8\Gamma(2-\frac{n}{2})_1F_2\Big(2-\frac{n}{2};4,3-\frac{n}{2};-\frac{a^2}{4}\Big)-$$
$$-2^n\Gamma(4-n)\ _2F_3\Big(2-\frac{n}{2},\frac{5}{2}-\frac{n}{2};4,3-\frac{n}{2},3-\frac{n}{2};-\frac{a^2}{4}\Big)\Big] \tag{2.4}$$

For $Re(a) \geq 0 \wedge Re(n) < 6 \wedge (Im(a) > 0 \vee Re(a) > 0)$

$$\int_0^1 \frac{sin^{-1}x\ J_4(ax)}{x^n}dx = \frac{1}{3072(n-5)}\pi a^4\Big[3\ 2^n\Gamma(6-n)\ _2F_3\Big(\frac{5}{2}-\frac{n}{2},3-\frac{n}{2};5,\frac{7}{2}-\frac{n}{2},\frac{7}{2}-\frac{n}{2};-\frac{a^2}{4}\Big)-$$
$$-4_1F_2\Big(\frac{5}{2}-\frac{n}{2};5,\frac{7}{2}-\frac{n}{2};-\frac{a^2}{4}\Big)\Big] \tag{2.5}$$

For $Re(a) \geq 0 \wedge Re(n) < 7 \wedge (Im(a) > 0 \vee Re(a) > 0)$

$$\int_0^1 \frac{sin^{-1}x\ J_5(ax)}{x^n}dx = \frac{1}{4096}\pi a^5\Big[32\ \Gamma(3-\frac{n}{2})_1F_2\Big(3-\frac{n}{2};6,4-\frac{n}{2};-\frac{a^2}{4}\Big)-$$
$$-2^n\ \Gamma(6-n)\ _2F_3\Big(3-\frac{n}{2},\frac{7}{2}-\frac{n}{2};6,4-\frac{n}{2},4-\frac{n}{2};-\frac{a^2}{4}\Big)\Big] \tag{2.6}$$

For $Re(a) \geq 0 \wedge Re(n) < 8 \wedge (Im(a) > 0 \vee Re(a) > 0)$

$$\int_0^1 \frac{sin^{-1}x\ J_6(ax)}{x^n}dx =$$
$$\frac{1}{737280(n-7)}\pi a^6\Big[45\ 2^n\Gamma(8-n)\ _2F_3\Big(\frac{7}{2}-\frac{n}{2},4-\frac{n}{2};7,\frac{9}{2}-\frac{n}{2},\frac{9}{2}-\frac{n}{2};-\frac{a^2}{4}\Big)-$$
$$-8_1F_2\Big(\frac{7}{2}-\frac{n}{2};7,\frac{9}{2}-\frac{n}{2};-\frac{a^2}{4}\Big)\Big] \tag{2.7}$$

For $Re(a) \geq 0 \wedge Re(n) < 9 \wedge (Im(a) > 0 \vee Re(a) > 0)$

$$\int_0^1 \frac{sin^{-1}x\ J_7(ax)}{x^n}dx = \frac{1}{65536}\pi a^7\Big[128\ \Gamma(4-\frac{n}{2})_1F_2\Big(4-\frac{n}{2};8,5-\frac{n}{2};-\frac{a^2}{4}\Big)-$$
$$-2^n\ \Gamma(8-n)\ _2F_3\Big(4-\frac{n}{2},\frac{9}{2}-\frac{n}{2};8,5-\frac{n}{2},5-\frac{n}{2};-\frac{a^2}{4}\Big)\Big] \tag{2.8}$$

For $Re(a) \geq 0 \wedge Re(n) < 10 \wedge (Im(a) > 0 \vee Re(a) > 0)$

$$\int_0^1 \frac{sin^{-1}x\ J_8(ax)}{x^n}dx = \frac{1}{262144}\pi a^8\Big[-2^n\Gamma(9-n)\ _2F_3\Big(\frac{9}{2}-\frac{n}{2},5-\frac{n}{2};9,\frac{11}{2}-\frac{n}{2},\frac{11}{2}-\frac{n}{2};-\frac{a^2}{4}\Big)-$$
$$-\frac{4\ _1F_2\Big(\frac{9}{2}-\frac{n}{2};9,\frac{11}{2}-\frac{n}{2};-\frac{a^2}{4}\Big)}{315(n-9)}\Big] \tag{2.9}$$

For $Re(a) \geq 0 \wedge (Im(a) > 0 \vee Re(a) > 0) \wedge Re(n) > -2$

$$\int_0^1 sin^{-1}x\ J_0(ax)x^n dx = \frac{1}{4(n+1)}\ \pi\Big[2\ {}_1F_2\Big(\frac{1}{2}+\frac{n}{2};1,\frac{3}{2}+\frac{n}{2};-\frac{a^2}{4}\Big)- -2^{-n}\ {}_2F_3\Big(\frac{n+1}{2},\frac{n+2}{2};1,\frac{n+3}{2},\frac{n+3}{2};-\frac{a^2}{4}\Big)\Big] \quad (2.10)$$

