On Hyers-Ulam and Hyers-Ulam-Rassias stability of Second Order Linear Dynamic Equations on Time Scales

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Abstract

In this paper, we investigate new sufficient conditions for Hyers-Ulam and Hyers-Ulam-Rassias stability of second order linear dynamic equations on time scales of the form

\[ \psi^2(t) + p(t)\psi(t) - f(t) = 0, \quad t \in [a, b]_T = [a, b] \cap \mathbb{T}, \]

where \( \mathbb{T} \) is a time scale and \( f \) is rd-continuous from \([a, b]_T\) to a Banach space \( X \). Our results depend on creating an equivalent integral equation and using the fixed point theorem.

Keywords: time scales, second order linear dynamic equations on time scales, Hyers-Ulam stability and Hyers-Ulam-Rassias stability.

1. PRELIMINARIES AND INTRODUCTION

In 1940, Ulam presented the following problem related to the stability of functional equations: "give conditions in order for a linear mapping near an approximately linear mapping to exist". See [21]. The case of approximately additive mappings was solved by Hyers [7] who proved that the Cauchy equation is stable in Banach spaces.
Since then, this type of stability founded by Ulam and Hyers, famed for Hyers-Ulam stability. Recently, there has been hundreds papers appeared concerning Hyers-Ulam stability due to its applications in control theory and numerical analysis etc. In 1978, Rassias [15] extended Hyers–Ulam stability concept and called it Hyers–Ulam–Rassias stability. For more details, we refer the reader to the monograph of Jung [9].

In 1998, Alsina and Ger [5] were first authors who investigated the Hyers-Ulam stability of differential equations. This result has been generalized by Miura, Takahasi and Choda [10], by Miura [11], and by Miura, Takahasi and Miyajima [12], [13]. Popa proved that Hyers-Ulam stability of linear recurrence with constant coefficients [14]. Many articles, dealing with Hyers-Ulam stability, were edited by Rassias [16]. Wang, Zhou and Sun introduced the Hyers-Ulam stability of linear differential equations of first order [22]. In 2012 Anderson, Gates and Heuer [1] extended the work of Li and Shen [18, 19] to prove the Hyers-Ulam stability of the scalar second order linear non-homogeneous dynamic equation on bounded time scales. They obtained their results via a related Riccati dynamic equation. Also in 2012 András and Mészáros studied the Ulam-Hyers stability of some linear and nonlinear dynamic equations and integral equations on time scales based on the theory of Picard operators [2]. Hamza and Yassen extended the work of Douglas, Gates and Heuer, and investigated Hyers-Ulam stability of abstract second order linear dynamic equations on unbounded time scales [6]. In 2017, Shen established Ulam stability of first order linear dynamic equations and its adjoint equation on time scales by using the integrating factor method [20]. Recently there has been a great interest in studying stability of dynamic equations on time scales.

In this paper, we investigate new sufficient conditions for Hyers-Ulam and Hyers-Ulam-Rassias stability of second order linear dynamic equations on time scales of the form

$$\psi^\Delta^2(t) + p(t)\psi(t) - f(t) = 0, \quad t \in [a, b]_T$$

where $\mathbb{T}$ is a time scale, $p \in C_{rd}([a, b]_\mathbb{T}, \mathbb{R})$ and $f \in C_{rd}([a, b]_\mathbb{T}, X)$. Here, $[a, b]_\mathbb{T}$ is the time scale

$$[a, b]_\mathbb{T} := [a, b] \cap \mathbb{T}.$$ 

Our results depend basically on finding an equivalent integral equation to equation (1.1). The main result of the paper is that a sufficient condition for equation (1.1) to have Hyers-Ulam stability is the existence of a unique solution $\psi$ satisfying the initial conditions $\psi^\Delta^i(a) = a_i, \ i = 0, 1$ for any initial values $a_0, a_1 \in X$.

For the terminology and notations used here, we refer the reader to the very interesting monographs of Bohner and Peterson [3] and [4]. We start the paper by introducing
some of the basic definitions and notations of the calculus of time scales.