For $Re(a) \geq 0 \wedge (Im(a) > 0 \vee Re(a) > 0) \wedge Re(n) > -3$

$$\int_0^1 sin^{-1}x\ J_1(ax)x^n dx = \pi\ a\ 2^{-n-4}\Big[2^n\ n\ \Gamma(\frac{n}{2})_1F_2\Big(1+\frac{n}{2};2,2+\frac{n}{2};-\frac{a^2}{4}\Big)- -\Gamma(n+2)_2F_3\Big(\frac{n+2}{2},\frac{n+3}{2};2,\frac{n+4}{2},\frac{n+4}{2};-\frac{a^2}{4}\Big)\Big] \quad (2.11)$$

For $Re(a) \geq 0 \wedge (Im(a) > 0 \vee Re(a) > 0) \wedge Re(n) > -4$

$$\int_0^1 sin^{-1}x\ J_2(ax)x^n dx = \pi\ a^2\ 2^{-n-6}\Big[2^{n+2}\ \Gamma(\frac{n+3}{2})_1F_2\Big(\frac{n+3}{2};3,\frac{n+5}{2};-\frac{a^2}{4}\Big)- -\Gamma(n+3)_2F_3\Big(\frac{n+3}{2},\frac{n+4}{2};3,\frac{n+5}{2},\frac{n+5}{2};-\frac{a^2}{4}\Big)\Big] \quad (2.12)$$

For $Re(a) \geq 0 \wedge (Im(a) > 0 \vee Re(a) > 0) \wedge Re(n) > -5$

$$\int_0^1 sin^{-1}x\ J_3(ax)x^n dx = \pi\ a^3\ 2^{-n-8}\Big[2^{n+3}\ \Gamma(\frac{n+4}{2})_1F_2\Big(\frac{n+4}{2};4,\frac{n+6}{2};-\frac{a^2}{4}\Big)- -\Gamma(n+4)_2F_3\Big(\frac{n+4}{2},\frac{n+5}{2};4,\frac{n+6}{2},\frac{n+6}{2};-\frac{a^2}{4}\Big)\Big] \quad (2.13)$$

For $Re(a) \geq 0 \wedge (Im(a) > 0 \vee Re(a) > 0) \wedge Re(n) > -6$

$$\int_0^1 sin^{-1}x\ J_4(ax)x^n dx = \pi\ a^4\ 2^{-n-10}\Big[2^{n+4}\ \Gamma(\frac{n+5}{2})_1F_2\Big(\frac{n+5}{2};5,\frac{n+7}{2};-\frac{a^2}{4}\Big)- -\Gamma(n+5)_2F_3\Big(\frac{n+5}{2},\frac{n+6}{2};5,\frac{n+7}{2},\frac{n+7}{2};-\frac{a^2}{4}\Big)\Big] \quad (2.14)$$

For $Re(a) \geq 0 \wedge (Im(a) > 0 \vee Re(a) > 0) \wedge Re(n) > -8$

$$\int_0^1 sin^{-1}x\ J_6(ax)x^n dx = \pi\ a^6\ 2^{-n-14}\Big[2^{n+6}\ \Gamma(\frac{n+7}{2})_1F_2\Big(\frac{n+7}{2};7,\frac{n+9}{2};-\frac{a^2}{4}\Big)- -\Gamma(n+7)_2F_3\Big(\frac{n+7}{2},\frac{n+8}{2};7,\frac{n+9}{2},\frac{n+9}{2};-\frac{a^2}{4}\Big)\Big] \quad (2.15)$$

For $Re(a) \geq 0 \wedge (Im(a) > 0 \vee Re(a) > 0) \wedge Re(n) > -9$

$$\int_0^1 sin^{-1}x\ J_7(ax)x^n dx = \pi\ a^7\ 2^{-n-16}\Big[2^{n+7}\ \Gamma(\frac{n+8}{2})_1F_2\Big(\frac{n+8}{2};8,\frac{n+10}{2};-\frac{a^2}{4}\Big)- -\Gamma(n+8)_2F_3\Big(\frac{n+8}{2},\frac{n+9}{2};8,\frac{n+10}{2},\frac{n+10}{2};-\frac{a^2}{4}\Big)\Big] \quad (2.16)$$

For $Re(a) \geq 0 \wedge (Im(a) > 0 \vee Re(a) > 0) \wedge Re(n) > -10$

$$\int_0^1 sin^{-1}x\ J_8(ax)x^n dx = \pi\ a^8\ 2^{-n-18}\Big[2^{n+8}\ \Gamma(\frac{n+9}{2})_1F_2\Big(\frac{n+9}{2};9,\frac{n+11}{2};-\frac{a^2}{4}\Big)-$$
$$-\Gamma(n+9)_2F_3\Big(\frac{n+9}{2},\frac{n+10}{2};9,\frac{n+11}{2},\frac{n+11}{2};-\frac{a^2}{4}\Big)\Big] \tag{2.17}$$

3. GENERALIZED FORMULA OF MAIN FORMULAE

For $Re(a) > 0 \vee (Im(a) > 0 \wedge Re(a) \geq 0)$

$$\int_0^1 \frac{sin^{-1}x\ J_n(ax)}{x^n}dx = \pi a^n\ 2^{-n-2}\Big[\sqrt{\pi}\ _1F_2\Big(\frac{1}{2};\frac{3}{2},n+1;-\frac{a^2}{4}\Big)-{}_2F_3\Big(\frac{1}{2},1;\frac{3}{2},\frac{3}{2},n+1;-\frac{a^2}{4}\Big)\Big] \tag{3.1}$$

For $Re(a) \geq 0 \wedge (Im(a) > 0 \vee Re(a) > 0) \wedge Re(n) > -1$

$$\int_0^1 x^n sin^{-1}x\ J_n(ax)dx = \sqrt{\pi}a^n\ 2^{-n-2}\ \Gamma(\frac{1}{2}+n)\Big[\sqrt{\pi}\ _1F_2\Big(n+\frac{1}{2};n+\frac{3}{2},n+1;-\frac{a^2}{4}\Big)-$$
$$-{}_1F_2\Big(n+\frac{1}{2};n+\frac{3}{2},n+\frac{3}{2};-\frac{a^2}{4}\Big)\Big] \tag{3.2}$$

4. SPECIAL CASES OF THE FORMULA

For $(Re(a) \geq 0 \wedge (Im(a) > 0) \vee Re(a) > 0$

$$\int_0^1 xsin^{-1}x\ J_0(ax)dx = \frac{\pi[J_1(a) - J_0(\frac{a}{2})J_1(\frac{a}{2})]}{2a} \tag{4.1}$$

$$\int_0^1 xsin^{-1}x\ J_1(ax)dx = \frac{-4Si(a)+\pi^2\ a\{H_0(a)\ J_1(a) - H_1(a)\ J_0(a)\} + 4\ sin\ a}{4a^2} \tag{4.2}$$

$$\int_0^1 x^2sin^{-1}x\ J_1(ax)dx = \frac{\pi[a\ J_0^2(\frac{a}{2}) - 4J_1(\frac{a}{2})J_0(\frac{a}{2}) - a\ J_1^2(\frac{a}{2}) + 2a\ J_2(a)]}{4a^2} \tag{4.3}$$

$$\int_0^1 x^2sin^{-1}x\ J_3(ax)dx = \frac{1}{4a^3}\pi[-(a^2-16)\ J_0^2(\frac{a}{2}) + 12aJ_1(\frac{a}{2})J_0(\frac{a}{2})+$$
$$+2(a^2-8)J_0(a) + a(J_1^2(\frac{a}{2}) - 12\ J_1(a))] \tag{4.4}$$

$$\int_0^1 x^4sin^{-1}x\ J_3(ax)dx = \frac{1}{4a^4}\pi[2a^3J_4(a) - (a^2-24)aJ_0^2\Big(\frac{a}{2}\Big) + (a^2-44)aJ_1^2\Big(\frac{a}{2}\Big)$$
$$+6(3a^2-16)J_1\Big(\frac{a}{2}\Big)J_0\Big(\frac{a}{2}\Big)] \tag{4.5}$$

$$\int_0^1 x^5sin^{-1}x\ J_4(ax)dx = \frac{1}{4a^5}\pi[2a^4J_5(a)+(-13a^2+192)aJ_0^2\Big(\frac{a}{2}\Big)+5(3a^2-80)aJ_1^2\Big(\frac{a}{2}\Big)-$$
$$-2(a^4-88a^2+384)J_1\Big(\frac{a}{2}\Big)J_0\Big(\frac{a}{2}\Big)] \tag{4.6}$$

$$\int_0^1 x^6 sin^{-1}x\ J_5(ax)dx = \frac{1}{4a^6}\pi[2a^5 J_6(a) + (a^4 - 166a^2 + 1920)aJ_0^2\Big(\frac{a}{2}\Big) -$$
$$-(a^4 - 208a^2 + 4384)aJ_1^2\Big(\frac{a}{2}\Big) - 40(a^4 - 50a^2 + 192)J_1\Big(\frac{a}{2}\Big)J_0\Big(\frac{a}{2}\Big)] \quad (4.7)$$