**Definition 1.1.** A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$.

**Definition 1.2.** Let $\mathbb{T}$ be a time scale. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

In this definition we put $\inf \emptyset = \sup \mathbb{T}$.

**Definition 1.3.** The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t.$$  

From now on, $X$ denotes a Banach space with a norm $\| \|$.  

**Definition 1.4.**

1. For a function $f : \mathbb{T} \to X$, $f^\sigma(t)$ is understood to mean $f(\sigma(t))$.

2. A function $f : \mathbb{T} \to X$ is said to be right-dense continuous or rd-continuous provided $f$ is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points $t$ in $\mathbb{T}$.

   The set of all rd-continuous functions $f : \mathbb{T} \to X$ will be denoted by $C_{rd}(\mathbb{T}, X)$.

3. Assume $f : \mathbb{T} \to X$, and let $t \in \mathbb{T}^k$. The delta-derivative of $f$ at $t$, denoted $f^\Delta(t)$, is defined to be the element of $X$ with the property that given any $\epsilon > 0$, there is a neighborhood $U$ of $t$ such that

$$\| [f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s] \| \leq \epsilon |\sigma(t) - s|, \quad \forall s \in U.$$

If $f^\Delta(t)$ exists we say $f$ is delta-differentiable at $t$, and we say $f^\Delta : \mathbb{T}^k \to X$ is the delta-derivative of $f$ on $\mathbb{T}^k$. For the notion $\mathbb{T}^k$, see [3] page 2. We denote by

$$f^\Delta^\sigma = (f^\Delta)^\sigma \text{ and } f^{\sigma \Delta} = (f^\sigma)^\Delta.$$  

Throughout the rest of the article, we denote by

$$C_{rd}^1([a, b]_\mathbb{T}, X) = \{ f : [a, b]_\mathbb{T} \to X | f^\Delta \text{ exists and rd-continuous } \},$$

and

$$C_{rd}^2([a, b]_\mathbb{T}, X) = \{ f : [a, b]_\mathbb{T} \to X | f^\Delta, f^{\Delta^2} \text{ exist and rd-continuous } \}.$$  

As usual for a bounded function $f$ from $\mathbb{T}$ to $X$, we denote by

$$\| f \|_{\infty} = \sup_{t \in \mathbb{T}} \| f(t) \|.$$
2. MAIN RESULTS

In this section, assume that $p \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$ and $f \in C_{rd}([a, b]_{\mathbb{T}}, X)$. We investigate Hyers-Ulam and Hyers-Ulam-Rassias stability of equation (1.1). First we recall the concept of Hyers-Ulam and Hyers-Ulam-Rassias stability. See [9].

**Definition 2.1.** (Hyers-Ulam stability)
We say that equation (1.1) has Hyers-Ulam stability if for any $\epsilon > 0$ and any $\psi \in C^2_{rd}([a, b]_{\mathbb{T}}, X)$ satisfies

$$
\|\psi^2(t) + p(t)\psi(t) - f(t)\| < \epsilon, \quad t \in [a, b]_{\mathbb{T}},
$$

there exists a solution $\phi \in C^2_{rd}([a, b]_{\mathbb{T}}, X)$ of equation (1.1) such that

$$
\|\psi(t) - \phi(t)\| < L\epsilon, \quad t \in [a, b]_{\mathbb{T}},
$$

for some $L > 0$.

**Definition 2.2.** (Hyers-Ulam-Rassias stability)
Let $C$ be a family of positive rd-continuous functions on $[a, b]_{\mathbb{T}}$. We say that equation (1.1) has Hyers-Ulam-Rassias stability of type $C$, if for any $\omega \in C$ and any $\psi \in C^2_{rd}([a, b]_{\mathbb{T}}, X)$ that satisfies

$$
\|\psi^2(t) + p(t)\psi(t) - f(t)\| < \omega(t), \quad t \in [a, b]_{\mathbb{T}},
$$

there exists a solution $\phi \in C^2_{rd}([a, b]_{\mathbb{T}}, X)$ of equation (1.1) such that

$$
\|\psi(t) - \phi(t)\| < L\omega(t), \quad t \in [a, b]_{\mathbb{T}},
$$

for some $L > 0$.

We need the following lemma in proving our results.

**Lemma 2.3.** $\psi$ is a solution of equation (1.1) if and only if $\psi$ satisfies the integral equation

$$
\psi(t) = a_0 + a_1(t - a) + \int_a^t (s - t + \mu(s))(p(s)\psi(s) - f(s))\Delta s.
$$

(2.1)

for some constants $a_0, a_1 \in X$.

**Proof.** Assume that $\psi$ satisfies the integral equation (2.1). We denote by

$$
M(t) = \int_a^t (s - t + \mu(s))(p(s)\psi(s) - f(s))\Delta s.
$$
By Theorem 1.117 in [3], we conclude that

\[ M^\Delta(t) = -\int_a^t (p(s)\psi(s) - f(s))\Delta s, \]

and

\[ M^\Delta^2(t) = -(p(t)\psi(t) - f(t)). \]

This implies that \( \psi^{\Delta^2}(t) = -(p(t)\psi(t) - f(t)) \).

To prove the other direction, assume \( \psi \) is a solution of equation (1.1). We denote by

\[ g(t) = p(t)\psi(t) - f(t), \]

\[ G(t) = \int_a^t g(s) \Delta s, \]

and

\[ L(t) = \int_a^t G(s) \Delta s. \]

Simple calculations show, by integrating two times both sides of (1.1), that

\[ \psi(t) = a_0 + a_1(t - a) - L(t). \]

Here \( a_i = \psi^{\Delta^i}(a), i = 0, 1 \). It is readily seen that, \( M(t) = -L(t) \) for every \( t \). Indeed, we have

\[ L^\Delta(t) = G(t) \]
\[ = \int_a^t g(s) \Delta s \]
\[ = -M^\Delta(t). \] (2.2)

Consequently, \( M(t) = -L(t) + C, t \in [a, \infty) \cap T \). We have \( C = M(a) + L(a) = 0 \). Therefore \( \psi \), satisfies equation (2.1).

**Corollary 2.4.** For any two elements \( a_0, a_1 \in X \), equation (1.1) has at most one solution satisfying \( \psi^{\Delta^i}(a) = a_i, i = 0, 1 \).

**Proof.** Assume that \( \psi_1 \) and \( \psi_2 \) are solutions of equation (1.1) which satisfy same initial conditions. Then both of them satisfy equation (2.1). This implies that

\[ \|\psi_1(t) - \psi_2(t)\| \leq \int_a^t |s - t + \mu(s)||p(s)||\psi_1(s) - \psi_2(s)||\Delta s. \]

By Grönwall inequality [3], \( \psi_1 = \psi_2 \).
Remark 2.5. It is well known that if equation (1.1) is regressive, that is $1 + \mu^2(t)p(t) \neq 0$ for all $t \in [a, b]_T$, then it has a unique solution $x$ that satisfies the initial conditions $x^{\Delta^i}(a) = a_i, i = 0, 1$ for every $a_0, a_1 \in X$. See [3].

Another sufficient condition for the existence of a unique solution of equation (1.1) can be stated in the following theorem.

Theorem 2.6. If there is $\alpha \in (0, 1)$ such that
\[
\int_a^t |p(s)| \Delta s \leq \frac{\alpha}{b - a + ||\mu||_\infty}, t \in [a, b]_T,
\]
then equation (1.1) has a unique solution $\psi$ that satisfies the initial conditions $\psi^{\Delta^i}(a) = a_i, i = 0, 1$ for any $a_0, a_1 \in X$.