$$\int_0^1 x^7 sin^{-1}x\ J_6(ax)dx = \frac{1}{2a^7}\pi[a^6 J_7(a) + (13a^4 - 1164a^2 + 11520)aJ_0^2\Big(\frac{a}{2}\Big) -$$
$$-14(a^4 - 110a^2 + 2016)aJ_1^2\Big(\frac{a}{2}\Big) + (a^6 - 347a^4 + 13152a^2 - 46080)J_1\Big(\frac{a}{2}\Big)J_0\Big(\frac{a}{2}\Big)] \quad (4.8)$$

$$\int_0^1 \frac{sin^{-1}x\ J_0(ax)}{x}dx =$$
$$\frac{1}{32}\pi\Big[a^2\Big\{{}_3F_4\Big(1,1,\frac{3}{2};2,2,2,2;-\frac{a^2}{4}\Big) - 2\ {}_2F_3\Big(1,1;2,2,2;-\frac{a^2}{4}\Big)\Big\} + 8\log\ 4\Big] \quad (4.9)$$

$$\int_0^1 \frac{sin^{-1}x\ J_1(ax)}{x^2}dx =$$
$$\frac{1}{128}\pi\ a\Big[a^2\Big\{{}_3F_4\Big(1,1,\frac{3}{2};2,2,2,3;-\frac{a^2}{4}\Big) - 2\ {}_2F_3\Big(1,1;2,2,3;-\frac{a^2}{4}\Big)\Big\} + 16\log\ 4\Big] \quad (4.10)$$

$$\int_0^1 \frac{sin^{-1}x\ J_4(ax)}{x^3}dx = \frac{1}{60a^2}\pi\Big[a^2(a^2+4)J_0^2\Big(\frac{a}{2}\Big) - 2a(a^2+16)J_1\Big(\frac{a}{2}\Big)J_0\Big(\frac{a}{2}\Big) +$$
$$+(a^4 + 2a^2 + 64)J_1^2\Big(\frac{a}{2}\Big) - 30a\ J_3(a)\Big] \quad (4.11)$$

$$\int_0^1 \frac{sin^{-1}x\ J_5(ax)}{x^4}dx = \frac{1}{420a^3}\pi\Big[-210a^2 J_4(a) + (a^4 + 5a^2 + 144)a^2 J_0^2\Big(\frac{a}{2}\Big) -$$
$$-2a(a^4 + 2a^2 + 576)J_1\Big(\frac{a}{2}\Big)J_0\Big(\frac{a}{2}\Big) + (a^6 + 3a^4 - 64a^2 + 2304)J_1^2\Big(\frac{a}{2}\Big)\Big] \quad (4.12)$$

$$\int_0^1 \frac{sin^{-1}x\ J_6(ax)}{x^5}dx = \frac{1}{3780a^4}\pi\Big[-1890a^3 J_5(a) + (a^6 + 6a^4 + 72a^2 + 9216)a^2 J_0^2\Big(\frac{a}{2}\Big) -$$
$$-2a(a^6 + 3a^4 - 864a^2 + 36864)J_1\Big(\frac{a}{2}\Big)J_0\Big(\frac{a}{2}\Big) + (a^8 + 4a^6 + 72a^4 - 8064a^2 + 147456)J_1^2\Big(\frac{a}{2}\Big)\Big] \quad (4.13)$$

$$\int_0^1 \frac{sin^{-1}x\ J_7(ax)}{x^6}dx =$$
$$\frac{1}{41580a^5}\pi\Big[-20790a^4 J_6(a) + (a^8 + 7a^6 + 114a^4 - 6912a^2 + 921600)a^2 J_0^2\Big(\frac{a}{2}\Big) -$$
$$-2a(a^8 + 4a^6 + 120a^4 - 142848a^2 + 3686400)J_1\Big(\frac{a}{2}\Big)J_0\Big(\frac{a}{2}\Big) +$$
$$+(a^{10} + 5a^8 + 112a^6 + 13536a^4 - 1032192a^2 + 14745600)J_1^2\Big(\frac{a}{2}\Big)\Big] \quad (4.13)$$

$$\int_0^1 \frac{sin^{-1}x\ J_8(ax)}{x^7}dx = \frac{1}{540540a^6}\pi\Big[-270270a^5J_7(a)+(a^{10}+8a^8+165a^6+6336a^4-2304000a^2+$$
$$+132710400)a^2J_0^2\Big(\frac{a}{2}\Big)-2a(a^{10}+5a^8+168a^6+140544a^4-25804800a^2+530841600)J_1\Big(\frac{a}{2}\Big)J_0\Big(\frac{a}{2}\Big)+$$
$$+(a^{12}+6a^{10}+161a^8+5904a^6+3096576a^4-169574400a^2+2123366400)J_1^2\Big(\frac{a}{2}\Big)\Big] \quad (4.14)$$

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