Proof. Fix $a_0, a_1 \in X$. Define the operator $T : C_{rd}([a, b]_T, X) \to C_{rd}([a, b]_T, X)$ by
\[
T\psi(t) = a_0 + a_1(t - a) + \int_a^t (s - t + \mu(s))(p(s)\psi(s) - f(s))\Delta s.
\]
For $\phi, \psi \in C_{rd}([a, b]_T, X)$, we have
\[
\|T\psi(t) - T\phi(t)\| \leq (b - a + ||\mu||_\infty)\|\psi - \phi\|_\infty \int_a^t |p(s)| \Delta s
\]
\[
\leq \alpha\|\psi - \phi\|_\infty, \quad t \in [a, b]_T.
\]
This implies that $T$ is contraction. Therefore $T$ has a unique fixed point $\psi$ which is the solution of the integral equation (2.1) satisfying the initial conditions. \qed

The next theorem indicates the equivalence between the existence of a solution of the scalar homogeneous second order equation and the existence of a solution $z$ of the corresponding Riccati equation. See also [3].

Theorem 2.7. Assume that Riccati equation
\[
z^\Delta(1 - \mu(t)z(t)) - z^2(t) = p(t), \quad t \in [a, b]_T
\]
associated with the scalar equation
\[
\psi^{\Delta^2}(t) + p(t)\psi(t) = 0, \quad t \in [a, b]_T,
\]
has a solution $z$ that satisfies $1 - \mu(t)z(t) \neq 0, t \in [a, b]_T$. Then equation (2.5) has a solution. Conversely, if equation (2.5) has a solution with no zeros, then equation (2.4) has a solution.
Proof. Let \( z \) be a solution of (2.4) that satisfies \( 1 - \mu(t)z(t) \neq 0, t \in [a, b]_\mathbb{T} \). The function
\[
x(t) = e_{-z}(t, a)
\]
is a solution of the dynamic equation
\[
x^\Delta(t) = -z(t)x(t).
\]
In fact, for \( t \in [a, b]_\mathbb{T} \), we have
\[
x^\Delta\Delta(t) + p(t)x(t) = z^2(t)x(t) - z^\Delta(t)x(\sigma(t)) + p(t)x(t)
\]
\[
= z^2(t)x(t) - z^\Delta(t)(x(t) + \mu(t)x^\Delta(t)) + p(t)x(t)
\]
\[
= z^2(t)x(t) - z^\Delta(t)(x(t) - \mu(t)z(t)x(t)) + p(t)x(t)
\]
\[
= x(t) \left( z^2(t) - z^\Delta(t)(1 - \mu(t)z(t)) + p(t) \right)
\]
\[
= 0.
\]
Conversely, assume that \( x \) is a scalar solution of equation (2.5) with no zeros. Define \( z \) by
\[
z(t) = -\frac{x^\Delta(t)}{x(t)}, t \in [a, b]_\mathbb{T}.
\]
Then
\[
z^\Delta(t) = -\frac{x(t)x^\Delta\Delta(t) - (x^\Delta(t))^2}{x(t)x(\sigma(t))}.
\]
Simple calculations show that
\[
z^\Delta(t)(1 - \mu(t)z(t)) - z^2(t) = -\frac{x^\Delta\Delta(t)}{x(t)} = p(t), t \in [a, b]_\mathbb{T}.
\]
\[
\square
\]
In the following result we establish a new sufficient condition for the existence of a unique solution \( x \) of the scalar equation (2.5) that satisfies the initial conditions \( x^\Delta_i(a) = a_i, i = 0, 1 \) for any \( a_0, a_1 \in \mathbb{R} \).

**Theorem 2.8.** Assume that Riccati equation (2.4) has a solution \( z \) that satisfies \( 1 - \mu(t)z(t) \neq 0, t \in [a, b]_\mathbb{T} \). Then equation (2.5) has a unique solution \( x \) that satisfies the initial conditions \( x^\Delta_i(a) = a_i, i = 0, 1 \) for any \( a_0, a_1 \in \mathbb{R} \).

**Proof.** In view of Theorem 2.7, assume that \( x(t) = e_{-z}(t, a) \) is a solution of equation (2.5). We investigate another solution \( y \) of the form
\[
y(t) = u(t)x(t),
\]
where \( u \) is a scalar function which will be chosen such that \( \{x, y\} \) is a fundamental set for equation (2.5). We have
\[
y^\Delta(t) = u(t)x^\Delta(t) + u^\Delta(t)x^\sigma(t),
\]
and consequently
\[
y^{\Delta^2}(t) + p(t)y(t) = u(t)x^{\Delta^2}(t) + u^\Delta(t)x^{\Delta^\sigma}(t) + u^\Delta(t)x^\sigma(t) + p(t)u(t)x(t)
\]
\[
= u^{\Delta^2}(t)x^{\sigma^2}(t) + u^\Delta(t)x^{\Delta^\sigma}(t) + u^\Delta(t)x^{\sigma}(t). \tag{2.8}
\]
Thus \( y \) is a solution of equation (2.5) if and only if \( u \) satisfies the following equation
\[
u^\Delta(t)e^{\sigma^2}(t,a) + u^\Delta(t)(e^{\Delta^\sigma}(t,a) + e^\sigma(t,a)) = 0.
\]
Setting \( q(t) = \frac{e^{\Delta^\sigma}(t,a) + e^{\sigma\Delta}(t,a)}{e^{\sigma}(t,a)} \) and \( v = u^\Delta \), the previous equation yields
\[
v^\Delta(t) + q(t)v(t) = 0,
\]
whose solution is given by
\[
v(t) = e^{-q(t,a)}.
\]
Hence
\[
u(t) = \int_a^t e^{-q(s,a)} \Delta s.
\]
The Wronskian of \( x \) and \( y = ux \) is given by
\[
W(x,y)(t) = e^{-z}(t,a)u(t)e^\Delta(t,a) + u^\Delta(t)e^{\sigma}(t,a) - e^{-z}(t,a)u(t)e^\Delta(t,a)
\]
\[
= e^{-z}(t,a)e^\Delta(t,a) - e^{-z}(t,a)u(t)e^\Delta(t)
\]
\[
= e^{-z}(t,a)e^\Delta(t,a)e^{-q(t,a)}.
\]
\[
(2.9)
\]
It follows that the Wronskian is positive and consequently \( \{x, y\} \) is a fundamental set of equation (2.5). Then for any \( a_0, a_1 \in \mathbb{R} \), there exist \( c_1, c_2 \in \mathbb{R} \) such that the solution \( \psi(t) = c_1x(t) + c_2y(t) \) of equation (2.5) satisfies the initial conditions \( \psi^\Delta(a) = a_i, i = 0, 1 \). \( \square \)

The following result establishes a new sufficient condition for the Hyers-Ulam stability of equation (1.1).

**Theorem 2.9.** Assume that for any \( a_0, a_1 \in X \) equation (1.1) has a unique solution \( \phi \) that satisfies \( \phi^\Delta(a) = a_i, i = 0, 1 \). Then equation (1.1) has Hyers-Ulam stability.
Proof. Let $\epsilon > 0$ and $\psi \in C^2_{rd}([a, b]_T, X)$ satisfies

$$\|\psi^2(t) + p(t)\psi(t) - f(t)\| < \epsilon, \ t \in [a, b]_T.$$ 

Set $h(t) = \psi^2(t) + p(t)\psi(t) - f(t)$. Then $\psi$ satisfies the equation

$$\psi^2(t) + p(t)\psi(t) = f(t) + h(t), \ t \in [a, b]_T.$$ 

Let $a_i = \psi^i(a), i = 0, 1$. Hence $\psi$ satisfies

$$\psi(t) = a_0 + a_1(t - a) + \int_a^t [s - t + \mu(s)][p(s)\psi(s) - f(s) - h(s)] \Delta s.$$ 

There exists a unique solution $\phi$ of equation (1.1) satisfying $\phi^i(a) = a_i, i = 0, 1$. Equivalently, $\phi$ satisfies the integral equation

$$\phi(t) = a_0 + a_1(t - a) + \int_a^t [s - t + \mu(s)][p(s)\phi(s) - f(s)] \Delta s. \quad (2.10)$$

We have

$$\|\psi(t) - \phi(t)\| \leq \int_a^t |s - t + \mu(s)||h(s)||\Delta s + \int_a^t |s - t + \mu(s)||p(s)(\psi(s) - \phi(s))||\Delta s$$

$$\leq (b - a)(b - a + \|\mu\|_\infty)\epsilon + (b - a + \|\mu\|_\infty)\|p\|_\infty \int_a^t \|\psi(s) - \phi(s)||\Delta s.$$  

(2.11)

We denote by $K = (b - a + \|\mu\|_\infty)\|p\|_\infty$ and $M = \sup \{e_K(t, a) : t \in [a, b]_T\}$. Inequality (2.11) yields

$$\|\psi(t) - \phi(t)\| \leq \frac{(b - a)K}{\|p\|_\infty}\epsilon + K\int_a^t \|\psi(s) - \phi(s)||\Delta s.$$  

(2.12)

By Grönwal’s inequality [3], we deduce that

$$\|\psi(t) - \phi(t)\| \leq \frac{(b - a)KM}{\|p\|_\infty}\epsilon, \ t \in [a, b]_T.$$  

(2.13)

Therefore, equation (1.1) is Hyers-Ulam stable.

Since a regressive equation has a unique solution satisfying any initial conditions, [3], we get the following result

**Theorem 2.10.** If equation (1.1) is regressive, then it has Hyers-Ulam stability.
We combine theorems 2.6 and 2.9, to obtain a new sufficient condition for Hyers-Ulam stability of equation (1.1).

**Theorem 2.11.** If there is \( \alpha \in (0, 1) \) such that
\[
\int_a^t |p(s)| \Delta s \leq \frac{\alpha}{b - a + \|\mu\|_\infty}, \quad t \in [a, b]_T,
\]
then equation (1.1) has Hyers-Ulam stability.

We combine theorems 2.8 and 2.9 to obtain another sufficient condition for the Hyers-Ulam stability of the scalar equation (2.5).

**Theorem 2.12.** The scalar equation (2.5) has Hyers-Ulam stability if the corresponding Ricatti equation (2.4) has a solution \( z \) that satisfies \( 1 - \mu(t)z(t) \neq 0, t \in [a, b]_T \).

The following results are concerning with Hyers-Ulam-Rassias stability. Throughout the rest of the paper, we denote by
\[
\mathcal{M} = \{ \omega \in C_{rd}([a, b]_T, \mathbb{R}) : \omega \text{ is positive, } \int_a^t \omega^2(s) \Delta s \leq \omega^2(t) \}. \tag{2.14}
\]

**Theorem 2.13.** Assume that for any \( a_0, a_1 \in X \) equation (1.1) has a unique solution \( \phi \) that satisfies \( \phi^{\Delta^i}(a) = a_i, i = 0, 1 \). Then equation (1.1) has Hyers-Ulam-Rassias stability of type \( \mathcal{M} \).

**Proof.** Let \( \omega \in \mathcal{M} \) and \( \psi \in C^2_{rd}([a, b]_T, X) \) satisfies
\[
\|\psi^{\Delta^2}(t) + p(t)\psi(t) - f(t)\| < \omega(t), \quad t \in [a, b]_T.
\]
Set \( h(t) = \psi^{\Delta^2}(t) + p(t)\psi(t) - f(t) \). Then \( \psi \) satisfies the equation
\[
\psi^{\Delta^2}(t) + p(t)\psi(t) = f(t) + h(t), \quad t \in [a, b]_T.
\]

Hence it satisfies the integral equation
\[
\psi(t) = a_0 + a_1(t - a) + \int_a^t [s - t + \mu(s)][p(s)\psi(s) - f(s) - h(s)] \Delta s,
\]
where \( a_i = \psi^{\Delta^i}(a), i = 0, 1 \). There exists a unique solution \( \phi \) of equation (1.1). Equivalently, \( \phi \) satisfies the integral equation
\[
\phi(t) = a_0 + a_1(t - a) + \int_a^t [s - t + \mu(s)][p(s)\phi(s) - f(s)] \Delta s. \tag{2.15}
\]
In view of Hölder inequality [3], we have
\[
\|\psi(t) - \phi(t)\| \leq \int_a^t |s - t + \mu(s)||h(s)|\Delta s + \int_a^t |s - t + \mu(s)||p(s)||\psi(s) - \phi(s)|\Delta s
\]
\[
\leq (b - a + \|\mu\|_\infty) \int_a^t \omega(s)\Delta s + (b - a + \|\mu\|_\infty)\|p\|_\infty \int_a^t \|\psi(s) - \phi(s)\|\Delta s
\]
\[
\leq \sqrt{b - a} \left( b - a + \|\mu\|_\infty \right) \omega(t) + (b - a + \|\mu\|_\infty)\|p\|_\infty \int_a^t \|\psi(s) - \phi(s)\|\Delta s.
\]
(2.16)

We denote by
\[
K = (b - a + \|\mu\|_\infty)\|p\|_\infty,
\]
and
\[
L = \sup_{t \in [a,b]_T} \left( \int_a^b e_K^2(t, \sigma(s))\Delta s \right)^{\frac{1}{2}}.
\]

Inequality (2.16) yields
\[
\|\psi(t) - \phi(t)\| \leq \frac{\sqrt{b - a}K}{\|p\|_\infty} \omega(t) + K \int_a^t \|\psi(s) - \phi(s)\|\Delta s.
\]
(2.17)

By Grönwal’s inequality [3], we deduce that
\[
\|\psi(t) - \phi(t)\| \leq \frac{\sqrt{b - a}K}{\|p\|_\infty} \omega(t) + \frac{\sqrt{b - a}K^2}{\|p\|_\infty} \int_a^t e_K(t, \sigma(s))\omega(s)\Delta s,
\]
\[\quad t \in [a,b]_T.\]
(2.18)

Again by Hölder inequality, it follows that
\[
\|\psi(t) - \phi(t)\| \leq \frac{\sqrt{b - a}K}{\|p\|_\infty} \omega(t) + \frac{\sqrt{b - a}K^2L}{\|p\|_\infty} \left( \int_a^t \omega^2(s)\Delta s \right)^{\frac{1}{2}},
\]
\[\quad t \in [a,b]_T.\]
(2.19)

This implies that
\[
\|\psi(t) - \phi(t)\| \leq \frac{\sqrt{b - a}K}{\|p\|_\infty} \left( 1 + KL \right) \omega(t),
\]
\[\quad t \in [a,b]_T.\]
(2.20)

Therefore, equation (1.1) is Hyers-Ulam-Rassias stable of type $\mathcal{M}$. \hfill \Box

**Theorem 2.14.** If equation (1.1) is regressive, then it has Hyers-Ulam-Rassias stability of type $\mathcal{M}$.

We combine theorem 2.6 and Theorem 2.13, to obtain a new sufficient condition for Hyers-Ulam-Rassias stability of equation (1.1).
**Theorem 2.15.** If there is $\alpha \in (0, 1)$ such that

$$
\int_a^t |p(s)| \Delta s \leq \frac{\alpha}{b - a + \|\mu\|_\infty}, \quad t \in [a, b)_T,
$$

then equation (1.1) has Hyers-Ulam-Rassias stability of type $\mathcal{M}$.

We combine theorems 2.8 and 2.13 to get another sufficient condition for Hyers-Ulam-Rassias stability of the scalar equation (2.5).

**Theorem 2.16.** The scalar equation (2.5) has Hyers-Ulam-Rassias stability of type $\mathcal{M}$ if the corresponding Ricatti equation (2.4) has a solution $z$ that satisfies $1 - \mu(t)z(t) \neq 0, t \in [a, b]_T$.

**Remark 2.17.** Theorems 2.13-2.16 hold if we replace $\mathcal{M}$ by

$$
\mathcal{K} = \{\omega \in C_{rd}([a, b]_T, \mathbb{R}) : \omega \text{ is non-negative, } \int_a^t \omega(s) \Delta s \leq \omega(t)\}. \quad (2.21)
$$

**REFERENCES**


